



XLVI. On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables

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upon in Lieut.-Col. Sabine's valuable Toronto report contained in the last Number of the Phil. Mag. p. 20, I beg to refer to my papers in the Phil. Mag. for June and July 1842, in which the Plymouth and Leith observations showed that *if the barometer or thermometer be observed four times a day at six-hour intervals, COMMENCING AT ANY CONVENIENT HOUR, the result of an annual series would all but equal the mean of one or two-hourly observations.* These conclusions are confirmed by Col. Sabine's communication; whence, denoting by

4 t the sums of the observations at 2, ^h 8, ^h 14, ^h 20, by
 4 t' 4, 10, 16, 22, and
 4 t'' 6, 12, 18, 0,

results the following Table :—

	Place.	t .	t' .	t'' .	Mean value.	Mean range.
Temperature	Toronto ...	44.29	44.56	44.35	44.40	11.35
	Greenwich .	49.50	49.70	49.50	49.60	10.80
Vapour pressure (inches).	Toronto260	.259	.254	.259	.057
	Greenwich .	.327	.327	.327	.326	.047
Gaseous pressure 29 inches.	Toronto349	.350	.348	.349	.067
	Greenwich .	.504	.506	.505	.505	.045
Force of wind, lbs. on 1 inch square.	Toronto ...	161½	159	146	155½	133
	Greenwich .	107	107¾	105¾	106¾	42½

London, February 17, 1845.

P.S. If one of a party of travellers ascending a high mountain await at one of the stations the return of the others, his six-hourly observations on the thermometer and barometer would give the *mean* daily height of these instruments, and hence the altitude of the station; whence by the method of differences, the exact altitude of the summit would always be ascertainable.

XLVI. *On a Connection between the General Theory of Normal Couples and the Theory of Complete Quadratic Functions of Two Variables.* By JOHN T. GRAVES, M.A., F.R.S., Examiner in Laws and Jurisprudence in the University of London.*

THE remarkable researches of Sir W. Hamilton on *Quaternions*, followed by Prof. De Morgan's very able and

* Communicated by the Author. The continuation of Sir W. R. Hamilton's paper on Quaternions will be found in our last Number, p. 220.

original paper on *Triplets* (published in the Cambridge Philosophical Transactions), have induced me to recur to the examination of *Couples*. In the present paper I shall confine myself to the algebraical consideration of the subject.

Couples are of the form $ix + jy$; the variables x and y are called the *constituents* of the couple, the constants i and j its *characteristic coefficients*. The couples that I treat of are supposed to follow the analogy of ordinary algebra, subject to the *characteristic rule*, that "if two couples are equal, their constituents are separately equal." We are precluded from supposing the existence of any linear relation between the coefficients. If we have the equation

$$(1.) \quad ix + jy = ix_1 + jy_1,$$

we are to assume $x = x_1$, $y = y_1$, and are not allowed to establish the equation $i = j \cdot \frac{y_1 - y}{x - x_1}$, if $\frac{y_1 - y}{x - x_1}$ have a determinate value. Hence, if we have $ix + jy = 0$, we are bound to assume $x = 0$, $y = 0$. This view of couples is derived from the ordinary algebra of imaginary quantities. If

$$(2.) \quad x + \sqrt{-1}y = x_1 + \sqrt{-1}y_1,$$

we have $x = x_1$ and $y = y_1$.

We are to assume such rules that functions of couples may be couples. The product of two couples is a couple. If we follow the rules of algebra, we shall have

$$(3.) \quad (ix + jy)(ix_1 + jy_1) = i^2xx_1 + ij(xy_1 + yx_1) + j^2yy_1.$$

Sir W. Hamilton, in quaternions, by a happy deviation from the rules of ordinary algebra, makes $ij = -ji$, but we are at present treating of couples which obey ordinary algebraic rules. Possibly science may gain more by the introduction of *anomalous* couples, but I confine myself at present to *normal* couples.

In order that $i^2xx_1 + ij(xy_1 + yx_1) + j^2yy_1$, which is the product of two couples, may be itself a couple, we must define i^2 , ij and j^2 in such a way as to reduce the product to the form $ix_2 + jy_2$. This cannot be effected unless we suppose

$$\left. \begin{aligned} (4.) \quad i^2 &= i\alpha + j\beta \\ (5.) \quad ij &= i\kappa + j\kappa' \\ (6.) \quad j^2 &= i\beta' + j\alpha' \end{aligned} \right\}, \quad . \quad . \quad . \quad . \quad (A.)$$

$\alpha, \alpha', \beta, \beta', \kappa, \kappa'$ being constants capable of being connected by linear relations. They may be termed "the connected constants of multiplication."

If we assume

$$(7.) \quad (i x + j y) (i x_1 + j y_1) = i x_2 + j y_2,$$

we shall obtain from (3.) and the equations (A.),

$$(8.) \quad \left. \begin{aligned} x_2 &= \alpha x x_1 + \kappa (x y_1 + y x_1) + \beta' y y_1 \\ y_2 &= \alpha' y y_1 + \kappa' (x y_1 + y x_1) + \beta x x_1 \end{aligned} \right\}.$$

In normal couples, the equations

$$(9.) \quad i^2 \cdot j = i \cdot i j, \quad j^2 \cdot i = j \cdot i j,$$

hold good by virtue merely of the assumed rules of multiplication, without the necessity of assuming any linear relation between i and j . We proceed to investigate the relations which must subsist among the constants $\alpha, \alpha', \beta, \beta', \kappa, \kappa'$, in order to satisfy this condition.

Since $i^2 j = i \cdot i j = j \cdot i^2$, we have, by (5.) and (4.),

$$(10.) \quad i^2 \kappa + i j \kappa' = i j \alpha + j^2 \beta.$$

Since $j^2 i = j \cdot i j = i \cdot j^2$, we have, by (5.) and (6.),

$$(11.) \quad i j \kappa + j^2 \kappa' = i^2 \beta' + i j \alpha'.$$

Substituting in (10.) and (11.) the values of $i^2, i j$ and j^2 given by equations (A.), we get from (10.),

$$(12.) \quad (i \alpha + j \beta) \kappa + (i \kappa + j \kappa') \kappa' = (i \kappa + j \kappa') \alpha + (i \beta' + j \alpha') \beta.$$

Similarly, we get from (11.),

$$(13.) \quad (i \kappa + j \kappa') \kappa + (i \beta' + j \alpha') \kappa' = (i \alpha + j \beta) \beta' + (i \kappa + j \kappa') \alpha'.$$

Arranging (12.) and (13.), we obtain

$$(14.) \quad i (\kappa \kappa' - \beta \beta') + j (\kappa' (\kappa' - \alpha) - \beta (\alpha' - \kappa)) = 0,$$

$$(15.) \quad j (\kappa \kappa' - \beta \beta') + i (\beta' (\kappa' - \alpha) - \kappa (\alpha' - \kappa)) = 0.$$

It is evident that the two equations (14.) and (15.) will be satisfied, independently of any linear relation between i and j , if we suppose, in conformity with the characteristic rule of couples,

$$(16.) \quad \kappa \kappa' - \beta \beta' = \kappa' (\kappa' - \alpha) - \beta (\alpha' - \kappa) = \beta' (\kappa' - \alpha) - \kappa (\alpha' - \kappa) = 0.$$

The three equations (16.) are (as it is not difficult to perceive) equivalent to the two following:

$$(17.) \quad \frac{\kappa'}{\beta} = \frac{\alpha' - \kappa}{\kappa' - \alpha} = \frac{\beta'}{\kappa}.$$

The two equations (17.) express the relations which, in every system of normal couples, connect the constants of multiplication. It is easy to see that the equations (17.) will be satisfied by integer values of the constants, if we suppose that the three equal fractions in (17.) are each equal to $-\frac{p}{q}$; and

if we take

$$(18.) \quad \kappa' = p q, \beta = -q q, \alpha' - \kappa = p \eta, \kappa' - \alpha = -q \eta, \beta' = -p \theta, \kappa = q \theta.$$

By these assumptions we are enabled to make the six connected constants, $\alpha, \alpha', \beta, \beta', \kappa, \kappa'$, depend upon five arbitrary and unconnected constants, p, q, g, η, θ . Certain limited variations of sign might be allowed in our last assumptions consistently with the integer character of the constants of multiplication. Thus it would appear more natural to assume the equal fractions in (17.) each equal to $+\frac{p}{q}$ instead of $-\frac{p}{q}$, and to give *positive* values to every numerator and denominator, taking

$$(19.) \quad \kappa' = p g, \beta = q g, \alpha' - \kappa = p \eta, \kappa' - \alpha = q \eta, \beta' = p \theta, \kappa = q \theta.$$

To this there would be no objection on principle, and the results to which it would lead would be convertible with those which follow. The assumptions (19.) were, in fact, nearly those which I at first made, but I found upon trial that the signs of (18.) are those which give the greatest concinnity to the subsequent calculations.

By substituting in equations (A.), in place of the connected constants of multiplication, their values as determined by equations (18.), we get

$$\left. \begin{aligned} (20.) \quad i^2 &= i(p g + q \eta) - j q g \\ (21.) \quad i j &= i q \theta + j p g \\ (22.) \quad j^2 &= j(q \theta + p \eta) - i p \theta \end{aligned} \right\} \dots (B.)$$

From equations (B.) we get the following system of equations, defining a general rule of multiplication to which all normal couples will be found to conform:

$$\left. \begin{aligned} (23.) \quad (i p_1 + j q_1) (i p_2 + j q_2) &= i p_3 + j q_3 \\ (24.) \quad p_3 &= (p g + q \eta) \cdot p_1 p_2 + q \theta \cdot (p_1 q_2 + q_1 p_2) - p \theta \cdot q_1 q_2 \\ (25.) \quad q_3 &= (q \theta + p \eta) \cdot q_1 q_2 + p g \cdot (p_1 q_2 + q_1 p_2) - q g \cdot p_1 p_2 \end{aligned} \right\} (C.)$$

If we express the couple $i p + j q$ by the notation (p, q) , and arrange equations (24.) and (25.), the multiplication of normal couples will be expressed by the following system:

$$\left. \begin{aligned} (26.) \quad (p_1, q_1) (p_2, q_2) &= (p_3, q_3) \\ (27.) \quad p_3 &= p p_1 p_2 g + q p_1 p_2 \eta + (q p_1 q_2 + q q_1 p_2 - p q_1 q_2) \theta \\ (28.) \quad q_3 &= q q_1 q_2 \theta + p q_1 q_2 \eta + (p q_1 p_2 + p p_1 q_2 - q p_1 p_2) g \end{aligned} \right\} (D.)$$

A connection between the theory of multiplication of normal couples and the resolution into factors of quadratic functions of two variables, will appear from the investigation to which I now proceed. Thus the theory of couples becomes connected with that of quadratic equations. We may treat i and j as ordinary algebraic quantities, and endeavour to deduce their values in terms of p, g, η, θ , by means of the

equations (B.). Eliminating j between (20.) and (21.), we get

$$(29.) \quad j = -\frac{i^2 - i(p\varrho + q\eta)}{q\varrho} = \frac{i q \theta}{i - p \varrho}.$$

Hence we find upon arranging,

$$(30.) \quad i^2 - (2p\varrho + q\eta)i + (p^2\varrho + pq\eta + q^2\theta)\varrho = 0.$$

In like manner, by eliminating i between (21.) and (22.), we find

$$(31.) \quad j^2 - (2q\theta + p\eta)j + (p^2\varrho + pq\eta + q^2\theta)\theta = 0.$$

Upon solving (30.), we get

$$(32.) \quad i = \frac{2p\varrho + q\eta \pm q\sqrt{\eta^2 - 4\varrho\theta}}{2},$$

and similarly, upon solving (31.), we get

$$(33.) \quad j = \frac{2q\theta + p\eta \mp p\sqrt{\eta^2 - 4\varrho\theta}}{2}.$$

It will be found upon examination, that, in order to satisfy the equation (21.), we must make the positive sign of the radical in (32.) correspond to the negative sign in (33.), and *vice versa*.

Let the two values of i given by (32.) be denoted by i_1 and i_2 , and let the corresponding values of j be denoted by j_1 and j_2 . Let

$$(34.) \quad p^2\varrho + pq\eta + q^2\theta = u,$$

and let

$$(35.) \quad p_n^2\varrho + p_n q_n \eta + q_n^2\theta = u_n.$$

Then we shall have (as will be found upon trial) the three following connected equations:

$$\left. \begin{aligned} (36.) \quad i_1 i_2 &= u \varrho \\ (37.) \quad i_1 j_2 + j_1 i_2 &= u \eta \\ (38.) \quad j_1 j_2 &= u \theta \end{aligned} \right\} \dots \dots \dots (\text{E.})$$

Hence we obtain the leading theorem,

$$(39.) \quad (i_1 p_n + j_1 q_n)(i_2 p_n + j_2 q_n) = u u_n.$$

Further, we have, by the equations (C.), since both i_1, j_1 and i_2, j_2 satisfy the equations (B.),

$$\left. \begin{aligned} (40.) \quad (i_1 p_1 + j_1 q_1)(i_1 p_2 + j_1 q_2) &= i_1 p_3 + j_1 q_3 \\ (41.) \quad (i_2 p_1 + j_2 q_1)(i_2 p_2 + j_2 q_2) &= i_2 p_3 + j_2 q_3 \end{aligned} \right\}.$$

p_3 and q_3 having the same values as in (24.) and (25.), or, as in (27.) and (28.)

Multiplying together (40.) and (41.), we get by (39.),

$$(42.) \quad u u_1 \cdot u u_2 = u u_3;$$

and, dividing both sides of (42.) by u , we obtain, finally, the interesting result,

$$(43.) \quad u \cdot u_1 \cdot u_2 = u_3.$$

This theorem may be expressed as follows :

“The continued product of three functions of the form

$x^2 \varrho + xy\eta + y^2 \theta$
is of the same form.” x and y are supposed to be constant, and ϱ , η and θ to be variable.

In symbols at length, we have

$$(44.) \quad (x^2 \varrho + xy\eta + y^2 \theta) (x_1^2 \varrho + x_1 y_1 \eta + y_1^2 \theta) (x_2^2 \varrho + x_2 y_2 \eta + y_2^2 \theta) \\ = x_3^2 \varrho + x_3 y_3 \eta + y_3^2 \theta,$$

x_3 and y_3 having the values following :

$$(45.) \quad x_3 = x x_1 x_2 \varrho + y x_1 x_2 \eta + (y x_1 y_2 + y y_1 x_2 - x y_1 y_2) \theta.$$

$$(46.) \quad y_3 = y y_1 y_2 \theta + x y_1 y_2 \eta + (x y_1 x_2 + x x_1 y_2 - y x_1 x_2) \varrho.$$

This appears to be the most important quadratic *modulus* theorem which the extended theory of normal couples furnishes. It admits of a great number of interesting particular cases. For example, making x in (44.) equal to 1, $\varrho = 1$, $y = 0$, the first factor is reduced to 1, and we get the theorem,

“The product of two functions of the form

$x^2 + xy\eta + y^2 \theta$
is of the same form.”

If, further, we make $\eta = 0$ and $\theta = 1$, we get the well-known theorem,

“The product of two sums of two squares is a sum of two squares.”

P.S. “The product of two sums of eight squares is a sum of eight squares.” To Euler is due a corresponding theorem relating to sums of *four* squares, which was extended by Lagrange, who added coefficients to the squares. As Euler’s theorem is connected with Hamilton’s quaternions, so my theorem concerning sums of eight squares may be made the basis of a system of octads or *sets of eight*, and was actually so applied by me about Christmas 1843. I mention this in consequence of the idea suggested by Mr. Cayley in the last Number of this Magazine. But the full statement and proof of the theorem concerning sums of eight squares, and of several other new theorems connected with the doctrine of numbers, must be reserved for another time.

XLVII. *Account of a new Experiment in Electro-Static Induction.* By Prof. MATTEUCCI, in a Letter to Dr. FARADAY.

MY DEAR FARADAY,

I HOPE it will not be disagreeable to you if I describe an experiment which I performed in one of my late lectures on electro-static induction, in which I dwelt entirely upon