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LII. On systems of algebra involving more than one imaginary; and on equations of the fifth degree

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direct means, as to those which are opake, as Mr. Joule has done for magnetic substances. For the proofs drawn from diamagnetism and sonorous vibrations are only indirect, although the last appear to be tolerably conclusive.

I shall soon return to this subject, in reference to some researches on the relation between diamagnetism and the induced currents, on which I am at present engaged.

LII. *On Systems of Algebra involving more than one Imaginary; and on Equations of the Fifth Degree.* *By* JAMES COCKLE, *Esq., M.A., Barrister-at-Law**.

CONCEIVE two imaginaries, such that their respective squares are equal either to positive or to negative unity. Then the product of two linear functions of these imaginaries is not of the same form as its factors. The product of the imaginaries prevents this similarity, and obstructs the formation of a System of Triple Algebra on the basis just mentioned. But, if we invest the last-named product with the character of a third imaginary, and assume that its square is equal either to positive or to negative unity, four systems of Quadruple Algebra will present themselves, in each of which the product of two linear functions of the three imaginaries will, in general, have the same form as its factors.

Let α and β respectively represent the first and second, and γ the third imaginary. Then $\gamma=\alpha\beta$ or $\beta\alpha$. But as, in quadruple algebra, $\alpha\beta$ is not always equal to $\beta\alpha$, I shall select the former as the expression for γ.

Let A denote a linear function of α, β, and γ; in other language, let

$$A=w+\alpha x+\beta y+\gamma z,$$

then, in one of the four systems of quadruple algebra above alluded to, the expression A bears the name of a *quaternion*. In the remaining three systems the respective terms *tessarine*, *coquaternion*, and *cotessarine* may be applied to it. At least I have suggested such a nomenclature in No. 1360 of the Mechanics' Magazine†, where I have shown the existence of

* Communicated by the Author. In connexion with his paper published at pp. 406–410 of the preceding (34th) volume of this Journal, Mr. Cockle is desirous of referring the reader to two articles subsequently communicated by him to the Mechanics' Magazine, and which will be found at pp. 534 and 558, 559 of vol. 50 of that work.

† See pp. 197, 198 of the current (51st) volume of that work. I had, however, previously employed the term "Tessarine" both in that and in the present Journal.

these four systems, discussed them, and pointed out their characteristics. I here propose to advert for a moment to the same subject, to consider it under a slightly different aspect, and also to exhibit, for convenience of comparison, the modular expressions of all the systems. We have, then,

1. The Quaternion System of Sir W. R. Hamilton, in which

$$\alpha^2 = -1, \quad \beta^2 = -1, \text{ and } \alpha\beta = \gamma;$$

but in which also

$$\gamma^2 = -1,$$

contrary to what we should have inferred from the equations

$$\gamma^2 = \alpha^2\beta^2 = -1 \times -1 = 1;$$

hence, the quaternion system is abnormal, or does not obey the laws of ordinary algebra. The modulus of a quaternion is the positive square root of the expression

$$w^2 + x^2 + y^2 + z^2.$$

2. The Tessarine System—a normal system in which

$$\alpha^2 = -1, \quad \beta^2 = 1, \text{ and } \gamma^2 = -1 = \alpha^2\beta^2.$$

The true modulus of the tessarine A is the positive square root of

$$(w \pm y)^2 + (x \pm z)^2.$$

3. The Coquaternion System, in which

$$\alpha^2 = -1, \quad \beta^2 = 1, \text{ and } \gamma^2 = 1;$$

but the last relation is inconsistent with the conditions

$$\gamma^2 = \alpha^2\beta^2 = -1 \times 1 = -1,$$

and the coquaternion system is abnormal. The modulus of A, considered as a coquaternion, is the positive square root of

$$(w \pm y \pm z)^2 + x^2.$$

4. The Cotessarine System, in which

$$\alpha^2 = 1, \quad \beta^2 = 1, \quad \gamma^2 = 1 = \alpha^2\beta^2,$$

and which is a normal system, having for its modular form the positive square root of

$$(w \pm x \pm y \pm z)^2.$$

It is to be borne in mind, that in all the above systems $\gamma = \alpha\beta$; that, whenever the double sign ($\pm$) occurs, the sign of the term is indifferent and quite independent of that of the preceding or following term; that w, x, y, and z are real quantities, positive, negative, or zero; and that, in multiplying two expressions of the form A, the modulus of the product is the product of the moduli of the factors. It is for the purpose of analogy and of making the modulus positive, in all cases, that I have given a quadratic form to the formula employed in expressing the cotessarine modulus.

On Equations of the Fifth Degree.

Whether Mr. Jerrard has succeeded in pointing out a method of solving the algebraic equation of the fifth degree or not, his investigations at pp. 545–574 of vol. xxvi. (continued at p. 63 of vol. xxviii.) of the present Series of this Journal must ever be a subject of interest, and form an essential part of the theory of such equations. There are, however, one or two portions of his papers which seem to me involved in doubt and difficulty—difficulty which, in one case, he has himself adverted to and endeavoured to explain. Mr. Jerrard will pardon me if, with great hesitation, I venture to intimate an opinion that the position taken by him in his note [‡] to p. 572 of vol. xxvi. is untenable. By way of example, suppose that the square root of $x+h$ is the function to be expanded. The general form of the expansion* is,—

A Series of converging or diverging terms *plus* a Remainder.

Now, when the series is convergent, the remainder may, in all cases where *numerical* value is the subject of inquiry, be entirely neglected; but it does not the less constitute an essential part of the *symbolic* expansion. Hence I conceive that, in considering the expansion under a purely symbolic point of view, even the convergent development must be regarded as incomplete without the remainder, and so placed on the same footing as the divergent one. Considered thus, the convergent and divergent developments are deducible, the one from the other, by an interchange of x and h, and each admits of that interchange without alteration of *symbolic* value. And I think that we *necessarily* obtain an expression which admits of such interchange—at least in all cases where a strictly symbolical expansion is required; and, if I rightly understand Mr. Jerrard's argument, it is to such expansions that his remarks apply.

But, admitting for a moment that the convergent series with the remainder neglected is a *symbolic* expansion of the function,

* I have elsewhere (in the course of my *Horæ Algebraicæ*, Mechanics' Magazine, vol. xlvii. p. 150) suggested *contraction* as a term to denote the inverse of expansion. Would it be advisable to confine the terms expansion and contraction to *symbolic* operation, and to use the terms *involution* and *evolution* exclusively in reference to arithmetic or quasi-arithmetic operations, including them both under the common name *volution*?

I feel some doubt as to another portion of Mr. Jerrard's argument. At all events I think it would be very desirable to show clearly the solvibility of the equation by which W is to be determined. It is true that one of its roots appears to be a known and rational function of another of them, and that an equation among whose roots such a relation exists is supposed to be capable of solution by means of the process of Abel. But doubts—and doubts apparently well-founded—have arisen respecting the universality of that theorem. It is not my object to discuss them here; but I would refer the reader to the learned paper on the Calculus of Functions in the *Encyclopædia Metropolitana*, where, at pp. 327, 328, art. (90.), and at pp. 381, 382, arts. (302.) and (303.) of vol. ii. of the Pure Sciences, he will find remarks upon this question; and I would also call attention to the respective notes to arts. (90.) and (303.) just adverted to. There, the nature of the difficulty which militates against the generality of the theorem—a difficulty which, in the instance of functions of a degree so low as the third, is only obviated by our having complete solution of a cubic—is clearly exhibited.

Standing on the frontiers which separate solvible equations from those as yet unsolved, the biquadratic partakes of the nature of both. It resembles the one inasmuch as it is capable of finite algebraic solution; the other, in its incapability of finite algebraic solution in terms of irreducible biquadratic surds. The latter characteristic might perhaps be of service in the discussion of equations of the fifth degree, and in the manner which I suggested in my First Series of Notes on the Theory of Algebraic Equations, published in vol. xlvi. of the Mechanics' Magazine. The reader is referred to p. 125 of that volume, and to the condition mentioned at a subsequent page (180) of it.

2 Church-Yard Court, Temple,
November 1, 1849.

LIII. *Remarkable Solar Phænomenon seen at the Villa, Beeston near Nottingham, October 22, 1849.* By Edward Joseph Lowe, *Esq., F.R.A.S.**

On the above day a strange spectacle presented itself about the sun. The morning had been misty, and had cleared up about 22^h; but being engaged with some papers I did not look at the sun until 0^h 10^m, when a remarkable phænomenon was immediately discovered: it resembled a huge

* Communicated by the Author.