Philosophical Magazine Series 3

Taylor & Francis
Taylor & Francis Group

ISSN: 1941-5966 (Print) 1941-5974 (Online) Journal homepage: http://www.tandfonline.com/loi/tphm14

LIX. On the symbols of algebra, and on the Theory of Tesarines

James Cockle Esq. M.A.

To cite this article: James Cockle Esq. M.A. (1849) LIX. On the symbols of algebra, and on the Theory of Tesarines , Philosophical Magazine Series 3, 34:231, 406-410, DOI: 10.1080/14786444908646257

To link to this article: http://dx.doi.org/10.1080/14786444908646257

Published online: 30 Apr 2009.

LIX. *On the Symbols of Algebra, and on the Theory of Tessarines. By* JAMES COCKLE, *Esq., M.A., of Trinity College, Cambridge, and Barrister-at-Law of the Middle Temple*.*

AT page 436 of the last (33rd) volume of this Journal, I proposed to include under the genus *imaginary* two independent species of quantity which I distinguished by the respective terms *unreal* and *impossible*. But, if we extend the meaning of the word "imaginary" so as to make it comprehend all quantity that is not *real*, a third species of quantity, for which I would suggest the name of *ideal*, must be added to the two already included in the common genus of imaginaries. And perhaps it will be conducive to distinctness of conception and to convenience if we admit a *fourth* species of imaginaries, which I propose to call *typal*, from their squares, &c. running into certain *types*.

The peculiar symbols of the quaternion theory of Sir W. R. Hamilton, of the octave theory of Mr. John T. Graves and Mr. Cayley, of the triple algebra of Professor De Morgan, and of the pluquaternion theory of the Rev. T. P. Kirkman, all belong to the species *ideal*. It may be stated as the characteristic of ideal quantities, that, in their combinations, they do not follow the laws of that ordinary algebra to which Mr. De Morgan applies the term Double Algebra. The imaginaries in some of the systems of Mr. Cayley's theory of couples (the systems marked C, D, C′, and D′, Phil. Mag., S. 3, vol. xxvii. pp. 39–40) are *ideal*. And so, in general, are those of the last two systems (E and E′, *Ib.* p. 40) in Mr. Cayley's paper. Mr. J. T. Graves would call the couples involved in the above systems *anomalous*.

But in other of the systems of Mr. Cayley (those marked A, B, A′, B′, *Ib.* p. 38 *et seq.*) the imaginaries are *typal*; and this is also the case in Mr. J. T. Graves's theory of couples (*Ib.* vol. xxvi.). Borrowing a term from the learned writer last mentioned, we may call all the couples mentioned in this paragraph *normal*. The characteristic of *typal* quantities is, that, although in their laws of combination they follow the rules of ordinary algebra, yet the types or conditions by which they are defined are not consistent with that algebra. A *typal* differs from an *impossible* quantity in this: that the ordinary algebra *forces* impossible quantity upon our notice, and defines it by means as purely algebraic as those by which unreal quantity is defined; while on the other hand, typal (as well as

* Communicated by Dr. John Cockle, F.R.C.S.

ideal) quantity is the offspring of arbitrary ultra-algebraic definition*.

I should propose to apply the term *hyper-algebraic* to typal and ideal quantities, and to confine it to those quantities. Ought we to apply the word *hyper-algebraic* to impossible quantity? I think not. If I may be permitted to use the term *possible* so as to include under it not only real quantities but also the *unreal* quantities of ordinary algebra, I would suggest that it is only in respect of certain anomalous results† (results, however, that do not defy explanation ‡) that impossible differs from possible quantity, and consequently that impossible quantity must be regarded as *algebraic*. It is unquestionably

* I must not be understood as desiring to underrate these symbolic children of definition. So far from it, I think it within the limits of possibility so to define symbols as that they may have their prototypes in nature, and serve to expound other of her phænomena than those to which symbols have been yet applied. For instance, might not arbitrary symbols be made the representatives of *chemical* phænomena? Mr. Boole's Mathematical Analysis of Logic is a step out of the beaten track which symbolic science has hitherto persevered in; and although to pass from mental to chemical phænomena may not authorise us to hope that such sciences as chemistry may be rendered symbolic, yet I cannot help thinking, that by a proper notation for *affinity*, &c. chemical decompositions might be represented: at any rate it may be worth a trial.

† Vide *suprà*, pp. 41, 42. It is not a little singular, that if we abandon the principle laid down, *suprà*, p. 39, note †, we have

$$(1+\surd j)(1-\surd j)=1-\surd j^2=1-1=0;$$

and also that if we preserve that principle, we derive from the equation (1.), *suprà*, p. 39, the following:

$$(-1)^4=(+\surd j)^4=(\surd j\times\surd j)\times(\surd j\times\surd j)=j\times j,$$

or

$$1=j^2.$$

On the other hand, although from the relation

$$(1+k)^2=2k=2ij$$

we may deduce

$$\frac{1}{4}(1-i+j+k)^2=j,$$

still, seeing, from the relation (*suprà*, p. 40)

$$+\surd j=-1,$$

the anomalous nature of the evolution of impossibles, we must not attempt to express $\surd j$ as a linear function of i, j, k.

I may add, that in previous papers (*suprà*, pp. 45, and 135,) the radius of the (larger) sphere is supposed to be unity.

‡ See paragraph 8 (and 10) of the Rev. Prof. Charles Graves's paper on Triple Algebra (*suprà*, p. 119–126). I propose to call Mr. C. Graves's system a *trinar*, and Mr. De Morgan's a *ternary* algebra; the latter form of name being given to the quadratic system, in which the *square* of the imaginary is *negative*. And hence the respective terms *trine* and *ternion* suggest themselves as *distinctive*.

algebraic in its origin, nor is it to be considered as other than algebraic in its laws of combination with other symbols; and if in first approaching the subject of impossible quantity some mysteries present themselves and some difficulties arise, it may be worth considering whether our estimate of the range of ordinary algebra has not been too limited, and whether its own inherent powers, when sufficiently developed, may not explain the mystery and clear away the difficulty. With these views it will not excite surprise if I state here, that I regard paragraph 8 of the Rev. Charles Graves's paper on Triple Algebra (*suprà*, pp. 123, 124) as a valuable contribution to *ordinary* algebra.

I take the opportunity of adding one or two remarks on tessarines, premising that I shall use $i' j'$ and k' to denote the imaginaries which enter into those expressions.

1. The product of two tessarines of the form

$$-\frac{xz}{y}+i'x+j'y+k'z$$

is of the same form, and the *moduli* and *amplitudes* of the factors and product are related in the same manner, and the latter may be constructed as readily, as if the factors and the product were quaternions.

2. The product of the two tessarines

$$i'x_1+j'y_1+k'z_1, \text{ and } i'x_2+j'y_2+k'z_2,$$

will be of the form

$$i'x_3+j'y_3+k'z_3,$$

provided that

$$y_1y_2-x_1x_2-z_1z_2=0. \quad . \quad . \quad . \quad . \quad . \quad (\varpi.)$$

But, if the two systems of values x_1, y_1, z_1, and x_2, y_2, z_2, respectively satisfy the condition

$$y^2-x^2-z^2=0, \quad . \quad . \quad . \quad . \quad . \quad (\text{w.})$$

and if, moreover,

$$\frac{x_1}{z_1}=\frac{x_2}{z_2},$$

then (ϖ.) may be satisfied*. But (w.) represents a right-

* If we multiply the two equations that result from substituting the two systems of values of x y and z respectively, we shall have after reduction, &c.

$$(y_1y_2-x_1x_2-z_1z_2)(y_1y_2+x_1x_2+z_1z_2)=0,$$

which suggests matter for future observation. With reference to the subject of Imaginary Geometry (*suprà*, p. 132–135), and indeed of analytical geometry in general, I may remark that I propose to call the real primary axis (that of x) the *axe*, the unreal secondary axis (that of y) the *perpe*, and the impossible tertiary axis (that of z) the *norme*. Thus, in the equation angled cone, whose axis is the axis of y, and whose vertex is the origin, the axes being rectangular. Hence, taking $i'x+j'y+k'z$ to denote a point whose rectangular co-ordinates are x, y, z, we see that if two points be taken in the same generatrix of the cone (w.), their tessarine product, considering the vertex as origin, will be the point* whose co-ordinates are

$$x_3=y_1z_2+z_1y_2, \quad y_3=-x_1z_2-z_1x_2, \quad z_3=x_1y_2+y_1x_2.$$

(1.) (*suprà*, p. 133) A is the axe, B the perpe, and C the norme. Let me add that the equations

$$a-x=0, \quad a^2+y^2=0, \quad \sqrt{a}+\sqrt{z}=0,$$

(of which the respective solutions are

$$x=a, \quad y=i'a, \quad z=j'a,)$$

denote, when considered separately, three points, each at a distance a from the origin, but in axes at right angles to each other.

* My friend Professor Davies has (*suprà*, p. 37, and vol. xxix. p. 171–175) intimated or expressed an opinion adverse to the interpretability of the symbol $\sqrt{-1}$ in geometry. If eminence in geometric science can confer a right, not only to express such an opinion, but to have that expression duly weighed, then I think that there are few, if any, English geometers who possess those rights more unquestionably. I must however confess that I do not see the force of the reasoning employed in Mr. Davies's proposition (*Ib.* p. 174). The inconsistency alluded to in paragraph 4 of the proposition could never arise—at least I am unable to perceive how it could. In saying that a rectangle is equal to the product of its sides, we mean that the *numbers of linear units* in the sides, when multiplied together, give a number equal to the *number of square units* in the rectangle. But the *signs* are not elements in the consideration when we multiply the sides. Unless I am mistaken, the inconsistency in question must arise in some such manner as the following:—Take A in BB′; and let it be required that the rectangle B′A × AB shall equal half the square on BB′. We should then (as to this *vide suprà*, p. 43) find $AC=\pm a\sqrt{-1}$, and that would be a solution of the problem. I consider, then, that my much-valued friend's argument rests solely on the inductive ground previously assigned (vol. xxix. p. 172). But there are strong inductive reasons on the other side of the question. The symbol $\sqrt{-1}$ appears to indicate that more *dimensions* of the subject-matter must be taken into consideration than are stated in the data of a question. If we are dealing with a *two-dimensioned* subject, and meet with the symbol $\sqrt{-1}$, that symbol indicates impossibility or not according to the fact of our having or not having two dimensions given in our data. Thus, in that spherical geometry with which Mr. Davies's name must ever be associated, and in which a point may be determined by its longitude (ϕ) and its latitude (ψ) (see Camb. Math. Journ., vol. i. p. 193; ii. p. 37, &c.), such an expression as

$$\cos\phi + \cos\psi\sqrt{-1}$$

is unmeaning; while in plane analytical geometry the expression $x+y\sqrt{-1}$ indicates possibility *in space*. I may here observe, that if we regard space under a purely graphic aspect, and consider all lines drawn through a point as determined by their inclination to two fixed lines, then we have a strictly *two-dimensioned* science. With reference to the subject of space, I would add that "Symbolical Geometry" may be made to take an almost infinite

This will readily be verified by multiplication, or by substitution in the formulæ of page 437 of the last volume of this Journal. So, we might have considered the conditions requisite in order that the product

$$(w_1+i'x_1+j'y_1)(w_2+i'x_2+j'y_2)$$

may have the same form as its factors; and the same of other forms. These concluding remarks are not offered as continuations of the *general* development of the tessarine theory, but as individual instances not perhaps altogether unworthy of notice.

2 Church Yard Court, Temple,
May 8, 1849.

LX. *Remarkable Solar Halos and Mock Suns seen at the Observatory of* Henry Lawson, *Esq.*, *F.R.S. &c.*, *Lansdown Crescent*, *Bath*. *By* Edward Joseph Lowe, *Esq.*, *F.R.A.S.*

[With a Plate.]

1849. FEBRUARY 12ᵈ and 13ᵈ. From 12ᵈ 22½ʰ a halo, A, Plate I. fig. 1, was formed encircling the sun, S, apparently in a very thin vapour, for the sky was quite blue. All the morning there had been a dense ground-fog of some extent.

$23^h\,30^m$. The horizontal diameter of the halo A 45° and its vertical diameter 47°. The portion of the heavens enclosed within this circle was many shades darker than that without, especially so near the circle, for it gradually became lighter near the sun. The sun was also surrounded with a burr, and had his rays carried out considerably, at times 20°. The inner edge of the halo was well-defined, but the outer edge very confused; the breadth about 1°.

$23^h\,40^m$. Two mock suns, B and C, fig. 2, faintly visible, partaking of the same tint as the halo, viz. *pale yellow*, were formed in the edge of the halo A on the sun's horizontal level,

---

variety of forms. Thus, let A, B, C, D be four points in space, then the equation

$$D-C=B-A \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (a.)$$

may be used to indicate either that the lines DC and BA are equal and *parallel*, as is done in the system of Sir W. R. Hamilton; or we may use (*a.*) to denote that those lines are equal and directed to some one point in space; and, in this latter system, (when the last-mentioned point is at an *infinite* distance,) we may be reconducted to the former one. And other systems might be devised, each geometrical system having probably a corresponding one in analysis, and *vice versâ*.