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III. On a new imaginary in algebra

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III. *On a new Imaginary in Algebra. By* JAMES COCKLE, *Esq., M.A., of Trinity College, Cambridge, and Barrister-at-Law of the Middle Temple*.*

ALGEBRA may be regarded under the triple aspect presented by the words Identity, Equivalence, and Impossibility†; but the latter view will fall more particularly within the scope of the present observations. The ordinary algebra, it is true, takes cognizance, not only of negative and unreal quantities, but sometimes of questions involving impossibility. This impossibility is indicated either by contradictory arithmetical results or, occasionally, by the symbol *infinity*‡. But, neither in the one case nor the other, does the indication of impossibility furnish us with the elements of a calculus. Unreal results, on the contrary, although not subjects of conception, like number, nor directly interpretable, like negative quantities, are yet not only indirectly interpretable, but also important instruments of investigation. Why is this? My answer is,—because impossibility has never yet been symbolized. And I would add that, before this is done, we ought to

* Communicated by T. S. Davies, Esq., F.R.S.L. & Ed., &c., who requests us to annex the following note.

[It will of course be understood that I do not pledge myself to an agreement with Mr. Cockle's views on the *geometrical* signification of his i, j, k. With respect to them as algebraical symbols, I would not here offer an opinion: but with respect to the geometrical interpretation, I take a totally different view, as will be inferred from a short paper of mine printed in vol. xxix. (pp. 171–175) of the Philosophical Magazine, under the signature "Shadow."

Mr. Cockle has undoubtedly the weight of cotemporary scientific authority on his side of the question; and, indeed, I believe I stand nearly alone in the view I take of these questions. It would, however, be as unphilosophical a mode of searching for the truth as it would be disingenuous in the discussion, to suppress the expression of all views which differ from those which I may happen to entertain. It is always a matter of far less moment WHO *is right* than WHAT *is true.*

Mr. Cockle very kindly put his paper into my hands for perusal before he printed it; and I have much pleasure in fulfilling his request by forwarding it to you.

I may add, that it is much to be desired that the *history* of the attempts that have been made to give an explanation of the *symbols of incongruity* should be published. A strict discrimination between the views of different algebraists and geometers might prevent the waste of much valuable time and power; for there can be no doubt that a large portion of the speculations which have been put forward in recent times are essentially identical with much earlier ones.

Little Heath, Charlton, Nov. 25, 1848. T. S. D.]

† Phil. Mag. S. 3, vol. xxxii. p. 352.

‡ Peacock, Report on Analysis (Third Report of British Association), pp. 237–238.

hesitate in saying, either that impossibility is incapable of being rendered subservient to the purposes of algebra, or even that it is absolutely uninterpretable. A symbol for impossibility is not only desirable, but actually necessary, provided that we wish to classify with accuracy the various subjects of algebraic research, and to distinguish those which are unreal from those which are impossible. We might adopt an arbitrary symbol to denote impossibility; but a deduced symbol is preferable, for it gives to our investigations the character of true developments of ordinary algebra. Such a deduced symbol I have obtained by means of a surd equation *. The next step is, to ascertain the fundamental properties of the new symbol, its origin and nature being duly considered. As we might expect, *à priori*, anomalous results offer themselves in the course of our progress with these inquiries; such results require at least an attempt at explanation. The geometrical interpretation (or rather capability of interpretation) of the new symbol is another point not unworthy of consideration, and the same may be said of the employment of that symbol in analytical discussions. I purpose, then, in this paper, to treat,—1, of the Utility of the new Symbol; 2, of the Value of its Square; 3, of a certain Anomalous Result; 4, of the Interpretation of the Symbol in Geometry; and, lastly, of its proposed Employment in Analysis.

1. *Of the Utility of the new Symbol.*

I shall show that there are relations, which cannot be expressed by means of the ordinary algebraic symbols. Hence, if it be of any importance to express such relations, a new notation must be adopted. That it is of importance, those who value logical precision and accurate classification will, if I am not mistaken, be disposed to admit, on such grounds alone, and quite irrespectively of any ulterior applications of which the new symbol may be susceptible. It will be observed, in what follows, that I have in certain cases employed an accented zero ($0'$). I have done this in order to distinguish what is, in fact, an absolute negation of existence, from the

* This symbol possesses the character which we might, almost, have anticipated for it *à priori*. In Universal Arithmetic, a negative quantity is *impossible* (Peacock, Report, p. 189), but its square is possible. So, even when negative quantities are recognized, a negative square is contradictory, and unreal quantities are *impossible*, but their squares are possible. In like manner, the square of a Pure Impossible—of a quantity taken as simultaneously positive and negative—is to be treated as possible. The contradiction vanishes on squaring, and there is a striking analogy with the other cases.

ordinary (unaccented) zero which represents a certain state of quantity*.

Let

$$0' = 1 + \sqrt{j}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

multiply both sides of this equation by $1 - \sqrt{j}$, and we have

$$0' \times (1 - \sqrt{j}) = (1 + \sqrt{j})(1 - \sqrt{j}); \quad . \quad . \quad . \quad (2.)$$

but

$$1 - \sqrt{j} = 2 - (1 + \sqrt{j}) = 2 - 0'\,;$$

hence, substituting on the left-hand side of (2.), and multiplying together the factors which compose its right, we obtain

$$0' \times (2 - 0') = 1 - j. \quad . \quad . \quad . \quad . \quad . \quad (3.)\dagger$$

In (1.) substitute unity for j; the result is

$$0' = 1 + 1\,;$$

hence j is not equal to unity, and consequently $1 - j$ is not equal to zero. Acting upon this, let us proceed to obtain, from the symbols of ordinary algebra, the most general expression for a quantity different from zero. We may attain our end as follows. Let a, w, and x be any real quantities whatever, positive, negative or zero, and let $i = \sqrt{-1}$; also suppose that

$$W = w - a + \left(\frac{a-a}{w-a}\right)\left(\frac{a-a}{x-a}\right),$$

and

$$X = x - a + \left(\frac{a-a}{w-a}\right)\left(\frac{a-a}{x-a}\right);$$

then

$$W + iX$$

is the most general expression of ordinary algebra for a quantity different from zero, and may be made to take any given value whatever, excepting zero. Now, as we have seen,

$$1 - j = \textit{some quantity other than zero}\,;$$

hence, those who would maintain the adequacy of the ordinary notation to express any relation whatever, possible or impossible, must sustain the equation

$$1 - j = W + iX\,;$$

* Peacock (Third Report of British Association), pp. 232, 233, &c., and also p. 268. The accented zero is discontinuous; and a remark of Professor J. R. Young, on impossible equations, in the Mechanics' Magazine (vol. xlix. p. 463) suggests the characteristic that, to $0'$, the right-hand side of (1.) can make *no approximation* in terms of the ordinary symbols of algebra.

† The product $\sqrt{j} \times \sqrt{j}$ has the double form j and $\sqrt{j^2}$; I here take the former value, on the principle that $\sqrt{-1} \times \sqrt{-1}$ is equal to -1.

but this equation cannot be sustained; for, if we deduce from it the value of j, and substitute that value in (1.), we arrive at

$$0' = 1 + \sqrt{1 - \mathrm{W} - i\mathrm{X}},$$

in place of which we may write

$$0' = 1 + \alpha + i\beta;$$

this last equation gives $\beta = 0$, whence we infer that $\mathrm{X} = 0$; consequently, α being real, and the positive square root being taken, we arrive at

$$0' = 1 + \alpha = \textit{unity together with a positive number};$$

which cannot be. Hence j is not of the form $\mathrm{W} + i\mathrm{X}$, and yet it is different from zero, as will be seen on substituting zero for j in (1.). In short, j is a quantity *sui generis*—an impossible quantity—a quantity the very conception of the existence of which involves the equation (3.). And, of this last-mentioned equation, it may be added, that it is to be regarded as one of the principal symbolic decompositions which the theory of j involves, and is not to be confounded with the apparently similar equation

$$0 \times (2 - 0) = \mathrm{W} + i\mathrm{X}.$$

Thus there are impossibilities not capable of being expressed either by zero, or by quantities of the form $\mathrm{W} + i\mathrm{X}$, unlimited as are the values which may be given to W and X. And, although infinity be among such values, it becomes necessary to have recourse to the new symbol j to indicate the impossibility implied in (1.). And j would be useful, were it only to indicate such impossibilities.

2. *Of the Value of its Square.*

We are now arrived at a topic, the discussion of which will, perhaps, assist in showing that the apparent difficulties, connected with the theory of j, are not such as to justify us in rejecting it as unsuited to the purposes of analysis, or as incapable of becoming the symbol of a calculus. In the fact, that j has a real square, we have something like a key to the method of rendering available this anomalous symbol. Starting from our fundamental equation, slightly changed in its form, we have

$$+\sqrt{j} = -1;$$

on cubing both sides of this equation, we obtain

$$+\sqrt{j}^3 = -1,$$

whence* we infer that

$$j^2 = 1.$$

* The present case is distinguishable from that alluded to in the last note.

Is j then identical with either of the ordinary square roots of unity? No. In what respect does it differ from those roots? This I proceed to show, and as follows. Unity may be regarded as the square, either of positive, or of negative unity: and, as we regard it from one or the other point of view, we must write its square roots thus:—

$$+\sqrt{(+1)^2}, \text{ or } -\sqrt{(-1)^2},$$

both of which are included in the expression

$$\pm\sqrt{(\pm 1)^2}.$$

Let us now reverse the signs under the radical; then the last expression becomes

$$\pm\sqrt{(\mp 1)^2}.$$

This last is a contradictory and, consequently, impossible expression, which takes the following two impossible forms,

$$+\sqrt{(-1)^2} \text{ and } -\sqrt{(+1)^2}$$

which I shall represent by j' and j'' respectively. Now, if we square $\pm\sqrt{(\mp 1)^2}$, it becomes $(\mp 1)^2$, the contradiction being eliminated by involution. Hence we infer that

$$j'^2 = j''^2 = 1 = j^2;$$

and it is not difficult to see that j, j' and j'' are values of impossible square roots of unity*. To indicate, however, that the discrepancy between the signs without and within the radical cannot be eliminated by merely changing the sign prefixed to the radical, I shall use additional brackets, and suppose that the following equations hold; viz.

$$\pm j' = \pm[+\sqrt{(-1)^2}], \text{ and } \pm j'' = \pm[-\sqrt{(+1)^2}].$$

Seeing thus the contradictory nature of the symbols j, j', j'', we must not be surprised at finding ourselves, very early in our inquiries on the subject, face to face with such a result as the equation (3.) given above. We see, however, that contradictions may vanish and available results follow.

3. *Of a certain Anomalous Result.*

In the case of the equation (3.), we have seen that there is an anomaly, inherent in the very supposition of impossible quantity, which does not occur in treating of real or unreal quan-

* I think that the following relations hold, viz.—

$$j' = -j'', \quad j'j'' = -1.$$

I shall not here attempt to discuss the relation of j' and j'' to j. The former quantities are only introduced here to illustrate what is meant by an *impossible* square-root of unity.

tity. That equation is to be considered, rather as an evidence of the *nature* of the quantity which I am discussing, than as a guide (or impediment) to us in its symbolic application; and although it merits further consideration, yet I do not feel called upon to bestow that consideration here*, inasmuch as in the theory of tessarines j is not affected with a radical sign, and it consequently becomes unnecessary, for the purpose which I have in view, to enter upon the subject of equations expressed by means of radicals. But I am about to point out another anomaly, the reverse of that which occurs in (3.): it is that on the supposition that j^2 is equal to unity,

$$(1+j)(1-j)=1-1=0;$$

that is to say, the (unaccented) zero may be considered as the product of two impossible factors, neither † of which vanishes. It can, however, be at once shown that this anomaly cannot lead us into error; for assuming the equation

$$a^2-b^2=(a+jb)(a-jb), \quad . \quad . \quad . \quad . \quad (4.)$$

and bearing in mind that a tessarine cannot vanish unless all its constituents are zero, we see that neither $a+jb$ nor $a-jb$ can vanish, unless $a=0$ and $b=0$. Suppose that $a=0=b$, then the equation (4.) becomes an identical one, and no error is introduced. On the other hand, imagine that a^2-b^2 should vanish from a becoming equal to b (both a and b being different from zero), then the right-hand side of (4.) would become

$$(a+ja)(a-ja);$$

but, bearing in mind the fundamental property of tessarines, we should be in no danger of inferring that one of these factors must be zero, and consequently we should introduce no error into our investigations. It may be said, Is zero, then, decomposable into non-vanishing factors? Impossible. I reply, true, the factors *are* impossible: they are so by their origin and nature.

4. *Of the Interpretation of the Symbol in Geometry.*

In this field, I am about to indicate what (I hope) will be my

* If for no other purpose, the accent on the zero is useful for the purpose of denoting an impossible equation. If the accented zero is different from the arithmetical zero (and there are indications of a difference), what is the square $(0'^2)$? a *negation of a negation*? I think that we must regard $0'^n$ as *identical* with $0'$, when n is positive, and, consequently, $0'^{-n}$ as identical with $0'^{-1}$, and then n is negative. Zero is not supposed to be included in these values of n.

† If one such factor be zero, the other must be infinite. But this is inconsistent with the forms of the factors. It is on a consideration of this kind that my theory of congeneric surd equations is based. (See Mechanics' Magazine, vols. xlvii., xlviii. and xlix.)

future course, rather than actually to commence it. But I will not disguise the end which I am desirous of attaining—that of vindicating, for ordinary algebra, a claim to the power of representing space of three dimensions, as completely as, by the aid of unreal quantities, it can denote any conditions whatever *in plano*, and any modification of such conditions. By way of illustration, let there be given two points, A and B, and let it be required to find a third point, C, such, that the rectangle AC × CB may be equal to half the square described upon AB. Now, if we proceed on the supposition that C is somewhere in the straight line which passes through A and B, we must suppose that C lies between A and B, for otherwise the rectangle would obviously be greater than the square. Bearing this in mind, let $AB=2a$, and $AC=x$; then the *quæsitum* of the problem gives the equation

$$x(2a-x)=2a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

or

$$x^2-2ax=-2a^2,$$

whence

$$x=a\pm a\sqrt{-1}.$$

Hence, the problem is an impossible one, if we regard C as lying in AB. But, if we interpret the symbol $\sqrt{-1}$ as meaning perpendicularity*—in which case we must regard (5.) as the representation of the problem in its most general form—we have the following construction. Along AB take $AD=\frac{1}{2}AB$; from D draw DC perpendicular to AB, and equal to DA; then, C is the point required. The point C, so obtained, evidently fulfils the condition of the question. In fact, any line in a plane being given, as an axis, we may represent any point whatever, in the plane, by the formula $p+q\sqrt{-1}$, p and q being real. It is to be remarked, however, that, a line being given in space, when the symbol $\sqrt{-1}$ occurs in our researches, we may (as in the above problem) draw our perpendicular in whatever direction we please, provided only it be in a plane perpendicular to the primitive axis. But, the perpendicular once drawn, we have a *determined plane* to which, I apprehend, all our interpretations of $\sqrt{-1}$ are to be confined; for I conceive that, consistently with ordinary algebra, we cannot, on $\sqrt{-1}$ occurring a second time in our investigations, take that symbol to denote a line perpendicular to, or making any angle with, the former perpendicular. Perhaps I have said enough to show what my own views are, as to the applications of the symbol $\sqrt{-1}$, and the limitations to be imposed on its inter-

* This has been done by Hamilton, Warren, and others.

pretation. Supposing, then, i and its interpretation to be admitted into geometrical inquiries, the question comes, how can j enter into such inquiries when it can never enter into the rational equations in which such inquiries usually result? The answer is, that geometrical conditions are not necessarily reducible to the form of rational equations. Consider, by way of example, the equation (5.). This equation, after reduction to the usual form, may be resolved into congeneric surd factors; and, the geometrical meanings of a and x being lines, we may express the evanescence of those factors in geometrical language. Such evanescence may be possible or impossible; if the latter, then j forces itself into our investigations, and the next point is, how to interpret its occurrence. I think that the following are the considerations which ought to guide us in this interpretation. Imagine three points A, B, and C. If these points are in a straight line, their relations may be represented by real quantities, and the only *determined* space before us is a *line*. But suppose that these points are required to fulfill conditions inconsistent with the hypothesis of their being in the same straight line; then (as will be clearly seen on referring to the above problem) a *plane* is determined, and unreal quantities introduced; and, supposing that a problem respecting three points admits of solution, the most general geometrical entity that can be determined by it is a plane. If then we arrive at an *impossible* quantity, as the result of our geometrical inquiries respecting the possibility of a supposed relation between three points, we may be sure that the relation cannot exist. Were the relation possible, it would be possible in a plane; and i is quite adequate to express (in combination with real quantities) any possible relative position of three points. Let us take a step further, and suppose that we are discussing the position of four points A, B, C, D. Take AB as the primitive axis, and let $\text{AB}=2a$; then the expression $p+iq$ may be made to represent either C or D (or both, provided that all the four points are in the same plane). Thus we shall have A represented by zero, B by $2a$, C by $c+ie$, and D by $d+if$. But, suppose that D is out of the plane of A, B and C, then, from what has been before observed, $d+if$ will cease to represent D, for i means perpendicularity to AB in the plane of A, B and C. Hence, in any inquiry respecting four points, the occurrence of j would not be conclusive as to the impossibility of the problem, but only as to the fact that the points cannot be in the same plane. (I assume, of course, that, in forming the condition or conditions of the problem, all the points are represented by different values of the expression $p+iq$.) In such a case, then, j would (provided the problem were possible) be the sign of perpendicularity to the plane ABC, and the transition, from regarding j as the sign of impossibility, to viewing it as the symbol of perpendicularity, is by no means difficult*. As may be inferred, from what I said in opening the question of interpretation, I am not prepared to complete this view of the question in the present paper; but I cannot refrain from remarking, that j is not an unreal root of unity, and that, although it may indicate perpendicularity, yet that we must *envisage* it in a manner different from that in which we regard i. In fact j indicates perpendicularity *to a plane* as i does *to a line*; $j^2=1$ and $i^2=-1$. We may realize the distinction, geometrically, as follows. On the semidiameter of a sphere conceive another sphere described. Let the point of contact of the spheres be considered as the pole of both. Conceive two points, one at the centre of the larger sphere, and the other situate anywhere on its equator; let the first point revolve in a meridian of the smaller sphere, and the second in the equator of the greater, and let the angular velocity of the first point be double that of the second. We need only consider the relative positions of the points when the first point is either at the centre of the greater sphere, or at the common pole of both the spheres. It will be seen that the phases (so to say) of the points correspond to those of the above quantities j^2 and i^2; and also that the first point represents direction perpendicular to a plane, and the second, direction perpendicular to a line†. I mention this because it might be supposed that j is only a second perpendicular to the primitive axis, and, consequently, that it is only a second unreal root of unity‡.

5. *Of its proposed Employment in Analysis.*

Should the admissibility of the new symbol j be established

* The transition from unreal to impossible quantities will, perhaps, be best exemplified by the two following problems:—(1.) Find three equidistant points, and, (2.), Find four equidistant points. The first may be expressed by means of a quadratic with unreal roots; the unreal quantities arising from one of the points being out of the *line* joining the other two. So, unless I am mistaken, the solution of the last may be exhibited by means of impossible quantities, which take their origin from the fourth point being out of the *plane* of the other three.

† This remark will perhaps be better expressed when we have substituted, for the lesser sphere, a prolate spheroid, and then diminished indefinitely the minor axis of the spheroid. Its major axis is, of course, to remain unaltered, and equal to, and coincident with, the axial semidiameter of the larger sphere.

‡ The symbol j will enter into geometrical inquiries in the following manner. Suppose that we arrive at such an equation as

$$\sqrt{r(+1)^2}+\sqrt{s(+1)^2}=0;$$

then, $s=jr$ is the solution.

its use would not be confined to the discussion of the functions which I have proposed to call tessarines. It would, as it seems to me, be capable of other applications, and would tend to generalize all processes to which the imaginary symbol $\sqrt{-1}$ has been yet applied. Thus, by way of illustration, consider the expression

$$\varepsilon^{w \pm ix \pm jy \pm kz}.$$

If we vary, in every possible way, the order of the signs in the above; add the different values, so obtained, together; and expand the sum; it will be seen that the result is free from imaginaries. Hence, that sum may, in all cases, be used instead of the resulting series. The finite expression for the series, so obtained, would in some instances be found useful.

It now only remains to make one remark respecting the notation which I have adopted. In taking i to represent $\sqrt{-1}$, I think that I acted under an impression that it had been so used anterior to the quaternion theory*. The use of j and k followed that of i, and seemed to offer an easier and better mode of comparing results with that theory than I should otherwise have had. In order, however, in future to avoid confusion, and the misapprehensions which may arise from employing like symbols for unlike purposes, I shall use α, β, and γ in place of i, j and k respectively. Under this notation, a tessarine (t) will be written

$$w + \alpha x + \beta y + \gamma z,$$

where

$$\alpha^2 = -1, \quad \alpha\beta = \gamma, \quad -\beta^2 = \gamma^2, \quad \gamma\alpha = -\beta;$$

and also, if the view taken in this paper be correct,

$$\beta^2 = 1, \text{ and } \beta\gamma = \alpha.$$

In my endeavours to bring space of three dimensions under the dominion of a new species of ordinary algebra, I may perhaps be permitted to disclaim anything like dogmatism. I should wish all my views respecting the new symbol to be regarded in the light of suggestions. And if, regarded as suggestions, they should have the effect of directing attention to the theory of congeneric surd equations, they will not be without their utility.

2 Church-Yard Court, Temple,
November 23, 1848.

Postscript, 7th December 1848. Perhaps I shall be permitted to add the following few lines on the subject:—

* Sir W. R. Hamilton has noticed this in the Phil. Mag. S. 3, vol. xxv.

6. *Of the Modular Relations of Tessarines.*

Let θ be the *amplitude*, ϕ the *colatitude*, ψ the *longitude*, and μ the *modulus* of a tessarine (t). To these quantities (which are identical with the corresponding ones in the quaternion theory, and which may, without confusion, be adopted from that theory) must be added another, which I propose to call the *submodulus*, and to denote by ν. The submodulus is defined by the equation

$$\nu^2 = wy + xz,$$

and the submodulus of the product (t'') of two tessarine factors (t and t') is determined by the relation

$$\nu''^2 = (\mu\nu')^2 + (\mu'\nu)^2,$$

and the modulus of that product by

$$\mu''^2 = (\mu\mu')^2 - (2\nu\nu')^2.$$

We have, further,

$$ww'' + xx'' + yy'' + zz'' = w'\mu^2 + 2y'\nu^2,$$

and

$$w'w'' + xx'' + yy'' + zz'' = w\mu'^2 + 2y\nu'^2.$$

The equation for the submodulus may also be expressed as follows:—

$$\nu^2 = \mu^2 \sin\theta \sin\phi \,(\cos\theta \cos\psi + \sin\theta \sin\psi \cos\phi).$$

The construction of this last equation, and of the preceding ones, I hope to discuss on another occasion. And there is a surface—that defined by the equation

$$\mu y \cos\chi + xz = 0$$

(where χ is supposed constant),—which appears to merit attention in connexion with the theory of tessarines.

Second Postscript, 14th December 1848. I have the satisfaction of adding, that the hope above expressed has not been disappointed, and that I have solved the problem of the *four equidistant points* in the manner I proposed. I have determined the position of the fourth point by an expression of the form

$$A + iB + jC;$$

where C is to be measured in a direction perpendicular to the *plane* in which the first, second, and third points are situate. The four points are, of course, at the angles of a regular tetrahedron. I hope to obtain for the solution a place in this Journal.