XI. On quaternions; or on a new system of imaginaries in algebra

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XI. *On Quaternions; or on a New System of Imaginaries in Algebra.* By Sir WILLIAM ROWAN HAMILTON, *LL.D., V.P.R.I.A., F.R.A.S., Corresponding Member of the Institute of France, &c., Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

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62. THE equations (85.), (90.), and (111.), of articles 56, 57, and 60, give

$$T(\rho-\lambda)=T(\rho-\mu')=b; \quad . \quad . \quad . \quad . \quad (113.)$$

and

$$T(\rho-\mu)=T(\rho-\lambda')=b; \quad . \quad . \quad . \quad . \quad (114.)$$

whence, by the meanings of the signs employed, the two following mutually connected constructions may be derived, for *geometrically generating an ellipsoid from a rhombus of constant perimeter*, or for geometrically describing an arbitrary curve on the surface of such an ellipsoid by the motion of a corner of such a rhombus, which the writer supposes to be new.

1st Generation. Let a rhombus LEM'E', of which each side preserves constantly a fixed length $=b$, but of which the angles vary, move so that the two opposite corners L, M' traverse two fixed and mutually intersecting straight lines AB, AB', (the point L moving along the line AB, and the point M' along AB',) while the diagonal LM', connecting these two opposite corners of the rhombus, remains constantly parallel to a third fixed right line AC (in the plane of the two former right lines); then, *according to whatever arbitrary law the plane of the rhombus may turn*, during this motion, *its two remaining corners* E, E' *will describe curves upon the surface of a fixed ellipsoid*; which surface is thus the *locus of all the pairs of curves* that can be described by this first mode of generation.

2nd Generation. Let now *another rhombus*, L'E''ME''', with the *same constant perimeter* $=4\,b$, move so that its opposite corners L', M traverse the *same two fixed lines* AB, AB', as before, but in such a manner that the diagonal L'M, connecting these two corners, remains parallel (not to the third fixed line AC, but) to a *fourth fixed line* AC'; then, whatever may be the arbitrary law according to which the plane of this new rhombus turns, provided that the angles BAB', CAC', between the first and second, and between the third and fourth fixed lines, have one *common bisector*, the *two remaining corners* E'', E''' *of this second rhombus will describe curves upon the surface of the same fixed ellipsoid*, as that determined by the former generation: which surface is thus the *locus of all the new pairs of curves*, described in this second mode, as it was just now seen to be the locus of all the old pairs of curves, obtained in the first mode of description.

63. The ellipsoid (with three unequal axes), thus generated, is therefore the *common locus of the four curves*, described by the four points E E′ E″ E‴; of which four curves, the first and third may be made to coincide with *any arbitrary curves on that ellipsoid*; but the second and fourth become determined, when the first and third have been chosen. And in this new *system of two connected constructions for generating an ellipsoid*, as well as in that other construction* which was given in article 61 for a *system of two reciprocal ellipsoids*, the two former fixed lines, AB, AB′, are the *axes of two cylinders of revolution*, circumscribed about the ellipsoid which is the locus of the point E; while the two latter fixed lines, AC, AC′, are the *two cyclic normals* (or the normals to the two planes of circular section) of that ellipsoid. The common (internal and external) bisectors, at the centre A, of the angles BAB′, CAC′, made by the first and second, and by the third and fourth fixed lines, coincide in direction with the *greatest and least axes* of the ellipsoid; and the constant length b, of the side of either rhombus, is the length of the *mean semiaxis*. The diagonal LM′ of the first rhombus is the *axis of a first circle* on the ellipsoid, of which circle a diameter coincides with the second diagonal EE′ of the same rhombus; and, in like manner, the diagonal L′M of the second rhombus is the *axis of a second circle* on the same ellipsoid, belonging to the second (or *subcontrary*) system of circular sections of that surface: while the other diagonal E″E‴, of the same second rhombus, is a diameter of the same second circle. In the quaternion analysis employed, the first of these two circular sections of the ellipsoid corresponds to the equations (113.); and the second circular section is represented by the equations (114.), of the foregoing article.

64. We may also present the interpretation of those quaternion equations, or the recent double construction of the ellipsoid, in the following other way, which also appears to be new; although the writer is aware that there would be no difficulty in proving its correctness, or in deducing it anew, either by the method of co-ordinates, or in a more purely geometrical mode. *Conceive two equal spheres to slide within two cylinders* (of revolution, whose axes intersect each other, and of which each touches its own sphere along a great circle of contact), *in such a manner that the right line joining the centres of the spheres shall be parallel to a fixed right line*; then *the locus*

* See Phil. Mag. for May 1848; or Proceedings of Royal Irish Academy for November 1847.

of the varying circle in which the two spheres intersect each other will be an ellipsoid, inscribed at once in both the cylinders, so as to touch one cylinder along one ellipse of contact, and the other cylinder along another such ellipse. And the *same* ellipsoid may be generated as the locus of *another* varying circle, which shall be the intersection of *two other equal spheres sliding within the same two cylinders of revolution*, but with a connecting line of centres which now moves parallel to *another fixed right line*; provided that the angle between these two fixed lines, and the angle between the axes of the two cylinders, have both one common pair of (internal and external) bisectors, which will then coincide in direction with the greatest and least axes of the ellipsoid, while the diameter of each of the *four sliding spheres* is equal to the mean axis. In fact, we have only to conceive (with the recent significations of the letters), that four spheres, with the same common radius $=b$, are described about the points L, M$'$, and L$'$, M, as centres; for then the first pair of spheres will cross each other in that circular section of the ellipsoid which has EE$'$ for a diameter; and the second pair of spheres will cross in the circle of which the diameter is E$''$E$'''$; after which the other conclusions above stated will follow, from principles already laid down.

[To be continued.]

XII. *Proceedings of Learned Societies.*

ROYAL SOCIETY.

[Continued from vol. xxxii. p. 541.]

March 23, 1848. "OBSERVATIONS on some Belemnites and other fossil remains of Cephalopoda, discovered by Mr. Reginald Neville Mantell, C.E., in the Oxford Clay, near Trowbridge in Wiltshire." By Gideon Algernon Mantell, Esq., LL.D., F.R.S., Vice-President of the Geological Society.

The author states, that a line of railway now in progress of construction to connect the large manufacturing town of Trowbridge with the Great Western, being part of the Wilts, Somerset, and Weymouth line, traverses extensive beds of the Oxford clay of the same geological character as those at Christian-Malford in the same county, which furnished the remarkable fossil cephalopods described by Mr. Channing Pearce under the name of *Belemnoteuthis*, and by Professor Owen (in a memoir which received the award of a Royal Medal of this Society), as the animals to which the fossils commonly known by the name of *Belemnites* belong.

The son of the author, Mr. R. N. Mantell, being engaged in these works under the eminent engineer Mr. Brunel, availed himself of the opportunity to form an extensive and highly interesting collection of