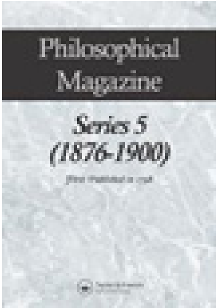

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XXXIII. On maintained vibrations

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XXXIII. *On Maintained Vibrations.* *By* Lord RAYLEIGH, *F.R.S., Professor of Experimental Physics in the University of Cambridge**.

WHEN a vibrating system is subject to dissipative forces, the vibrations cannot be permanent, since they are dependent upon an initial store of energy which suffers gra dual exhaustion. In the usual equation

$$\frac{d^2\theta}{dt^2}+\kappa\frac{d\theta}{dt}+n^2\theta=0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

κ is positive, and the solution indicates the progressive decay of the vibrations in accordance with the exponential law. In order that the vibrations may be maintained, the vibrating body must be in connexion with a source of energy. This condition being satisfied, two principal classes of maintained vibrations may be distinguished. In the first class the magnitude of the force acting upon the body in virtue of its connexion with the source of energy is proportional to the amplitude, and its phase depends in an approximately constant manner upon the phase of the vibration itself; in the second class the body is subject to influences whose phase is independently determined.

The first class is by far the more extensive, and includes

* Communicated by the Author.

vibrations maintained by wind (organ-pipes, harmonium-reeds, æolian harps, &c.), by heat (singing flames, Rijke's tubes, &c.), by friction (violin-strings, finger-glasses, &c.), as well as the slower vibrations of clock-pendulums and of electromagnetic tuning-forks. When the amplitude is small, the force acting upon the body may be divided into two parts, one proportional to the displacement θ (or to the acceleration), the second proportional to the velocity $d\theta/dt$. The inclusion of these forces does not alter the *form* of (1). By the first part (proportional to θ) the pitch is modified, and by the second the coefficient of decay*. If the altered κ be still positive, vibrations gradually die down; but if the effect of the included forces be to render the complete value of κ negative, vibrations tend on the contrary to increase. The only case in which according to (1) a steady vibration is possible, is when the complete value of κ is zero. If this condition be satisfied, a vibration of any amplitude is permanently maintained.

When κ is negative, so that small vibrations tend to increase, a point is of course soon reached after which the approximate equations cease to be applicable. We may form an idea of the state of things which then arises by adding to equation (1) a term proportional to a higher power of the velocity. Let us take

$$\frac{d^2\theta}{dt^2}+\kappa\frac{d\theta}{dt}+\kappa'\left(\frac{d\theta}{dt}\right)^3+n^2\theta=0, \quad . \quad . \quad . \quad . \quad (2)$$

in which κ and κ' are supposed to be small. The approximate solution of (2) is

$$\theta=\mathrm{A}\sin nt+\frac{\kappa' n\mathrm{A}^3}{32}\cos 3nt, \quad . \quad . \quad . \quad . \quad (3)$$

in which A is given by

$$\kappa+\tfrac{3}{4}\kappa' n^2\mathrm{A}^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (4) we see that no steady vibration is possible unless κ and κ' have different signs. If κ and κ' be both positive, the vibration in all cases dies down; while if κ and κ' be both negative, the vibration (according to (2)) increases without limit. If κ be negative and κ' positive, the vibration becomes steady and assumes the amplitude determined by (4). A smaller vibration increases up to this point, and a larger vibration falls down to it. If, on the other hand, κ be positive, while κ' is negative, the steady vibration abstractedly possible is

* For more detailed application of this principle to certain cases of maintained vibrations, see Proceedings of the Royal Institution, March 15, 1878.

unstable, a departure in either direction from the amplitude given by (4) tending always to increase.

Of the second class the vibrations commonly known as *forced* have the first claim upon our attention. The theory of these vibrations has long been well understood, and depends upon the solution of the differential equation formed by writing as the right-hand member of (1) P cos pt in place of zero. The period of steady vibration is coincident with that of the force, and independent of the natural period of vibration; but the amplitude of vibration is greatly increased by a near agreement between the two periods. In all cases the amplitude is definite and is proportional to the magnitude of the impressed force. When the force, though strictly periodic, is not of the simple harmonic type, vibrations may be maintained by its operation whose period is a submultiple of the principal period.

There is also another kind of maintained vibration which from one point of view may be considered to be forced, inasmuch as the period is imposed from without, but which differs from the kind just referred to in that the imposed periodic variations do not tend directly to displace the body from its configuration of equilibrium. Probably the best-known example of this kind of action is that form of Melde's experiment in which a fine string is maintained in transverse vibration by connecting one of its extremities with the vibrating prong of a massive tuning-fork, *the direction of motion of the point of attachment being parallel to the length of the string**. The effect of the motion is to render the tension of the string periodically variable; and at first sight there is nothing to cause the string to depart from its equilibrium condition of straightness. It is known, however, that under these circumstances the equilibrium position may become unstable, and that the string may settle down into a state of permanent and vigorous vibration, *whose period is the double of that of the point of attachment*†.

The theory of vibrations of this kind presents some points of difficulty, and does not appear to have been treated hitherto. In the present investigation we shall start from the assumption that a steady vibration is in progress, and inquire under what circumstances the assumed state of things is possible.

If the force of restitution, or "spring," of a body susceptible of vibration be subject to an imposed periodic variation,

* When the direction of motion is transverse, the case falls under the head of ordinary forced vibrations.

† See Tyndall's 'Sound,' 3rd ed. ch. iii. § 7, where will also be found a general explanation of the mode of action.

the differential equation becomes

$$\frac{d^2\theta}{dt^2}+\kappa\frac{d\theta}{dt}+(n^2-2\alpha\sin 2pt)\theta=0, \quad . \quad . \quad . \quad (5)$$

in which κ and α are supposed to be small. A similar equation would apply approximately in the case of a periodic variation in the effective mass of the body. The motion expressed by the solution of (5) can only be regular when it keeps perfect time with the imposed variations. It will appear that the necessary conditions cannot be satisfied rigorously by any simple harmonic vibration; but we may assume

$$\theta=A_1\sin pt+B_1\cos pt+A_3\sin 3pt+B_3\cos 3pt$$
$$+A_5\sin 5pt+\ldots, \quad . \quad . \quad . \quad (6)$$

in which it is not necessary to provide for sines and cosines of even multiples of pt. If the assumption is justifiable, the series in (6) must be convergent. Substituting in the differential equation, and equating to zero the coefficients of $\sin pt$, $\cos pt$, &c., we find

$$A_1(n^2-p^2)-\kappa pB_1-\alpha B_1-\alpha B_3=0,$$
$$B_1(n^2-p^2)+\kappa pA_1-\alpha A_1-\alpha A_3=0,$$

$$A_3(n^2-9p^2)-3\kappa pB_3-\alpha B_1+\alpha B_5=0,$$
$$B_3(n^2-9p^2)+3\kappa pA_3+\alpha A_1-A_5=0,$$

$$A_5(n^2-25p^2)-5\kappa pB_5-\alpha B_3+\alpha B_7=0.$$
$$B_5(n^2-25p^2)+5\kappa pA_5+\alpha A_3-\alpha A_7=0,$$

. . . .

These equations show that relatively to A_1, B_1, A_3, B_3 are of the order α; that relatively to A_3, B_3, A_5, B_5 are of the order α, and so on. If we omit A_3, B_3 in the first pair of equations, we find as a first approximation,

$$A_1(n^2-p^2)-(\kappa p+\alpha)B_1=0,$$

$$A_1(\kappa p-\alpha)+(n^2-p^2)B_1=0;$$

whence

$$\frac{B_1}{A_1}=\frac{n^2-p^2}{\kappa p+\alpha}=\frac{\alpha-\kappa p}{n^2-p^2}=\frac{\sqrt{(\alpha-\kappa p)}}{\sqrt{(\alpha+\kappa p)}}, \quad . \quad . \quad . \quad (7)$$

and

$$(n^2-p^2)^2=\alpha^2-\kappa^2p^2. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus, if α be given, the value of p necessary for a regular motion is definite; and p having this value, the regular motion is

$$\theta=P\sin(pt+\epsilon),$$

in which ϵ, being equal to $\tan^{-1}(B_1/A_1)$, is also definite. On the other hand, as is evident at once from the linearity of the original equation, there is nothing to limit the amplitude of vibration.

These characteristics are preserved however far it may be necessary to pursue the approximation. If A_{2m+1}, B_{2m+1}, may be neglected, the first m pairs of equations determine the *ratios* of all the coefficients, leaving the absolute magnitude open; and they provide further an equation connecting p and α, by which the pitch is determined.

For the second approximation the second pair of equations gives

$$A_3 = \frac{\alpha B_1}{n^2 - 9p^2}, \qquad B_3 = -\frac{\alpha A_1}{n^2 - 9p^2},$$

whence

$$\theta = P \sin(pt + \epsilon) + \frac{\alpha P}{9p^2 - n^2} \cos(3pt + \epsilon); \quad . \quad (9)$$

and from the first pair

$$\tan \epsilon = \left\{ n^2 - p^2 + \frac{\alpha^2}{n^2 - 9p^2} \right\} \div (\alpha + \kappa p), \quad . \quad (10)$$

while p is determined by

$$(n^2 - p^2)^2 - \frac{\alpha^4}{(n^2 - 9p^2)^2} = \alpha^2 - \kappa^2 p^2. \quad . \quad . \quad . \quad . \quad (11)$$

Returning to the first approximation, we see from (8) that the solution is only possible under the condition that $\alpha > \kappa p$. If $\alpha = \kappa p$, then $p = n$; i.e. the imposed variation in the "spring" must be exactly twice as quick as the natural vibration of the body would be in the absence of friction. From (7) it appears that in this case $\epsilon = 0$, which indicates that the spring is a minimum one eighth of a period *after* the body has passed its position of equilibrium, and a maximum one eighth of a period *before* such passage. Under these circumstances the greatest possible amount of energy is communicated to the system; and in the case contemplated it is just sufficient to balance the loss by dissipation, the adjustment being evidently independent of the amplitude.

If $\alpha < \kappa p$, sufficient energy cannot pass to maintain the motion, whatever may be the phase-relation; but if $\alpha > \kappa p$, the equality between energy supplied and energy dissipated may be attained by such an alteration of phase as shall diminish the former quantity to the required amount. The alteration of phase may for this purpose be indifferently in either direction; but if ϵ be positive, we must have

$$p^2 = n^2 - \sqrt{\{\alpha^2 - \kappa^2 p^2\}};$$

while if ϵ be negative,

$$p^2=n^2+\sqrt{\{\alpha^2-\kappa^2p^2\}}.$$

If α be very much greater than κp, $\epsilon=\pm\frac{1}{4}\pi$, which indicates that when the system passes through its position of equilibrium the spring is at its maximum or at its minimum.

The inference from the equations that the adjustment of pitch must be absolutely rigorous for steady vibration will be subject to some modification in practice; otherwise the experiment could not succeed. In most cases n^2 is to a certain extent a function of amplitude; so that if n^2 have very nearly the required value, complete coincidence is attainable, without other alteration in the conditions of the system, by the assumption of an amplitude of large and determinate amount.

When a particular solution of (5) has been found, it may be generalized by a known method. Thus, if $\theta=A\theta_1$, we have as the complete solution

$$\theta=A\theta_1+B\theta_1\int_0^t\theta_1^{-2}e^{-\kappa t}dt.$$

which may be put into the form

$$\theta=P\theta_1-B\theta_1\int_t^\infty\theta_1^{-2}e^{-\kappa t}dt. \quad . \quad . \quad . \quad (12)$$

When t is great, the second term diminishes rapidly, and the solution tends to assume the original form $\theta=P\theta_1$.

The number of cases falling under the present head which have been discovered and examined hitherto is not great. The mysterious *son rauque* of Savart, which sometimes accompanies the longitudinal vibrations of bars, and is attributed by Terquem to an associated transverse vibration, is doubtless of this character. Just as in Melde's experiment already spoken of, the periodic variations of tension accompanying the longitudinal vibrations will throw the bar into lateral vibration, if there happen to be a mode of such vibration whose pitch is nearly enough coincident with the *suboctave* of the principal note.

For a lecture illustration we may take a pendulum formed of a bar of soft iron and vibrating on knife-edges. Underneath the pendulum is placed symmetrically a vertical bar electromagnet, through which is caused to pass an electric current rendered intermittent by an interrupter whose frequency is twice that of the pendulum. The magnetic force does not tend to displace the pendulum from its equilibrium position, but produces the same sort of effect as if gravity were subject to a periodic variation.

A similar result is obtained by causing the point of support of the pendulum to vibrate in a *vertical* path. If we denote this motion by $\eta=\beta\sin 2pt$, the effect is as if gravity were variable by the term $4p^2\beta\sin 2pt$. Of the same nature are the crispations observed by Faraday and others on the surface of water which oscillates vertically. Faraday arrived experimentally at the conclusion that there were two complete vibrations of the support for each complete vibration of the liquid. This view has been contested by Matthiessen*, who maintains that the vibrations are isoperiodic. By observations, which I hope to find another opportunity of detailing, I have convinced myself that in this matter Faraday was perfectly correct. The vibrations of water standing upon a horizontal glass plate, which was attached to the centre of a vibrating iron bar, were at the rate of 15 per second when the vibrations of the bar were at the rate of 30 per second. The only difference of importance between this case and that of the pendulum is that, whatever may be the rate of vibration of the plate, there is always possible a free water-vibration of nearly the same frequency, and that consequently no special tuning is called for.

XXXIV. *On a Measurement of Wave-lengths in the Ultra-red Region of the Spectrum of the Sun. By* ERNST PRINGSHEIM†.

IN order to investigate the solar spectrum it is of special importance to know the wave-length of the extreme rays emitted by the sun, and thus to determine the extent of the entire spectrum. In order to determine the wave-length of the least-refrangible rays, Müller‡, and subsequently Lamansky§, observed with the aid of a thermopile the index of refraction of the extreme rays of a spectrum projected by a crown-glass or rock-salt prism, and from that index calculated the wave-length by means of an empirical formula, the correctness of which was controllable only within the limits of the visible rays. The untrustworthiness of this method is obvious; thus, from the same observation Müller calculated, by two different formulæ, for the extreme wave-length the values 0·00177 and 0·0048 millim.

A trustworthy determination of the wave-length is possible only with the aid of the interference of the rays; and this course was taken by Abney‖, who succeeded in photographing

* Pogg. *Ann.* cxli. 1870.
† Translated from Wiedemann's *Annalen*, 1883, No. 1, pp. 32–45.
‡ Pogg. *Ann.* cv. p. 352 (1858). § Ibid. cxlvi. 1872.
‖ Phil. Trans. 1880, pp. 653–667.