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XXII. *On Quaternions; or on a New System of Imaginaries in Algebra.* *By* Professor Sir William Rowan Hamilton, *LL.D., Corresponding Member of the Institute of France, and Royal Astronomer of Ireland.*

[Continued from p. 31.]

22. THE *geometrical considerations* of the foregoing article may often suggest *algebraical transformations* of functions of the new imaginaries which enter into the present theory. Thus, if we meet the function

$$\alpha\,\mathrm{S}\,.\,\alpha'\alpha'' - \alpha'\,\mathrm{S}\,.\,\alpha''\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

we may see, in the first place, that in the recent notation this function is algebraically a pure imaginary, or *vector form*, which may be constructed geometrically in this theory by a straight line having length and direction in space; because the three symbols α, α', α'' are supposed to be themselves such vector forms, or to admit of being constructed by three such lines; while $\mathrm{S}.\alpha'\alpha''$ and $\mathrm{S}.\alpha''\alpha$ are, in the same notation, two *scalar forms*, and denote some two real numbers, positive, negative, or zero. We may therefore equate the proposed function (1.) to a new small Greek letter, accented or unaccented, for example to α''', writing

$$\alpha''' = \alpha\,\mathrm{S}.\,\alpha'\alpha'' - \alpha'\,\mathrm{S}.\,\alpha''\alpha. \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Multiplying this equation by α'', and taking the scalar parts of the two members of the product, that is, operating on it by the characteristic $\mathrm{S}.\alpha''$; and observing that, by the properties of scalars,

$$\mathrm{S}.\,\alpha''\alpha\mathrm{S}.\,\alpha'\alpha'' = \mathrm{S}.\,\alpha''\alpha\,.\,\mathrm{S}.\,\alpha'\alpha''$$
$$= \mathrm{S}.\,\alpha''\alpha'\,.\,\mathrm{S}.\,\alpha''\alpha = \mathrm{S}.\,\alpha''\alpha'\,\mathrm{S}.\,\alpha''\alpha,$$

in which the notation $\mathrm{S}.\alpha''\alpha\mathrm{S}.\alpha'\alpha''$ is an abridgement for $\mathrm{S}(\alpha''\alpha\mathrm{S}.\alpha'\alpha'')$, and the notation $\mathrm{S}.\alpha''\alpha.\mathrm{S}.\alpha'\alpha''$ is abridged from $(\mathrm{S}.\alpha''\alpha).(\mathrm{S}.\alpha'\alpha'')$, while $\mathrm{S}.\alpha'\alpha''$ is a symbol equivalent to $\mathrm{S}(\alpha'\alpha'')$, and also, by article 20, to $\mathrm{S}(\alpha''\alpha')$, or to $\mathrm{S}.\alpha''\alpha'$, although $\alpha'\alpha''$ and $\alpha''\alpha'$ are not themselves equivalent symbols; we are conducted to the equation

$$\mathrm{S}.\,\alpha''\alpha''' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

which shows, by comparison with the general *equation of perpendicularity* assigned in the last article, that *the new vector α''' is perpendicular to the given vector α''*, or that these two vector forms represent two rectangular straight lines in space. Again, because the squares of vectors are scalars (being real, though negative numbers), we have

$$\alpha(\alpha\mathrm{S}.\alpha'\alpha'')\,.\,\alpha' = \alpha^2\alpha'\mathrm{S}.\alpha'\alpha'' = \alpha'(\alpha\mathrm{S}.\alpha'\alpha'')\,.\,\alpha,$$
$$\alpha'(\alpha'\mathrm{S}.\alpha''\alpha)\,.\,\alpha = \alpha'^2\alpha\mathrm{S}.\alpha''\alpha = \alpha(\alpha'\mathrm{S}.\alpha''\alpha)\,.\,\alpha';$$

therefore the equation (2.) gives also

$$\alpha \alpha''' \alpha' = \alpha' \alpha''' \alpha; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

a result which, when compared with the general *equation of coplanarity* assigned in the same preceding article, shows that *the new vector* α''' *is coplanar with the two other given vectors,* α and α'; it is therefore perpendicular to the vector of their product, $\mathrm{V}.\alpha\alpha'$, which is perpendicular to both those given vectors. We have therefore two known vectors, namely $\mathrm{V}.\alpha\alpha'$ and α'', to both of which the sought vector α''' is perpendicular; it is therefore parallel to, or coaxal with, the vector of the product of the two known vectors last mentioned, or is equal to this vector of their product, multiplied by some scalar coefficient x; so that we may write the transformed expression,

$$\alpha''' = x\,\mathrm{V}(\mathrm{V}.\alpha\alpha'.\alpha''). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

And because the function α''' is, by the equation (2.), homogeneous of the dimension unity with respect to each separately of the three vectors α, α', α'', while the function $\mathrm{V}(\mathrm{V}.\alpha\alpha'.\alpha'')$ is likewise homogeneous of the same dimension with respect to each of those three vectors, we see that the scalar coefficient x must be either an entirely constant number, or else a homogeneous function of the dimension zero, with respect to each of the same three vectors; we may therefore assign to these vectors any arbitrary lengths which may most facilitate the determination of this scalar coefficient x. Again, the two expressions (2.) and (5.) both vanish if α'' be perpendicular to the plane of α and α'; in order therefore to determine x, we are permitted to suppose that α, α', α'' are three coplanar vectors: and, by what was just now remarked, we may suppose their lengths to be each equal to the assumed unit of length. In this manner we are led to seek the value of x in the equation

$$x\,\mathrm{V}.\alpha\alpha'.\alpha'' = \alpha\,\mathrm{S}.\alpha'\alpha'' - \alpha'\,\mathrm{S}.\alpha''\alpha, \quad . \quad . \quad (6.)$$

under the conditions

$$\mathrm{S}.\alpha\alpha'\alpha'' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

and

$$\alpha^2 = \alpha'^2 = \alpha''^2 = -1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

so that α, α', α'' are here *three coplanar and imaginary units.* Multiplying each member of the equation (6.), as a multiplier, *into* $-\alpha''$ as a multiplicand, and taking the vector parts of the two products; observing also that

$$\mathrm{V}.\alpha'\alpha'' = -\mathrm{V}.\alpha''\alpha', \text{ and } -\mathrm{V}.\alpha\alpha'' = \mathrm{V}.\alpha''\alpha;$$

we obtain this other equation,

$$x\,\mathrm{V}.\alpha\alpha' = \mathrm{V}.\alpha''\alpha.\mathrm{S}.\alpha'\alpha'' - \mathrm{V}.\alpha''\alpha'.\mathrm{S}.\alpha''\alpha: \quad . \quad . \quad (9.)$$

in which the three vectors $V.\alpha''\alpha$, $V.\alpha''\alpha'$, $V.\alpha\alpha'$ are coaxal, being each perpendicular to the common plane of the three vectors α, α', α''; they bear therefore scalar ratios to each other, and are proportional (by the last article) to the areas of the parallelograms under the three pairs of unit-vectors, α'' and α, α'' and α', and α and α', respectively; that is, to the sines of the angles a, a', and $a'-a$, if a be the rotation from α'' to α, and a' the rotation from α'' to α', in the common plane of these three vectors. At the same time we have (by the principles of the same article) the expressions:

$$-S.\alpha''\alpha = \cos a; \quad S.\alpha''\alpha' = -\cos a';$$

so that the equation (9.) reduces itself to the following very simple form,

$$x \sin(a'-a) = \sin a' \cos a - \sin a \cos a', \quad . \quad . \quad (10.)$$

and gives immediately

$$x = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.)$$

Such then is the value of the coefficient x in the transformed expression (5.); and by comparing this expression with the proposed form (1.), we find that we may write, for *any three vectors*, α, α', α'', not necessarily subject to any conditions such as those of being equal in length and coplanar in direction (since those conditions were not used in *discovering the form* (5.), but only in *determining the value* (11.),) the following *general transformation*:

$$\alpha\, S.\alpha'\alpha'' - \alpha'\, S.\alpha''\alpha = V(V.\alpha\alpha'.\alpha''); \quad . \quad . \quad . \quad (12.)$$

which will be found to have extensive applications.

23. But although it is possible thus to employ geometrical considerations, in order to *suggest* and even to *demonstrate* the validity of many general transformations, yet it is always desirable to know how to obtain the same *symbolic results*, from the *laws of combination of the symbols*: nor ought the *calculus of quaternions* to be regarded as complete, till all such *equivalences of form* can be deduced from such symbolic laws, by the fewest and simplest principles. In the example of the foregoing article, the symbolic transformation may be effected in the following way.

When a scalar form is multiplied by a vector form, or a vector by a scalar, the product is a vector form; and the sum or difference of two such vector forms is itself a vector form; therefore the expression (1.) of the last article is a vector form, and may be equated as such to a small Greek letter; or in other words, the equation (2.) is allowed. But every vector form is equal to its own vector part, or undergoes no change of signification when it is operated on by the characteristic V;

we have therefore this other expression, after interchanging, as is allowed, the places of the two vector factors $\alpha' \alpha''$ of a binary product under the characteristic S,

$$\alpha''' = \mathrm{V}(\alpha\, \mathrm{S}.\, \alpha'' \alpha' - \alpha'\, \mathrm{S}.\, \alpha'' \alpha). \quad . \quad . \quad . \quad (1.)'$$

Substituting here for the characteristic S, that which is, by article 18, symbolically equivalent thereto, namely the characteristic $1 - \mathrm{V}$, and observing that

$$0 = \mathrm{V}(\alpha \alpha'' \alpha' - \alpha' \alpha'' \alpha), \quad . \quad . \quad . \quad . \quad . \quad (2.)'$$

because, by article 20, $\alpha \alpha'' \alpha' - \alpha' \alpha'' \alpha$ is a scalar form, we obtain this other expression,

$$\alpha''' = \mathrm{V}(\alpha'\, \mathrm{V}.\, \alpha'' \alpha - \alpha\, \mathrm{V}.\, \alpha'' \alpha'). \quad . \quad . \quad . \quad (3.)'$$

The expression (1.)' may be written under the form

$$\alpha''' = \mathrm{V}(\alpha\, \mathrm{S}.\, \alpha' \alpha'' - \alpha'\, \mathrm{S}.\, \alpha \alpha''); \quad . \quad . \quad . \quad (4.)'$$

and (3.)' under the form

$$\alpha''' = \mathrm{V}(\alpha\, \mathrm{V}.\, \alpha' \alpha'' - \alpha'\, \mathrm{V}.\, \alpha \alpha''), \quad . \quad . \quad . \quad . \quad (5.)'$$

obtained by interchanging the places of two vector-factors in each of two binary products under the sign V, and by then changing the signs of those two products; taking then the semisum of these two forms (4.)', (5.)', and using the symbolic relation of article 18, $\mathrm{S} + \mathrm{V} = 1$, we find

$$\alpha''' = \frac{1}{2}\, \mathrm{V}(\alpha \alpha' \alpha'' - \alpha' \alpha \alpha'')$$

$$= \mathrm{V}\left(\frac{1}{2}(\alpha \alpha' - \alpha' \alpha).\, \alpha''\right); \quad . \quad . \quad . \quad . \quad (6.)'$$

in which, by article 20, $\frac{1}{2}(\alpha \alpha' - \alpha' \alpha) = \mathrm{V}.\, \alpha \alpha'$; we have therefore finally

$$\alpha''' = \mathrm{V}(\mathrm{V}.\, \alpha \alpha'.\, \alpha''): \quad . \quad . \quad . \quad . \quad . \quad (7.)'$$

that is, we are conducted by this purely symbolical process, from laws of combination previously established, to the transformed expression (12.) of the last article.

24. A relation of the form (4.), art. 22, that is an *equation between the two ternary products of three vectors taken in two different and opposite orders*, or an evanescence of the scalar part of such a ternary product, may (and in fact does) present itself in several researches; and although we know, by art. 21, the *geometrical interpretation* of such a symbolic relation between three vector forms, namely that it is the condition of their representing *three coplanar lines*, which interpretation may suggest a transformation of one of them, as a *linear function with scalar coefficients*, of the two other vectors, because any one straight line in any given plane may be treated as the

diagonal of a parallelogram of which two adjacent sides have any two given directions in the same given plane; yet it is desirable, for the reason mentioned at the beginning of the last article, to know how to obtain the same general transformation of the same symbolic relation, without having recourse to geometrical considerations.

Suppose then that any research has conducted to the relation,

$$\alpha\,\alpha'\,\alpha'' - \alpha''\,\alpha'\,\alpha = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

which is not in this theory an identity, and which it is required to transform. [We propose for convenience to commence from time to time a new numbering of the *formulæ*, but shall take care to avoid all danger of confusion of reference, by naming, where it may be necessary, the *article* to which a formula belongs; and when no such reference to an article is made, the formula is to be understood to belong to the *current series* of formulæ, connected with the existing investigation.] By article 20, we may write the recent relation (1.) under the form,

$$\mathrm{S}.\,\alpha\,\alpha'\,\alpha'' = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

and because generally, for any three vectors, we have the formula (12.) of art. 22, if we make, in that formula, $\alpha'' = \mathrm{V}.\,\beta\beta'$, and observe that $\mathrm{S}(\mathrm{V}.\beta\beta'.\alpha) = \mathrm{S}(\alpha\,\mathrm{V}.\beta\beta') = \mathrm{S}.\alpha\beta\beta'$, we find, for *any four vectors* $\alpha\alpha'\,\beta\beta'$, the equation:

$$\mathrm{V}(\mathrm{V}.\alpha\alpha'.\mathrm{V}.\beta\beta') = \alpha\,\mathrm{S}.\alpha'\beta\beta' - \alpha'\,\mathrm{S}.\alpha\beta\beta'; \quad . \quad . \quad . \quad (3.)$$

making then, in this last equation, $\beta = \alpha'$, $\beta' = \alpha''$, we find, for *any three vectors*, $\alpha\,\alpha'\,\alpha''$, the formula:

$$\mathrm{V}(\mathrm{V}.\alpha\alpha'.\mathrm{V}.\alpha'\alpha'') = -\alpha'\,\mathrm{S}.\alpha\alpha'\alpha''. \quad . \quad . \quad . \quad . \quad (4.)$$

If then the scalar of the product $\alpha\,\alpha'\,\alpha''$ be equal to zero, that is, if the condition (2.) or (1.) of the present article be satisfied, the product of the two vectors $\mathrm{V}.\alpha\alpha'$ and $\mathrm{V}.\alpha'\alpha''$ is a scalar, and therefore the latter of these two vectors, or the opposite vector $\mathrm{V}.\alpha''\alpha'$, is in general equal to the former vector $\mathrm{V}.\,\alpha\,\alpha'$, multiplied by some scalar coefficient b; we may therefore write, under this condition (1.), the equation

$$\mathrm{V}.\,\alpha''\,\alpha' = b\,\mathrm{V}.\,\alpha\,\alpha', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

that is,

$$\mathrm{V}.\,(\alpha'' - b\,\alpha)\,\alpha' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

so that the one vector factor $\alpha'' - b\alpha$ of this last product must be equal to the other vector factor α' multiplied by some new scalar b'; and we may write the formula,

$$\alpha'' = b\,\alpha + b'\,\alpha', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

as a transformation of (1.) or of (2.). We may also write, more symmetrically, the equation

$$a\alpha + a'\alpha' + a''\alpha'' = 0, \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

introducing *three* scalar coefficients a, a', a'', which have however only *two arbitrary ratios*, as a symbolic transformation of the proposed equation $\alpha\alpha'\alpha'' - \alpha''\alpha'\alpha = 0$. And it is remarkable that while we have thus *lowered by two units the dimension of that proposed equation*, considered as involving three variable vectors α, α', α'', we have at the same time *introduced* (what may be regarded as) *two arbitrary constants*, namely the two ratios of a, a', a''. A converse process would have served to *eliminate two arbitrary constants*, such as these two ratios, or the two scalar coefficients b and b', from a linear equation of the form (8.) or (7.), between three variable vectors, at the same time *elevating the dimension of the equation by two units*, in the passage to the form (2.) or (1.). And the analogy of these two converse transformations to *integrations and differentiations of equations* will appear still more complete, if we attend to the intermediate stage (5.) of either transformation, which is of an *intermediate degree*, or dimension, and involves *one arbitrary constant b*; that is to say, *one more* than the equation of highest dimension (1.), and *one fewer* than the equation of lowest dimension (7.).

25. As the equation $S.\alpha\alpha'\alpha'' = 0$ has been seen to express that the three vectors $\alpha\alpha'\alpha''$ represent coplanar lines, or that any one of these three lines, for example the line represented by the vector α, is in the plane determined by the other two, when they diverge from a common origin; so, if we make for abridgement

$$\left.\begin{aligned} \beta &= V(V.\alpha\alpha'.V.\alpha'''\alpha^{\text{IV}}), \\ \beta' &= V(V.\alpha'\alpha''.V.\alpha^{\text{IV}}\alpha^{\text{V}}), \\ \beta'' &= V(V.\alpha''\alpha'''.V.\alpha^{\text{V}}\alpha), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1.)$$

the equation

$$S.\beta\beta'\beta'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

may easily be shown to express that *the six vectors* $\alpha\alpha'\alpha''\alpha'''\alpha^{\text{IV}}\alpha^{\text{V}}$ *are homoconic*, or represent *six edges of one cone of the second degree*, if they be supposed to be all drawn from one common origin of vectors. For if we regard the five vectors $\alpha'\alpha''\alpha'''\alpha^{\text{IV}}\alpha^{\text{V}}$ as given, and the remaining vector α as variable, then first the equation (2.) will give for the locus of this variable vector α, some cone of the second degree; because, by the definitions (1.) of β, β', β'', if we change α to $a\alpha$, a being any scalar, each of the two vectors β and β'' will also be multiplied by a, while β' will not be altered; and therefore the function $S.\beta\beta'\beta''$ will be multiplied by a^2, that is by the square of the scalar a, by which the vector α is multiplied. In the next place, this conical locus of α will contain the given vector α'; because if we suppose $\alpha = \alpha'$, we have $\beta = 0$, and the equation (2.) is satisfied: and in like manner the locus of α contains the vector α^{v}, because the supposition $\alpha = \alpha^{\text{v}}$ gives $\beta'' = 0$. In the third place, the cone contains α'' and α^{iv}; for if we suppose $\alpha = \alpha''$, then, by the principle contained in the formula (4.) of the last article, we have

$$\beta'' = -\text{V}(\text{V}.\alpha^{\text{v}}\alpha''.\text{V}.\alpha''\alpha''') = \alpha''\text{S}.\alpha^{\text{v}}\alpha''\alpha''';$$

and by the same principle, under the same condition,

$$\begin{aligned}\text{V}.\beta\beta' &= \text{V}(\text{V}(\text{V}.\alpha'''\alpha^{\text{iv}}.\text{V}.\alpha'\alpha'').\text{V}(\text{V}.\alpha'\alpha''.\text{V}.\alpha^{\text{iv}}\alpha^{\text{v}}))\\ &= -\text{V}.\alpha'\alpha''.\text{S}(\text{V}.\alpha'''\alpha^{\text{iv}}.\text{V}.\alpha'\alpha''.\text{V}.\alpha^{\text{iv}}\alpha^{\text{v}});\end{aligned}$$

but $\text{S}(\text{V}.\alpha'\alpha''.\alpha'') = \text{S}.\alpha'\alpha''\alpha'' = 0$; therefore $\text{S}.\beta\beta'\beta'' = \text{S}(\text{V}.\beta\beta'.\beta'') = 0$; and in like manner this last condition is satisfied, if $\alpha = \alpha^{\text{iv}}$, because β and $\text{V}.\beta'\beta''$ then differ only by scalar coefficients from α^{iv} and $\text{V}.\alpha^{\text{iv}}\alpha^{\text{v}}$, respectively, so that the scalar of their product is zero. Finally, the conical locus of α contains also the remaining vector α''', because if we suppose $\alpha = \alpha'''$, we have

$$\beta = \alpha'''\text{S}.\alpha'\alpha'''\alpha^{\text{iv}}, \quad \beta'' = \alpha'''\text{S}.\alpha''\alpha'''\alpha^{\text{v}},$$

and therefore in this case $\text{S}.\beta\beta'\beta'' = 0$, because the scalar of the product of α''' and $\beta'\alpha'''$ is zero. The locus of α is therefore a cone of the second degree, containing the five vectors $\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \alpha^{\text{v}}$; and in exactly the same manner it may be shown without difficulty that *whichever of the six vectors* $\alpha \ldots \alpha^{\text{v}}$ *may be regarded as the variable vector, its locus assigned by the equation* (2.), of the present article, *is a cone of the second degree, containing the five other vectors.* We may therefore say that this equation,

$$\text{S}.\beta\beta'\beta'' = 0,$$

when the symbols β, β', β'' have the meanings assigned by the definitions (1.), or (substituting for those symbols their values) we may say that the following equation

$$\left.\begin{aligned}\text{S}\{\text{V}(\text{V}.\alpha\alpha'.\text{V}.\alpha'''\alpha^{\text{iv}}).\text{V}(\text{V}.\alpha'\alpha''.\text{V}.\alpha^{\text{iv}}\alpha^{\text{v}}).\\ \text{V}(\text{V}.\alpha''\alpha'''.\text{V}.\alpha^{\text{v}}\alpha)\} = 0,\end{aligned}\right\} \quad . \quad . \quad (3.)$$

is the *equation of homoconicism*, or of *uniconality*, expressing that, when it is satisfied, one common cone of the second degree passes through all the six vectors $\alpha\alpha'\alpha''\alpha'''\alpha^{\text{iv}}\alpha^{\text{v}}$, and enabling us to deduce from it all the properties of this common cone.

26. The considerations employed in the foregoing article might leave a doubt whether *no other* cone of the same degree could pass through the same six vectors; to remove which doubt, by a method consistent with the spirit of the

present theory, we may introduce the following considerations respecting conical surfaces in general.

Whatever four vectors may be denoted by α, α', β, β', we have

$$\mathrm{V}(\mathrm{V}.\alpha\alpha'.\mathrm{V}.\beta\beta') + \mathrm{V}(\mathrm{V}.\beta\beta'.\mathrm{V}.\alpha\alpha') = 0; \quad . \quad (1.)$$

substituting then for the first of these two opposite vector functions the expression (3.) of art. 24, and for the second the expression formed from this by interchanging each α with the corresponding β, we find, for any four vectors,

$$\alpha\mathrm{S}.\alpha'\beta\beta' - \alpha'\mathrm{S}.\alpha\beta\beta' + \beta\mathrm{S}.\beta'\alpha\alpha' - \beta'\mathrm{S}.\beta\alpha\alpha' = 0. \quad . \quad (2.)$$

Again, it follows easily from principles and results already stated, that the scalar of the product of three vectors changes sign when any two of those three factors change places among themselves, so that

$$\left.\begin{array}{l}\mathrm{S}.\alpha\beta\gamma = -\mathrm{S}.\alpha\gamma\beta = \mathrm{S}.\gamma\alpha\beta \\ = -\mathrm{S}.\gamma\beta\alpha = \mathrm{S}.\beta\gamma\alpha = -\mathrm{S}.\beta\alpha\gamma.\end{array}\right\} \quad . \quad . \quad . \quad (3.)$$

Assuming therefore any three vectors ι, $\varkappa$, λ, of which the scalar of the product does not vanish, we may express any fourth vector α in terms of these three vectors, and of the scalars of the three products $\alpha\varkappa\lambda$, $\iota\alpha\lambda$, $\iota\varkappa\alpha$, by the formula:

$$\alpha\mathrm{S}.\iota\varkappa\lambda = \iota\mathrm{S}.\alpha\varkappa\lambda + \varkappa\mathrm{S}.\iota\alpha\lambda + \lambda\mathrm{S}.\iota\varkappa\alpha. \quad . \quad . \quad (4.)$$

Let α be supposed to be a vector function of one scalar variable t, which supposition may be expressed by writing the equation

$$\alpha = \phi(t); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

and make for abridgement

$$\frac{\mathrm{S}.\alpha\varkappa\lambda}{\mathrm{S}.\iota\varkappa\lambda} = f_1(t); \quad \frac{\mathrm{S}.\iota\alpha\lambda}{\mathrm{S}.\iota\varkappa\lambda} = f_2(t); \quad \frac{\mathrm{S}.\iota\varkappa\alpha}{\mathrm{S}.\iota\varkappa\lambda} = f_3(t); \quad . \quad (6.)$$

the forms of these three scalar functions $f_1 f_2 f_3$ depending on the form of the vector function ϕ, and on the three assumed vectors $\iota\varkappa\lambda$, and being connected with these and with each other by the relation

$$\phi(t) = \iota f_1(t) + \varkappa f_2(t) + \lambda f_3(t). \quad . \quad . \quad . \quad . \quad (7.)$$

Conceive t to be eliminated between the expressions for the two ratios of the three scalar functions $f_1 f_2 f_3$, and an equation of the form

$$\mathrm{F}(f_1(t),\ f_2(t),\ f_3(t)) = 0 \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

to be thus obtained, in which the function F is scalar (or real), and homogeneous; it will then be evident that while the equation (5.) may be regarded as the *equation of a curve in space* (equivalent to a system of three real equations between the three co-ordinates of a curve of double curvature andan auxiliary variable t, which latter variable may represent the *time*, in a motion along this curve), the *equation of the cone* which passes through this arbitrary curve, and has its vertex at the origin of vectors, is

$$F(S.\alpha\kappa\lambda, \quad S.\iota\alpha\lambda, \quad S.\iota\kappa\alpha) = 0. \quad . \quad . \quad . \quad . \quad (9.)$$

Such being a form in this theory for the equation of an *arbitrary conical surface*, we may write, in particular, as a *definition of the cone of the nth degree*, the equation:

$$\Sigma(A_{p,q,r}(S.\alpha\kappa\lambda)^p.(S.\iota\alpha\lambda)^q.(S.\iota\kappa\alpha)^r) = 0; \quad . \quad (10.)$$

p, q, r being any three whole numbers, positive or null, of which the sum is n; $A_{p,q,r}$ being a scalar function of these three numbers; and the summation indicated by Σ extending to all their systems of values consistent with the last-mentioned conditions, which may be written thus:—

$$\left.\begin{array}{c} \sin p\pi = \sin q\pi = \sin r\pi = 0; \\ p \geqq 0, \quad q \geqq 0, \quad r \geqq 0; \\ p + q + r = n. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (11.)$$

When $n = 2$, these conditions can be satisfied only by *six* systems of values of p, q, r; therefore, in this case, there enter only six coefficients A into the equation (10.); consequently *five* scalar ratios of these six coefficients are sufficient to particularize a cone of the second degree; and these can in general be found, by ordinary elimination between five equations of the first degree, when five particular vectors are given, such as α', α'', α''', α^{IV}, α^{V}, through which the cone is to pass, or which its surface must contain upon it. Hence, as indeed is known from other considerations, it is in general a determined problem to find the particular cone of the second degree which contains on its surface five given straight lines: and the general solution of this problem is contained in the equation of homoconicism, assigned in the preceding article. The proof there given that the six vectors $\alpha \ldots \alpha^{\text{V}}$ are homoconic, when they satisfy that equation, does not involve any property of conic sections, nor even any property of the circle: on the contrary, that equation having once been established, by the proof just now referred to, might be used as the basis of a complete theory of conic sections, and of cones of the second degree.

27. To justify this assertion, without at present attempting to effect the actual development of such a theory, it may be sufficient to deduce from the equation of homoconicism assigned in article 25, that great and fertile property of the circle, or of the cone with circular base, which was discovered by the genius of Pascal. And this deduction is easy; for the three

auxiliary vectors β, β', β'', introduced in the equations (1.) of the 25th article, are evidently, by the principles stated in other recent articles of this paper, the respective lines of intersection of three pairs of planes, as follows:—The planes of $\alpha\,\alpha'$ and $\alpha'''\,\alpha^{\text{iv}}$ intersect in β; those of $\alpha'\,\alpha''$ and $\alpha^{\text{iv}}\,\alpha^{\text{v}}$ in β'; and those of $\alpha''\,\alpha'''$ and $\alpha^{\text{v}}\,\alpha$ in β''; and the form (2.), art. 25, of the equation of homoconicism, expresses that these three lines, $\beta\,\beta'\,\beta''$, are coplanar. *If then a hexahedral angle be inscribed in a cone of the second degree, and if each of the six plane faces be prolonged* (if necessary) *so as to meet its opposite in a straight line, the three lines of meeting of opposite faces,* thus obtained, *will be situated in one common plane*: which is a form of the theorem of Pascal.

[To be continued.]

XXIII. *On the Influence exerted by Electricity, Platinum and Silver upon the Luminosity of Phosphorus.* By *Dr.* C. F. Schœnbein*.

SOME time ago I tried to show that the shining of phosphorus in atmospheric air is intimately connected with the formation of that highly oxidizing agent I have called ozone. The correctness of that view is confirmed by the fact that phosphorus never becomes luminous if the production of ozone be prevented, or that luminous phosphorus grows dark if the ozone be removed. It is well-known that phosphorus remains dark at low temperatures, and I have ascertained that under these circumstances no ozone is produced; my experiments have further shown that phosphorus still shines in ozonized air at a temperature at which in pure air phosphorus exhibits not the slightest emission of light.

According to the results of my former researches, ozone is formed during electrical discharges taking place in atmospheric air, the electrolysis of water and the action exerted by phosphorus upon moist mixtures of oxygen and nitrogen, oxygen and hydrogen, oxygen and carbonic acid gas.

These facts taken together led me to suspect that phosphorus might become luminous in atmospheric air within which electrical discharges had been effected, at a temperature at which phosphorus does not shine in common air. How far that conjecture is well-founded will appear from the facts I am going to state.

1. If, at a temperature of 2° to 5° R. below zero, a piece of phosphorus, about an inch long, and having a clean surface,

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