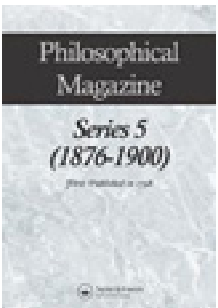

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X. On the electromagnetic theory of light

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X. *On the Electromagnetic Theory of Light.* *By* Lord Rayleigh, *F.R.S., Professor of Experimental Physics in the University of Cambridge**.

THE claims of the theory propounded by Maxwell, according to which light consists of a disturbance in a medium susceptible of dielectric polarization, are so considerable that it is desirable to extend its application as far as possible to various optical phenomena. The question of the velocity of propagation in vacuum and in singly or doubly refracting transparent dielectrics was considered by Maxwell himself; and the agreement with experiment, though far from perfect, is sufficiently encouraging. More recently it has been shown by Helmholtz †, Lorentz ‡, Fitzgerald §, and J. J. Thomson ||, that the same theory leads to Fresnel's expressions for the intensity of light reflected and refracted at the surface of separation of transparent media, and that the auxiliary hypotheses necessary in this part of the subject agree with those required to explain the laws of double refraction. In this respect the electromagnetic theory has a marked advantage over the older view, which assimilated luminous vibrations to the ordinary transverse vibrations of elastic solids. According to the latter, Fresnel's laws of double refraction, fully confirmed by modern

* Communicated by the Author.
† Crelle, Bd. lxxii., 1870.
‡ Schlömilch, xxii., 1877.
§ Phil. Trans. 1880.
|| Phil. Mag. April 1880.

observation*, require us to suppose that in a doubly-refracting crystal the rigidity of the medium varies with the direction of the strain; while, in order to explain the facts relating to the intensities of reflected light, we have to make the inconsistent assumption that the rigidity does not vary in passing from one medium to another. A further discussion of this subject will be found in papers published in the Philosophical Magazine during the year 1871.

If the dielectric medium be endowed with sensible conductivity, the electric vibrations will be damped; that is to say, the light will undergo absorption, with a rapidity which Maxwell has calculated. By supposing the conductivity to be so great that practically complete absorption takes place within a distance comparable with the wave-length, we may obtain a theory of metallic reflection which is not without interest, although the phenomena of abnormal dispersion show that it cannot be regarded as complete.

For an isotropic medium at rest we have the equations (Maxwell's 'Electricity and Magnetism,' §§ 591, 598, 607, 610, 611)

$$u=p+\frac{df}{dt}, \text{ \&c.}, \qquad (1)$$

$$f=\frac{K}{4\pi}P, \text{ \&c.}, \qquad (2)$$

$$p=CP, \text{ \&c.}, \qquad (3)$$

$$P=-\frac{dF}{dt}-\frac{d\Psi}{dx}, \text{ \&c.}, \qquad (4)$$

$$a=\frac{dH}{dy}-\frac{dG}{dz}, \text{ \&c.}, \qquad (5)$$

$$a=\mu\alpha, \text{ \&c.}, \qquad (6)$$

$$4\pi u=\frac{d\gamma}{dy}-\frac{d\beta}{dz}, \text{ \&c.}; \qquad (7)$$

in which f, g, h are the electric displacements, p, q, r the currents of conduction, u, v, w the total currents, P, Q, R the components of electromotive force, K the specific inductive capacity, C the conductivity, α, β, γ the components of magnetic force, a, b, c the components of magnetization, μ the magnetic capacity, F, G, H the components of electrokinetic momentum, and Ψ the electric potential.

* Glazebrook, Phil. Trans. 1879.

From (2), (4), and (5) we get

$$4\pi\left(\frac{d}{dy}\frac{f}{K}-\frac{d}{dx}\frac{g}{K}\right)=-\frac{d}{dt}\left(\frac{dF}{dy}-\frac{dG}{dx}\right)=\frac{dc}{dt}\ \&c. \quad . \quad (8)$$

In the case of K constant, equation (8) expresses that the electric displacement $\int(f\,dx+g\,dy)$ round a small circuit in the plane of xy corresponds to the electromotive force round the circuit, represented by dc/dt.

Again, from (1), (2), (3), (6), (7),

$$4\pi\left(\frac{df}{dt}+\frac{4\pi C}{K}f\right)=\frac{d}{dy}\frac{c}{\mu}-\frac{d}{dz}\frac{b}{\mu}\ \&c. \quad . \quad . \quad . \quad (9)$$

From equations (8) and (9) the problem of reflection can be investigated. In order to limit ourselves to plane waves of simple type, we shall suppose that K, μ, and C are independent of z, and that the electric and magnetic functions are independent of z and (as dependent upon the time) proportional to e^{int}. The two principal cases will be considered separately, (1) when the electric displacements are perpendicular to the plane of incidence, (2) when they are executed in that plane.

Case 1. This is defined by the conditions

$$f=0,\quad g=0,\ \text{and (accordingly)}\ c=0.$$

Thus

$$ina=-4\pi\frac{d}{dy}\frac{h}{K},\quad inb=4\pi\frac{d}{dx}\frac{h}{K}, \quad . \quad . \quad (10)$$

$$4\pi\left(in+\frac{4\pi C}{K}\right)h=\frac{d}{dx}\frac{b}{\mu}-\frac{d}{dy}\frac{a}{\mu}. \quad . \quad . \quad . \quad (11)$$

Eliminating a and b from (10) and (11), we get

$$\frac{d}{dx}\left(\frac{1}{\mu}\frac{d}{dx}\right)\frac{h}{K}+\frac{d}{dy}\left(\frac{1}{\mu}\frac{d}{dy}\right)\frac{h}{K}+n^2K\left(1-in\frac{4\pi C}{K}\right)\frac{h}{K}=0. \quad (12)$$

Case 2. Here the special conditions are

$$h=0,\quad a=0,\quad b=0.$$

We have

$$4\pi\left(\frac{d}{dy}\frac{f}{K}-\frac{d}{dx}\frac{g}{K}\right)=inc, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$4\pi\left(in+\frac{4\pi C}{K}\right)f=\frac{d}{dy}\left(\frac{c}{\mu}\right),\quad 4\pi\left(in+\frac{4\pi C}{K}\right)g=-\frac{d}{dx}\left(\frac{c}{\mu}\right); \quad (14)$$

whence by elimination of f and g,

$$\frac{d}{dx}\left\{\frac{1}{Kn^2(1-4\pi n^{-1}CK^{-1})}\frac{d}{dx}\left(\frac{c}{\mu}\right)\right\}$$

$$+\frac{d}{dy}\left\{\frac{1}{Kn^2(1-4\pi n^{-1}CK^{-1})}\frac{d}{dy}\left(\frac{c}{\mu}\right)\right\}+\mu\left(\frac{c}{\mu}\right)=0. \quad (15)$$

Equations (12) and (15) simplify considerably in their application to a uniform medium, assuming the common form

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+n^2\mu K(1-4\pi nCK^{-1})=0. \quad . \quad (16)$$

To express the boundary conditions let us suppose that $x=0$ is the surface of transition between two uniform media. From (12) we learn that the required conditions for case 1 are that

$$\frac{h}{K} \text{ and } \frac{1}{\mu}\frac{d}{dx}\left(\frac{h}{K}\right)$$

must be continuous.

In like manner, for case 2 we see from (15) that

$$\frac{c}{\mu} \text{ and } \frac{1}{K(1-4\pi n^{-1}CK^{-1})}\frac{d}{dx}\left(\frac{c}{\mu}\right)$$

must be continuous.

If the media are transparent, or but moderately opaque, we have to put $C=0$. The differential equation is of the form

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+n^2\mu K=0. \quad . \quad . \quad . \quad . \quad (17)$$

In case 1 the boundary conditions are the continuity of the dependent variable and of $\frac{1}{\mu}\frac{d}{dx}$, and in case 2 the continuity of the dependent variable and of $\frac{1}{K}\frac{d}{dx}$. Analytically, the results are thus of the same form in both cases. If θ and θ_1 are respectively the angles of incidence and refraction, the ratio of the reflected to the incident vibration is in case 1

$$\frac{\dfrac{\tan\theta_1}{\tan\theta}-\dfrac{\mu}{\mu_1}}{\dfrac{\tan\theta_1}{\tan\theta}+\dfrac{\mu}{\mu_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and in case 2

$$\frac{\dfrac{\tan\theta_1}{\tan\theta}-\dfrac{K}{K_1}}{\dfrac{\tan\theta_1}{\tan\theta}+\dfrac{K}{K_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

in which K, μ relate to the first, and K_1, μ_1 to the second medium; while the relation between θ_1 and θ is

$$K_1\mu_1 : K\mu = \sin^2\theta : \sin^2\theta_1. \quad . \quad . \quad . \quad . \quad (20)$$

As Helmholtz has remarked, Fresnel's formulæ may be obtained on two distinct suppositions. If $\mu_1=\mu$,

$$(18) = \frac{\sin(\theta_1-\theta)}{\sin(\theta_1+\theta)},$$

and

$$(19) = \frac{\tan(\theta_1-\theta)}{\tan(\theta_1+\theta)};$$

but if $K_1=K$, then (19) identifies itself with the sine-formula, and (18) with the tangent-formula. Electrical phenomena, however, lead us to prefer the former alternative, and thus to the assumption that the *electric* displacements are perpendicular to the plane of polarization. The formulæ for the refracted waves, which follow from those of the reflected waves in virtue of the principle of energy alone, do not call for detailed consideration.

In the problem of perpendicular incidence, we have from (12), if μ be constant and C zero,

$$\frac{d^2}{dx^2}\frac{h}{K}+n^2\mu K\left(\frac{h}{K}\right)=0. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

For an application of this equation to determine the influence of defective suddenness in the transition between two uniform media, the reader is referred to a paper in the eleventh volume of the Proceedings of the Mathematical Society.

In order to obtain a theory of metallic reflection, C must be considered to have a finite value in the second medium. The symbolical solution is not thereby altered from that applicable to transparent media, the effect of the finiteness of C being completely represented in both cases by the substitution of $K(1-i4\pi nCK^{-1})$ for K. Thus, if μ be constant, the formula for the amplitude and phase of the reflected wave in case 1 is to be found by transformation of (18), in which the imaginary angle of refraction θ_1 is connected with θ by the relation

$$K_1(1-i4\pi nCK_1^{-1}) : K = \sin^2\theta : \sin^2\theta_1. \quad . \quad . \quad (22)$$

In like manner the solution for case 2 is to be found by transformation of (19) under the same supposition.

With regard to the proposed transformations, the reader is referred to a paper by Eisenlohr* and to some remarks thereupon by myself†. The results are the formulæ published without proof by Cauchy. From the calculations of Eisenlohr it appears that Jamin's observations cannot be reconciled with the formulæ without supposing $K_1 : K$, *i. e.* the real part of the square of the complex refractive index, to be negative—a

* Pogg. *Ann.* t. civ. p. 368. † Phil. Mag. May 1872.

further proof that much remains to be done before the electrical theory of metallic reflection can be accepted as complete*.

The same fundamental equations (8) and (9) will now be applied to the problem of determining the effect on a train of plane waves of a small variation in the quantities K and μ which define the medium. A similar method will be adopted to that already used for light in a paper "On the Scattering of Light by small Particles"†, and in my book 'On the Theory of Sound,' § 296, the principle of which consists in an approximation depending upon the neglect of the higher powers of the small variations ΔK and $\Delta\mu$.

Let us suppose that a train of plane waves, in which the electric displacement is parallel to z, and magnetization parallel to y, propagates itself parallel to x undisturbed until it falls upon a region where the generally constant values of K and μ become $K+\Delta K$ and $\mu+\Delta\mu$. If ΔK and $\Delta\mu$ were zero, the wave would pass on as before; but under the circumstances secondary waves are generated, which diverge from the region of disturbance, and are ultimately, when ΔK and $\Delta\mu$ are small enough, proportional in magnitude to these quantities. As the expression of the primary waves we may take

$$h_0 = e^{int} e^{ikx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and corresponding thereto, by (8),

$$b_0 = 4\pi k n^{-1} K^{-1} e^{int} e^{ikx}, \quad . \quad . \quad . \quad . \quad (24)$$

in which, if λ denote the wave-length, $k=2\pi/\lambda$, and n/λ is the velocity of propagation $(K\mu)^{-\frac{1}{2}}$. The complete values of the functions being represented, as before, by f, g, h, a, b, c, we shall put

$$f = f_0 + f_1 + f_2 + \ldots \&c., \quad a = a_0 + a_1 + \ldots \&c.,$$

$f_0 \ldots a_0 \ldots$ being independent of ΔK and $\Delta\mu$, $f_1 \ldots a_1 \ldots$ being of the first order, $f_2 \ldots a_2 \ldots$ of the second order, and so on, in these quantities. In the actual case f_0, g_0, a_0, c_0 vanish, and only h_0 and b_0 are finite.

From (8) and (9) with $C=0$, we get

* July 15.—I see that Lorentz, in a pamphlet *Over de Theorie der Terugkaatsing en Breking van het Licht* (Arnhem, 1875), has developed a theory of metallic reflection similar to that indicated in the text, and has noticed the same difficulty in the application to experiment.

† Phil. Mag. June 1871.

$$\left.\begin{aligned}4\pi\left\{\frac{df}{dy}-\frac{dg}{dx}+\mathrm{K}\frac{d}{dy}(f\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d}{dx}(g\Delta\mathrm{K}^{-1})\right\}&=\mathrm{K}\frac{dc}{dt},\\4\pi\left\{\frac{dg}{dz}-\frac{dh}{dy}+\mathrm{K}\frac{d}{dz}(g\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d}{dy}(h\Delta\mathrm{K}^{-1})\right\}&=\mathrm{K}\frac{da}{dt},\\4\pi\left\{\frac{dh}{dx}-\frac{df}{dz}+\mathrm{K}\frac{d}{dx}(h\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d}{dz}(f\Delta\mathrm{K}^{-1})\right\}&=\mathrm{K}\frac{db}{dt}.\end{aligned}\right\}\quad(25)$$

$$\left.\begin{aligned}\frac{dc}{dy}-\frac{db}{dz}+\mu\frac{d}{dy}(c\Delta\mu^{-1})-\mu\frac{d}{dz}(b\Delta\mu^{-1})&=4\pi\mu\frac{df}{dt},\\\frac{da}{dz}-\frac{dc}{dx}+\mu\frac{d}{dz}(a\Delta\mu^{-1})-\mu\frac{d}{dx}(c\Delta\mu^{-1})&=4\pi\mu\frac{dg}{dt},\\\frac{db}{dx}-\frac{da}{dy}+\mu\frac{d}{dx}(b\Delta\mu^{-1})-\mu\frac{d}{dy}(a\Delta\mu^{-1})&=4\pi\mu\frac{dh}{dt}.\end{aligned}\right\}\quad .\quad(26)$$

By differentiation of the first equation of (26) and substitution from (25), we get, having regard to

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

which is a consequence of (1), (3), (7),

$$\begin{aligned}\mu\mathrm{K}\frac{d^2f}{dt^2}=\frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}+\frac{d^2f}{dz^2}+\mathrm{K}\left(\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)(f\Delta\mathrm{K}^{-1})\\-\mathrm{K}\frac{d^2}{dx\,dy}(g\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d^2}{dx\,dz}(h\Delta\mathrm{K}^{-1})\\+\frac{\mu\mathrm{K}}{4\pi}\frac{d^2}{dy\,dt}(\mathrm{C}\Delta\mu^{-1})-\frac{\mu\mathrm{K}}{4\pi}\frac{d^2}{dz\,dt}(b\Delta\mu^{-1}),\end{aligned}$$

or, remembering that the functions as dependent upon time vary as e^{int},

$$\begin{aligned}\nabla^2f+k^2f+\mathrm{K}\left(\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)(f\Delta\mathrm{K}^{-1})\\-\mathrm{K}\frac{d^2}{dx\,dy}(g\Delta\mathrm{K}^{-1})-\mathrm{K}\frac{d^2}{dx\,dz}(h\Delta\mathrm{K}^{-1})\\+\frac{in\mu\mathrm{K}}{4\pi}\frac{d}{dy}(c\Delta\mu^{-1})-\frac{in\mu\mathrm{K}}{4\pi}\frac{d}{dz}(b\Delta\mu^{-1})=0,\quad(28)\end{aligned}$$

with two similar equations in g and h.

Introducing now the expansion in powers of $\Delta\mathrm{K}$ and $\Delta\mu$, we get as the first approximation

$$\nabla^2f_1+k^2f_1-\mathrm{K}\frac{d^2}{dx\,dz}(h_0\Delta\mathrm{K}^{-1})-\frac{in\mu\mathrm{K}}{4\pi}\frac{d}{dz}(b_0\Delta\mu^{-1})=0,$$

or, on substitution for b_0 in terms of h_0 from (23), (24),

$$\nabla^2 f_1 + k^2 f_1 - \mathrm{K}\frac{d^2}{dx\,dz}(h_0\Delta\mathrm{K}^{-1}) - ik\mu\frac{d}{dz}(h_0\Delta\mu^{-1}) = 0, \quad (29)$$

and

$$\nabla^2 g_1 + k^2 g_1 - \mathrm{K}\frac{d^2}{dy\,dz}(h_0\Delta\mathrm{K}^{-1}) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\nabla^2 h_1 + k^2 h_1 + \mathrm{K}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)(h_0\Delta\mathrm{K}^{-1}) + ik\mu\frac{d}{dx}(h_0\Delta\mu^{-1}) = 0. \quad . \quad . \quad . \quad (31)$$

The solution of (29) is

$$f_1 = -\frac{\mathrm{K}}{4\pi}\iiint \frac{e^{-ikr}}{r}\frac{d^2}{dx\,dz}(h_0\Delta\mathrm{K}^{-1})\,dx\,dy\,dz - \frac{ik\mu}{4\pi}\iiint \frac{e^{-ikr}}{r}\frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\,dx\,dy\,dz, \quad . \quad . \quad (32)$$

where r, equal to $\sqrt{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}}$, is the distance of the element of volume $dx\,dy\,dz$ from the point α, β, γ at which f_1 is to be estimated.

In applying (32) to the calculation of a secondary wave at a distance from the region of disturbance, we may conveniently integrate it by parts. Thus,

$$f_1 = -\frac{\mathrm{K}}{4\pi}\iiint h_0\Delta\mathrm{K}^{-1}\frac{d^2}{dx\,dz}\left(\frac{e^{-ikr}}{r}\right)dx\,dy\,dz + \frac{ik\mu}{4\pi}\iiint h_0\Delta\mathrm{K}^{-1}\frac{d}{dz}\left(\frac{e^{-ikr}}{r}\right)dx\,dy\,dz.$$

From the general value of r,

$$\frac{d}{dz}\left(\frac{e^{-ikr}}{r}\right) = \frac{\gamma-z}{r}\,\frac{e^{-ikr}(1+ikr)}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\frac{d^2}{dx\,dz}\left(\frac{e^{-ikr}}{r}\right) = \frac{\alpha-x}{r}\,\frac{\gamma-z}{r}\,\frac{e^{-ikr}(3+3ikr-k^2r^2)}{r^3}. \quad . \quad . \quad (34)$$

If r be sufficiently great in comparison with λ, only the highest power of kr in the above expressions need be retained; and if r be also great in comparison with the dimensions of the region of disturbance, supposed to be situated about the origin of coordinates, $(\alpha-x)/r$ &c. may be replaced by α/r &c. Thus,

$$\frac{d}{dz}\left(\frac{e^{-ikr}}{r}\right) = \frac{\gamma}{r}\,\frac{ike^{-ikr}}{r};$$

$$\frac{d^2}{dz\,dx}\left(\frac{e^{-ikr}}{r}\right) = -\frac{\alpha\gamma}{r^2}\,\frac{k^2e^{-ikr}}{r};$$

and the expression for f_1 becomes

$$f=\frac{k^2}{4\pi r}\Big[\mathrm{K}\frac{\alpha\gamma}{r^2}\iiint h_0\Delta\mathrm{K}^{-1}e^{-ikr}\,dx\,dy\,dz$$
$$-\mu\frac{\gamma}{r}\iiint h_0\,\Delta\mu^{-1}e^{-ikr}\,dx\,dy\,dz\Big].$$

For the sake of brevity we will write this

$$f_1=\frac{k^2}{4\pi r}\Big[\mathrm{KP}\frac{\alpha\gamma}{r^2}-\mu\mathrm{Q}\frac{\gamma}{r}\Big], \quad . \quad . \quad . \quad (35)$$

where

$$\left.\begin{aligned}\mathrm{P}&=\iiint h_0\Delta\mathrm{K}^{-1}e^{-ikr}\,dx\,dy\,dz,\\ \mathrm{Q}&=\iiint h_0\Delta\mu^{-1}e^{-ikr}\,dx\,dy\,dz.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (36)$$

In like manner from (30) and (31),

$$g_1=\frac{k^2}{4\pi r}\Big[\mathrm{KP}\frac{\beta\gamma}{r^2}\Big], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$h_1=\frac{k^2}{4\pi r}\Big[-\mathrm{KP}\frac{\alpha^2+\beta^2}{r^2}+\mu\mathrm{Q}\frac{\alpha}{r}\Big]. \quad . \quad . \quad . \quad (38)$$

Equations (35), (37), (38) express the electric displacement in the secondary waves. Since $\alpha f+\beta g+\gamma h=0$, the displacement is perpendicular to the direction of the secondary ray. The general expression for the intensity is found by adding the squares of f, g, h; but it will be sufficient for our present purpose to limit ourselves to the case where the secondary ray is perpendicular to the primary ray, *i. e.* to the case $\alpha=0$. Then

$$f^2+g^2+h^2=\frac{k^4}{16\pi^2r^2}\Big[\mathrm{K}^2\mathrm{P}^2\frac{\beta^2}{r^2}+\mu^2\mathrm{Q}^2\frac{\gamma^2}{r^2}\Big]. \quad . \quad . \quad (39)$$

If P and Q are both finite, there is no direction along which the secondary light vanishes. We find by experiment, however, that the light scattered by small particles on which polarized light impinges does vanish in one direction perpendicular to the original ray; and thus either P or Q must vanish. Now, when the particles are very small, we have

$$\mathrm{P}=h_0\Delta\mathrm{K}^{-1}e^{-ikr}\iiint dx\,dy\,dz,\ \mathrm{Q}=h_0\Delta\mu^{-1}e^{-ikr}\iiint dx\,dy\,dz; \quad (40)$$

so that if P vanishes, $\Delta\mathrm{K}=0$; and if Q vanishes, $\Delta\mu=0$. The optical evidence that either $\Delta\mathrm{K}$ or $\Delta\mu$ vanishes is thus very strong; while electrical reasons lead us to conclude that it is $\Delta\mu$.

If we write T for the volume of the small particle, we get from (40), as the special forms of (35), (37), (38) applicable to this case,

$$f_1=\frac{\pi T}{\lambda^2 r}e^{i(nt-kr)}\left[K\Delta K^{-1}\frac{\alpha\gamma}{r^2}-\mu\Delta\mu^{-1}\frac{\gamma}{r}\right], \quad . \quad . \quad (41)$$

$$g_1=\frac{\pi T}{\lambda^2 r}e^{i(nt-kr)}\left[K\Delta K^{-1}\frac{\beta\gamma}{r^2}\right], \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$h_1=\frac{\pi T}{\lambda^2 r}e^{i(nt-kr)}\left[-K\Delta K^{-1}\frac{\alpha^2+\beta^2}{r^2}+\mu\Delta\mu^{-1}\frac{\alpha}{r}\right]. \quad (43)$$

If $\Delta\mu=0$, as we shall henceforward suppose, $f:g=\alpha:\beta$, showing that the electrical displacement is in the plane containing the secondary ray and the direction of primary electrical displacement (z), and

$$f_1^2+g_1^2+h_1^2\propto\frac{\alpha^2+\beta^2}{r^2};$$

so that the intensity is proportional to the square of the sine of the angle between the secondary ray and the direction of the primary electrical displacement. The blue colour of the light scattered from small particles is explained by the occurrence of λ^2 in the denominators of the expressions for f_1, g_1, h_1; but for further particulars on this subject the reader must be referred to my previous papers.

Equations (35), (36), &c. are rigorously applicable, however large the region of disturbance, if the square of ΔK may really be neglected. From them we see that, under the circumstances in question, each element of a homogeneous obstacle acts independently as a centre of disturbance, and that the aggregate effect in any direction depends upon the phases of the elementary secondary disturbances as affected by the situation of the element along the paths of the primary and of the secondary light. In fact,

$$P=\Delta K^{-1}e^{int}\iiint e^{ikx}e^{-ikr}\,dx\,dy\,dz.$$

If θ, ϕ be the angles defining (in the usual notation) the direction of the secondary ray, and r_0 correspond to the origin of coordinates, we have

$$P=\Delta K^{-1}e^{i(nt-kr_0)}\iiint e^{ik(x+x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)}\,dx\,dy\,dz; \quad (44)$$

and the question now before us for consideration is the value of the integral in (44) as dependent upon the size of the obstacle and the direction of the secondary ray. It is evident that the formulæ are applicable only when the whole retardation of the primary light in traversing the obstacle can be neglected in comparison with the wave-length; but if this condition be satisfied, there is no further limitation upon the size of the obstacle. In the case where the secondary ray forms the prolongation of the primary, or deviates sufficiently little from this direction, the exponential in (44) reduces to unity, signifying that every element of the obstacle acts alike, any retardation of phase at starting due to situation along the primary ray being balanced by an acceleration corresponding to a less distance to be travelled along the secondary ray. At a greater or less obliquity, according to the size of the obstacle, opposition of phase sets in; and at still greater obliquities the resultant can be found only by an exact integration. Its intensity is then less, and generally much less, than in the first case—a conclusion abundantly borne out by observation.

The simplest example of this kind is that afforded by an infinite cylinder (*e. g.* a fine spider-line), on which the light impinges perpendicularly to the axis, so that every thing takes places in two dimensions. This case is indeed not strictly covered by the preceding formulæ, on account of the infinite extension of the region of disturbance; but a moment's consideration will make it clear that each elementary column here acts according to the laws already described—that is to say, gives rise to a component disturbance whose phase is determined by the situation of the element along the primary and secondary rays. If the angle between the two rays be called χ, we have to consider the value of

$$\iint e^{ik(x+x\cos\chi+y\sin\chi)}\,dx\,dy.$$

Introducing polar coordinates r, θ, we find

$$x+x\cos\chi+y\sin\chi=2r\cos\tfrac{1}{2}\chi\ \cos(\theta-\tfrac{1}{2}\chi);$$

so that the integral

$$=\iint e^{ikr\,.\,2\cos\frac{1}{2}\chi\,.\,\cos\theta}\,r\,dr\,d\theta$$

$$=\int_0^a\int_0^{2\pi}\{\cos(2kr\cos\tfrac{1}{2}\chi\ \cos\theta)+i\sin(2kr\cos\tfrac{1}{2}\chi\ \cos\theta)\}\,r\,dr\,d\theta$$

$$=2\pi\int_0^a J_0(2kr\cos\tfrac{1}{2}\chi)\,r\,dr, \quad \ldots\ldots \quad (45)$$

J_0 denoting the Bessel's function of zero order.

The integration with respect to r indicated in (45) can be effected by known properties of Bessel's functions; and the result is expressible by a function of the first order. We get

$$\frac{\pi a}{k\cos\frac{1}{2}\chi}J_1(2ka\cos\tfrac{1}{2}\chi); \quad \ldots\ldots \quad (46)$$

and J_1 is defined by

$$J_1(z)=\frac{z}{2}\left(1-\frac{z^2}{2.4}+\frac{z^4}{2.4^2.6}-\frac{z^6}{2.4^2.6^2.8}+\ldots\right). \quad (47)$$

If $\cos\frac{1}{2}\chi=0$ (*i. e.* in the direction of original propagation), (46) becomes πa^2, every element of the area acting alike. This is the maximum value. When χ is such that

$$2ka\cos\tfrac{1}{2}\chi=\pi\times 1{\cdot}2197,$$

the secondary light vanishes, at a greater angle revives, then vanishes again, and so on, the angles being of course functions of the wave-length. If we conceive the cylinder to increase in size gradually from zero, the scattered light vanishes first in the backward direction $\chi=0$, in which direction evidently the greatest differences of phase occur. Every thing is determined by the course of the function J_1; and (46) within the limits of its application embodies the theory of Young's eriometer.

We will now consider the case of an obstacle in the form of a sphere. If z be a coordinate measured perpendicularly to the plane containing the primary and secondary rays, formula (46), multiplied by dz, will represent the effect of a slice of the sphere, whose radius is a and thickness dz, and what remains to be effected is merely the integration with respect to z. For this purpose we write $z=c\sin\phi$, $a=c\cos\phi$, where c is the radius of the sphere. The integral then takes the form

$$\frac{2\pi c^2}{k\cos\frac{1}{2}\chi}\int_0^{\frac{1}{2}\pi} J_1(2kc\cos\tfrac{1}{2}\chi\cos\phi)\cos^2\phi\, d\phi, \quad . \quad (48)$$

or, if we expand J_1 by (47), and integrate according to a known formula,

$$\frac{2\pi c^3}{3}\left\{2-\frac{m^2}{5}+\frac{m^4}{7.5.4}-\frac{m^6}{9.7.5.4.6}+\frac{m^8}{11.9.7.5.4.6.8}\ldots\right\}, \quad (49)^*$$

in which m is written for $2kc\cos\frac{1}{2}\chi$. It will be understood that (49), after multiplication by $e^{int}\Delta K^{-1}$, gives merely the value of P in (36), and that to find the complete expression for the secondary light in any direction other factors must be introduced in accordance with (35), (37), (38). The angle χ,

* July 15.—I find for the first root of (49), $m=4{\cdot}50$, giving as the smallest obliquity $(\pi-\chi)$ at which the secondary light vanishes,

$$\pi-\chi=2\sin^{-1}(4{\cdot}50/2kc).$$

being that included between the secondary ray and the axis of x, may be expressed by

$$\sin\chi = \sqrt{(\beta^2+\gamma^2)} \div r. \quad . \quad . \quad . \quad . \quad (50)$$

Our theory, as hitherto developed, shows that, whatever the shape and size of the particles, there is no scattered light in a direction parallel to the primary electric displacements, except such as may depend upon squares and higher powers of the difference of optical properties. In order to render an account of the "residual blue" observed by Tyndall when particles in their growth have reached a certain magnitude, it is necessary to pursue the approximation. By (28), with $\Delta\mu$ neglected, we have

$$\nabla^2 f_2 + k^2 f_2 + \mathrm{K}\left(\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)(f_1\Delta\mathrm{K}^{-1}) \\ -\mathrm{K}\frac{d^2}{dx\,dy}(g_1\Delta\mathrm{K}^{-1}) - \mathrm{K}\frac{d^2}{dx\,dz}(h_1\Delta\mathrm{K}^{-1}) = 0, \quad (51)$$

and two similar equations in g_2 and h_2. On the supposition that f_1, g_1, h_1 are known throughout the region of disturbance, these equations may be solved in the same way as (29), (30), and (31). For the sake of brevity we may confine ourselves to the particular direction for which the terms of the first order vanish. Thus at a sufficient distance r' along the axis of z,

$$f_2 = -\frac{k^2\mathrm{K}}{4\pi r'}\iiint f_1\Delta\mathrm{K}^{-1}e^{-ikr'}\,d\alpha\,d\beta\,d\gamma, \quad . \quad . \quad (52)$$

$$g_2 = -\frac{k^2\mathrm{K}}{4\pi r'}\iiint g_1\Delta\mathrm{K}^{-1}e^{-ikr'}\,d\alpha\,d\beta\,d\gamma, \quad . \quad . \quad (53)$$

$$h_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

We have now to find the values of f_1 and g_1 within the region of disturbance, to which of course (35) &c. are not applicable. In the general solution (32), h_0 is a function of x only; so that the elements of the integral vanish in the *interior* of a homogeneous obstacle, and we have only to deal with the surface. Integrating by parts across this surface, we find

$$f_1 = \frac{\mathrm{K}}{4\pi}\iiint\frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\frac{d}{dx}\left(\frac{e^{-ikr}}{r}\right)dx\,dy\,dz \\ = -\frac{\mathrm{K}}{4\pi}\frac{d}{d\alpha}\iiint\frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\,.\,\frac{e^{-ikr}}{r}\,dx\,dy\,dz, \quad (55)$$

r being a function of x and α only through $(\alpha - x)$. In like manner

$$g_1 = -\frac{\mathrm{K}}{4\pi}\frac{d}{d\beta}\iiint \frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\cdot\frac{e^{-ikr}}{r}\,dx\,dy\,dz. \quad . \quad . \quad (56)$$

In the case of a small homogeneous sphere, whose centre is taken as origin of coordinates, these formulæ lead to fairly simple results. The triple integral in (55), (56) may readily be exhibited in its real character of a surface-integral. Thus

$$\iiint \frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\frac{e^{-ikr}}{r}\,dx\,dy\,dz = -\Delta\mathrm{K}^{-1}\iint \frac{h_0 z}{c}\frac{e^{-ikr}}{r}\,d\mathrm{S}, \quad (57)$$

where $d\mathrm{S}$ is an element of the surface whose radius is c. This applies to a sphere of any size; but we have now to introduce an approximation depending on the supposition that kc is small. As far as the first power of kc,

$$-\Delta\mathrm{K}^{-1}\iint \frac{h_0 z}{c}\frac{e^{-ikr}}{r}\,d\mathrm{S} = -\Delta\mathrm{K}^{-1}\frac{e^{int}}{c}\iint\left(\frac{z+ikzx}{r} - ikz\right)d\mathrm{S}$$

$$= -\Delta\mathrm{K}^{-1}\frac{e^{int}}{c}\iint \frac{z+ikzx}{r}\,d\mathrm{S},$$

in which the double integral is the common potential of matter distributed over the spherical surface with density $(z+ikzx)$. Calling this for the moment V, we have (Thomson and Tait, 'Nat. Phil.' § 536) at any internal point (α, β, γ),

$$\mathrm{V} = 4\pi c(\gamma + \tfrac{1}{3}ik\gamma\alpha);$$

so that

$$\iiint \frac{d}{dz}(h_0\Delta\mathrm{K}^{-1})\frac{e^{-ikr'}}{r}\,dx\,dy\,dz$$

$$= -4\pi\Delta\mathrm{K}^{-1}e^{int}(\gamma + \tfrac{1}{3}ik\gamma\alpha). \quad . \quad (58)$$

Thus by (55), (56),

$$f_1 = \tfrac{1}{3}\mathrm{K}\Delta\mathrm{K}^{-1}ik\gamma e^{int}, \quad g_1 = 0. \quad . \quad . \quad . \quad . \quad (59)$$

We are now prepared to calculate f_2, g_2 from (52), (53). These formulæ apply to both directions along the axis of z; but in what follows it will be convenient to suppose that it is the positive direction which is under consideration. In this case, if ρ denote the distance from the centre of the sphere, $r' = \rho - \gamma$ and $e^{-ikr'} = e^{-ik\rho}(1 + ik\gamma)$ approximately; so that

$$f_2 = -\frac{k^2(\mathrm{K}\Delta\mathrm{K}^{-1})^2 e^{i(nt-k\rho)}}{12\pi\rho}\iiint ik\gamma(1+ik\gamma)\,d\alpha\,d\beta\,d\gamma$$

$$= \frac{k^4(\mathrm{K}\Delta\mathrm{K}^{-1})^2}{12\pi}\frac{e^{i(nt-k\rho)}}{\rho}\iiint \gamma^2\,d\alpha\,d\beta\,d\gamma;$$

or if, as before, T be the volume of the sphere,

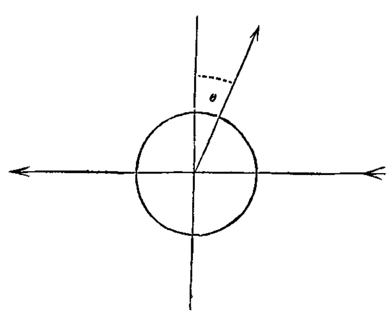
$$\left.\begin{aligned} f_2 &= \frac{\pi \mathrm{T}}{\lambda^2 \rho} e^{i(nt-k\rho)} (\mathrm{K}\Delta\mathrm{K}^{-1})^2 \frac{k^2c^2}{15}, \\ g_2 &= 0. \end{aligned}\right\} \quad . \quad . \quad (60)$$

Comparing (60) and (41), we see that the amplitude of the light scattered along z is not only of higher order in ΔK, but is also of the order k^2c^2 in comparison with that scattered in other directions. The incident light being white, the intensity of the component colours scattered along z varies as the inverse 8th power of the wave-length, so that the resultant light is a rich blue.

There is another point of importance to be noticed. Although when the terms of the second order are included the scattered light does not vanish along the axis of z, the peculiarity is not lost, but merely transferred to another direction. Putting together the terms of the first and second orders, we see that the scattered light will vanish in a direction in the plane of xz, inclined to z (towards x) at a small angle θ, such that

$$\theta = -\mathrm{K}\Delta\mathrm{K}^{-1}\frac{k^2c^2}{15} = \frac{\Delta\mathrm{K}}{\mathrm{K}}\frac{k^2c^2}{15}. \quad . \quad . \quad . \quad (61)$$

In the usual case of particles optically denser than the surrounding medium, ΔK is positive, from which we gather that the direction in which the scattered light vanishes to the second order of approximation is inclined backwards, so that the angle through which the light may be supposed to be bent by the action of the particle is *obtuse*.



The fact that, when the primary light is polarized, there is in one perpendicular direction no light scattered by very small particles, was stated by Stokes*; but it is, I believe, to Tyndall that we owe the observation that with somewhat larger particles the direction of minimum illumination becomes oblique. I do not find, however, any record of the direction of the obliquity (that is, of the sign of the small angle θ), and have therefore made a few observations for my own satisfaction.

In a darkened room a beam of sunlight was concentrated

* Phil. Trans. 1852, § 183.

by a large lens of 2 or 3 feet focus; and in the path of the light was placed a beaker glass, containing a dilute solution of hyposulphite of soda. On the addition of a small quantity of dilute sulphuric acid a precipitate of sulphur slowly forms, and during its growth manifests exceedingly well the phenomena under consideration. The more dilute the solutions, the slower is the progress of the precipitation. A strength such that there is a delay of four or five minutes before any effect is apparent, will be found suitable; but no great nicety of adjustment is necessary. By addition of ammonia in sufficient quantity to neutralize the acid, the precipitation may be arrested at any desired stage. More time is thus obtained to complete the examination; but the condition of things is not absolutely permanent, the already precipitated sulphur appearing to aggregate itself into large masses.

In the optical examination we may, if we prefer it, polarize the primary light; but it is usually more convenient to analyze the scattered light. In the early stages of the precipitation the polarization is complete in a perpendicular direction, and incomplete in oblique directions. After an interval the polarization begins to be incomplete in the perpendicular direction, the light which reaches the eye when the nicol is in the position of minimum transmission being of a beautiful blue, much richer than any thing that can be be seen in the earlier stages. This is the moment to examine whether there is a more complete polarization in a direction somewhat oblique; and it is found that with θ positive there is in fact an oblique direction of more complete polarization, while with θ negative the polarization is more imperfect than in the perpendicular direction itself.

The polarization in a distinctly oblique direction, however, is not perfect, a feature for which more than one reason may be put forward. In the first place, with a given size of particles, the direction of complete polarization indicated by (61) is a function of the colour of the light, the value of θ being three or four times as large for the violet as for the red end of the spectrum. The experiment is, in fact, much improved by passing the primary light through a coloured glass held in the window-shutter. Not only is the oblique direction of maximum polarization more definite and the polarization itself more complete, but the observation is easier than with white light, by the uniformity of the colour of the light scattered in various directions. If we begin with a blue glass, we may observe the gradually increasing obliquity of the direction of maximum polarization; and then by exchanging the blue glass for a red one, we may revert to the original condition of things, and observe the transition from perpendicularity to obliquity over again. The change in the wave-length of the light has the same effect as a change in the size of the particles; and the comparison gives curious information as to the rate of growth.

But even with homogeneous light it would be unreasonable to expect an oblique direction of perfect polarization. So long as the particles are all very small in comparison with the wave-length, there is complete polarization in the perpendicular direction; but when the size is such that obliquity sets in, the degree of obliquity will vary with the size of the particles, and the polarization will be complete only on the very unlikely condition that the size is the same for them all. It must not be forgotten, too, that a very moderate increase in dimensions may carry the particles beyond the reach of our approximations.

The fact that at this stage the polarization is a maximum when the angle through which the light is turned *exceeds* a right angle is the more worthy of note, as the opposite result would probably have been expected. By Brewster's law this angle in the case of a plate is *less* than a right angle; so that not only is the law of polarization for a very small particle different from that applicable to a plate, but the first effect of an increase of size is to augment the difference.

We must remember that our recent results are limited to particles of a spherical form. It is not difficult to see that, for elongated particles, the terms in $(\Delta K)^2$ may be of the same order with respect to kc as the principal term; so that if $(\Delta K)^2$ be sensible, mere smallness of the particle will not secure complete evanescence of scattered light along z. The general solution of the problem for an infinitesimal particle of arbitrary shape must raise the same difficulties as beset the general determination of the induced magnetism developed in a piece of soft iron when placed in a uniform field of force. In the case of an ellipsoidal particle the problem is soluble; but it is perhaps premature to enter upon it, until experiment has indicated the existence of phenomena likely to be explained thereby.

For an infinitesimal particle in the form of a sphere, we may readily obtain the complete solution without any approximation depending upon the smallness of ΔK. We know by the analogous theory of magnetism, that a dielectric sphere situated in a uniform field of electric force will undergo electric displacement of uniform amount, and in a direction parallel to that of the force. Thus the complete solution applicable to an infinitely small sphere is obtained from (29), (30), (31) by writing h for h_0; where by h is denoted the actual displacement (parallel to z) within the particle, and by h_0 the displacement in the enveloping medium under the same electric force. If K′ be the specific inductive capacity for the particle, the ratio of $h : h_0$ is $3K' : K'+2K$; and in this ratio the results expressed in (41), (42), (43) are to be increased. If we extract the factors $K\Delta K^{-1}$ which there occur, we get

$$\frac{3K'}{K'+2K}K\Delta K^{-1}=\frac{3K'K}{K'+2K}\left(\frac{1}{K'}-\frac{1}{K}\right)=-\frac{3(K'-K)}{K'+2K};$$

so that

$$f=-\frac{3(K'-K)}{K'+2K}\frac{\pi T}{\lambda^2 r}\frac{\alpha\gamma}{r^2}e^{i(nt-kr)}, \&c. \quad . \quad . \quad . \quad (62)$$

We learn from (62) that our former result as to the evanescence of the secondary light along z is true for an infinitely small spherical particle to *all* orders of ΔK.

We will now return to the two-dimension problem with the view of determining the disturbance resulting from the impact of plane waves upon a cylindrical obstacle whose axis is parallel to the plane of the waves. There are, as in the problem of reflection from plane surfaces, two principal cases—(1) when the electric displacements are parallel to the axis of the cylinder taken as axis of z, (2) when the electric displacements are perpendicular to this direction.

Case 1.—From (12), with $C=0$, $\mu=$ constant,

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\frac{h}{K}+n^2\mu K\frac{h}{K}=0;$$

or if, as before, $k=2\pi/\lambda$,

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+k^2\right)\frac{h}{K}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

in which k is constant in each medium, but changes as we pass from one medium to another. From (63) we see that the problem now before us is analytically identical with that treated in my book on Sound, § 343, to which I must refer for more detailed explanations. The incident plane waves are represented by

$$e^{int}e^{ikx}=e^{int}e^{ikr\cos\theta}$$

$$=e^{int}\{J_0(kr)+2iJ_1(kr)\cos\theta+\ldots+2i^mJ_m(kr)\cos m\theta+\ldots\}; \quad (64)$$

and we have to find for each value of m an internal motion finite at the centre, and an external motion representing a divergent wave, which shall in conjunction with (64) satisfy at the surface of the cylinder ($r=c$) the condition that the function and its differential coefficient with respect to r shall be continuous. The divergent wave is expressed by

$$B_0\psi_0 + B_1\psi_1 \cos\theta + B_2\psi_2 \cos 2\theta + \ldots,$$

where ψ_0, ψ_1, &c. are the functions of kr defined in § 341. The coefficients B are determined in accordance with

$$B_m \left\{ kc \frac{d\psi_m}{d\,.\,kc} J_m(k'c) - k'c\, \psi_m \frac{d}{d\,.\,k'c} J_m(k'c) \right\}$$
$$= 2i^m \{k'c\, J_m(kc) J_m'(k'c) - kc\, J_m(k'c) J_m'(kc)\},$$

except in the case of $m=0$, when $2i^m$ on the right-hand side is to be replaced by i^m. In working out the result we suppose kc and $k'c$ to be small; and we find approximately for the secondary disturbance corresponding to (64)

$$\psi = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \left[\frac{k'^2c^2 - k^2c^2}{2} - \frac{k^2c^2(k'^2c^2 - k^2c^2)}{8} \cos\theta\right]; \quad (65)$$

showing, as was to be expected, that the leading term is independent of θ.

For case 2, which is of greater interest, we have from (15),

$$\left(\frac{d}{dx}\frac{1}{k^2}\frac{d}{dx} + \frac{d}{dy}\frac{1}{k^2}\frac{d}{dy} + 1\right) c = 0. \quad . \quad . \quad (66)^*$$

This is of the same form as (63) within a uniform medium, but gives a different boundary condition at a surface of transition. In both cases the function itself is to be continuous; but in that with which we are now concerned the second condition requires the continuity of the differential coefficient *after division by* k^2. The equation for B_m is therefore

$$B_m \left\{ k'c \frac{d\psi_m}{d\,.\,kc} J_m(k'c) - kc\, \psi_m \frac{dJ_m(k'c)}{d\,.\,k'c} \right\}$$
$$= 2i_m \{kc\, J_m(kc)\, J_m'(k'c) - k'c\, J_m(k'c)\, J_m'(kc)\},$$

with the understanding that the 2 is to be omitted when $m=0$. Corresponding to the primary wave $e^{i(nt+kx)}$, we find as the expression of the secondary at a great distance from the cylinder,

$$\psi = \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{i(nt-kr)} \left[-\frac{k^2c^2}{16}(k^2c^2 - k'^2c^2) \right.$$
$$\left. - k^2c^2 \frac{k'^2 - k^2}{k'^2 + k^2} \cos\theta - \frac{1}{8} k^4c^4 \frac{k^2 - k'^2}{k^2 + k'^2} \cos 2\theta \right]. \quad . \quad (67)$$

The term in $\cos\theta$ is now the leading term; so that the second-

* In (66) c is the magnetic component, and not the radius of the cylinder. So many letters are employed in the electromagnetic theory, that it is difficult to hit upon a satisfactory notation.

ary disturbance approximately vanishes in the direction of the primary electrical displacements, agreeably with what has been proved before. It should be stated here that (67) is not complete to the order k^4c^4 in the term containing $\cos\theta$. The calculation of the part omitted is somewhat tedious in general; but if we introduce the supposition that the difference between k'^2 and k^2 is small, its effect is to bring in the factor $(1-\frac{1}{4}k^2c^2)$.

Extracting the factor (k'^2-k^2), we may conveniently write (67)

$$\psi=-k^2c\,\frac{k'^2-k^2}{k'^2+k^2}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}e^{i(nt-kr)}\left[\cos\theta-\frac{k'^2c^2+k^2c^2}{16}-\frac{k^2c^2}{8}\cos 2\theta\right], \quad . \quad (68)$$

in which

$$\cos\theta-\frac{k'^2c^2+k^2c^2}{16}-\frac{k^2c^2}{8}\cos 2\theta$$
$$=\cos\theta-\frac{k'^2c^2-k^2c^2}{16}-\frac{k^2c^2}{4}\cos^2\theta. \quad . \quad . \quad (69)$$

In the directions $\cos\theta=0$, the secondary light is thus not only of high order in kc, but is also of the second order in $(k'-k)$. For the direction in which the secondary light vanishes to the next approximation, we have

$$\tfrac{1}{2}\pi-\theta=\tfrac{1}{16}(k'^2c^2-k^2c^2)=\frac{k^2c^2}{16}\,\frac{\mathrm{K}'-\mathrm{K}}{\mathrm{K}}. \quad . \quad . \quad (70)$$

This corresponds to (61) for the sphere; and is true if kc, $k'c$ be small enough, whatever may be the relation of k' and k. For the cylinder, as for the sphere, the direction is such that the primary light would be bent through an angle *greater* than a right angle.

If we neglect the square of (k'^2-k^2), the complete expression corresponding to (69) is

$$\cos\theta(1-\tfrac{1}{4}k^2c^2)-\tfrac{1}{4}k^2c^2\cos^2\theta=\cos\theta[1-\tfrac{1}{4}k^2c^2-\tfrac{1}{4}k^2c^2\cos\theta].$$

This may be compared with the value obtained by the former method, viz. $\cos\theta\,\mathrm{J}_1(2kc\cos\frac{1}{2}\theta)\div kc\cos\frac{1}{2}\theta$, and will be found to agree with it as far as the square of kc.

If we suppose the cylinder to be extremely small, we may confine ourselves to the leading terms in (65) and (67). Let us compare the intensities of the secondary lights emitted in the two cases along $\theta=0$, *i. e.* directly backwards. From (65)

$$\psi\propto\frac{k'^2c^2-k^2c^2}{2},$$

while from (67)

$$\psi\propto-k^2c^2\,\frac{k'^2-k^2}{k'^2+k^2}.$$

The opposition of sign is apparent only, and relates to the different methods of measurement adopted in the two cases. In (65) the primary and secondary disturbances are represented by h/K, but in (67) by the magnetic function c. If we express the solution in the second case in terms of the electric function g, we shall find (see 13) that the ratio of c to g changes sign when we pass from the primary light propagated along $-x$ to the secondary light propagated along $+x$. The actual ratio of amplitudes in the two cases is thus $(k'^2+k^2)/2k^2$, or $(K'+K)/2K$. Unless the difference between K' and K be neglected, the two components of unpolarized light are scattered along this direction in different proportions, that component preponderating in which the electric displacement is parallel to the axis of the cylinder. The secondary light is therefore partially polarized in the plane perpendicular to the axis.

June 1881.

XI. *An Abstract of the Results obtained in a Recalculation of the Atomic Weights.* By FRANK WIGGLESWORTH CLARKE, *S.B., Professor of Chemistry in the University of Cincinnati**.

DURING the past three years I have been engaged upon a recalculation of all the atomic-weight determinations which have been published from the time of Berzelius's earlier investigations down to the present date. My purpose has been to reduce all similar series of experiments to common standards, to calculate the probable error of each series, to combine the results into general means, and then to deduce the atomic weights in such a way that each value should represent a fair average of all the trustworthy estimations. In other words, I have sought to bring together all the vast number of scattered details, and to derive from them a more consistent table of atomic weights than has hitherto been found in chemical literature. My complete work will appear in due time as a separate volume; my present intention is to give merely a summary of my methods, and my conclusions.

Taking hydrogen as unity, I necessarily began with the ratio between it and oxygen. This ratio has been determined accurately in only two ways:—first, by the synthesis of water over copper oxide; and secondly, from the relative density of the two gases. Ignoring earlier inexact experiments, we may consider only the data furnished by Dumas, by Erdmann and Marchand, and by Regnault. From Dumas's nineteen syntheses of water we get for oxygen values ranging

* Communicated by the Author.