membrane would have some resemblance to the curve of a catenary of uniform strength.

If ϕ is zero or negligible the equations in this paper reduce to the usual Poisson equations for thin plates. Now it follows from equation (12) that ϕ can be zero provided that

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} = 0. \quad . \quad . \quad . \quad (106)$$

But this is precisely the condition that the bent middle surface should be a developable surface. When ϕ is zero the middle surface is unstretched, and now we find from our equations, what is quite obvious from geometry, that a plane sheet can be bent into a developable surface without the stretching or shrinking of any of its elements.

If the expression on the left-hand side of equation (106) is small but not zero, it is still possible that ϕ be so small that Poisson's equations are approximately true. The examples we have worked out indicate that this condition is satisfied if w is everywhere small in comparison with the thickness of the plate. Since the expression in (106) is of the second order in w it is clear that ϕ depends on w^2 rather than on w. The condition that ϕ should be negligible will still be satisfied if the deflexion of the middle surface, measured from some developable surface, is small compared with the thickness of the plate. The third problem worked out above supplies an example of this.

X. *On the Transverse Vibrations of Bars of Uniform Cross-Section. By* Prof. S. P. Timoshenko *.

§ 1. In a paper recently published in this Magazine †, I have dealt with the corrections which must be introduced into the equation for transverse vibrations of a prismatic bar, viz.,

$$EI\frac{\partial^4 y}{\partial x^4} + \frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2} = 0, \quad . \quad . \quad . \quad . \quad (1)$$

in order that the effects of "rotatory inertia" and of the deflexion due to shear may be taken into account.

* Communicated by Mr. R. V. Southwell, M.A.

† Phil. Mag. vol. xli. pp. 744–746.

In equation (1)

EI denotes the flexural rigidity of the bar,

Ω the area of the cross-section

and $\frac{\rho}{g}$ the density of the material.

It was shown that the correction for shear, in a representative example, was four times as important as the correction for "rotatory inertia," and that both corrections are unimportant if the wave-length of the transverse vibrations is large in comparison with the dimensions of the cross-section.

In the present paper, an exact solution of the problem is given in the case of a beam of rectangular section, of which the breadth is great or small compared with the depth, so that the problem is virtually one of plane strain or of plane stress. The results are compared with those of my former paper, and confirm the conclusions which were there obtained.

§ 2. When the problem is one of plane strain (so that w is constant), we have to solve the equations *

$$\left.\begin{array}{l}(\lambda+2\mu)\nabla^2\Delta+\rho p^2\Delta = 0, \\ \text{and} \quad \mu\nabla^2\varpi+\rho p^2\varpi = 0, \\ \text{where } \Delta \text{ denotes the cubical dilatation } \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \\ \varpi \quad \text{,,} \quad \text{the rotation } \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right), \\ \text{and} \quad \nabla^2 \quad \text{,,} \quad \text{the operator } \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).\end{array}\right\} \quad (2)$$

Taking the x-axis in the direction of the central line; and choosing for Δ an even and for ϖ an uneven function of y, we may write

$$\left.\begin{aligned}\Delta &= \mathrm{A}\sin\alpha x\sinh my\cos pt, \\ 2\varpi &= \mathrm{B}\cos\alpha x\cosh ny\cos pt,\end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

where A and B are undetermined coefficients,

* A. E. H. Love, 'Theory of Elasticity,' §§ 14 (*d*) and 204.

$$\left.\begin{aligned} m &= \alpha\sqrt{1-\left(\frac{V}{V_1}\right)^2}, \\ n &= \alpha\sqrt{1-\left(\frac{V}{V_2}\right)^2}, \\ V &= \frac{p}{\alpha}, \text{ denotes the velocity of waves of transverse vibration,} \\ V_1 &= \sqrt{\frac{\lambda+2\mu}{\rho}}, \text{ denotes the velocity of waves of dilatation,} \\ \text{and}\quad V_2 &= \sqrt{\frac{\mu}{\rho}}, \text{ denotes the velocity of waves of distortion.} \end{aligned}\right\} \quad (4)$$

It is easily verified that equations (3) are satisfied by the expressions

$$\left.\begin{aligned} u &= \cos \alpha x\,(M \sinh my + N \sinh ny) \cos pt, \\ v &= \sin \alpha x\left(M\frac{m}{\alpha} \cosh my + N\frac{\alpha}{n} \cosh ny\right)\cos pt, \end{aligned}\right\}$$

and the conditions that the boundaries $(y = \pm c)$ are free from traction give us

$$\left.\begin{aligned} [(\lambda+2\mu)m^2-\lambda\alpha^2]\, M \sinh mc + 2\mu\alpha^2 N \sinh nc &= 0, \\ \text{and}\qquad 2mnM \cosh mc + (\alpha^2+n^2)\, N \cosh nc &= 0. \end{aligned}\right\}$$

Hence, by eliminating the ratio M/N, we obtain the "frequency equation" in the form *

$$4\mu\alpha^2 mn \tanh nc = (\alpha^2+n^2)\,[(\lambda+2\mu)m^2-\lambda\alpha^2] \tanh mc\,;$$

whence, denoting the length of the waves by l, and putting $V/V_1 = f$, $V/V_2 = h$, we have

$$4\sqrt{(1-f^2)(1-h^2)} \tanh\left(\frac{2\pi c}{l}\sqrt{1-h^2}\right)$$
$$= (2-h^2)^2 \tanh\left(\frac{2\pi c}{l}\sqrt{1-f^2}\right). \quad (5)$$

If V be given, the corresponding value of the ratio $l/2c$

* This "frequency equation" was found by Prof. P. Ehrenfest and myself in collaboration. The solution given in this paper is my own.

can be calculated from this equation. Some results are given in the table below *.

h =	0·5	0·7	0·9	0·9165	0·9192
$l/2c$ =	4·8	2·5	0·75	0·47	0·30

§ 3. In the case of long waves, the velocity V is inversely proportional to the length l, but as l diminishes it can be seen, from equation (5), to approach a limit which is lower than V_2, and can be found from the relation

$$4\sqrt{(1-f^2)(1-h^2)} = (2-h^2)^2.$$

This limit is the velocity of the "Rayleigh waves" †. If we put $\lambda=\mu$, we obtain, in the limit, $V=0·9194\,V_2$.

As the wave-length l increases, the arguments of the hyperbolic functions in equation (5) decrease, and we can employ the ascending series. If we limit our attention to the first three terms, we may write equation (5) in the form

$$a(\alpha c)^4+b(\alpha c)^2+d = 0, \quad . \quad . \qquad (6)$$

where

$$\left.\begin{aligned} a &= \tfrac{2}{15}[4(1-h^2)^3-(2-h^2)^2(1-f^2)^2], \\ b &= -\tfrac{1}{3}[4(1-h^2)^2-(2-h^2)^2(1-f^2)], \\ d &= -h^4. \end{aligned}\right\} \quad . \quad . \qquad (7)$$

When αc is very small, we can obtain a first approximation by neglecting the first term in (6) and putting

$$(\alpha c)^2 = -\frac{d}{b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

This approximation will be

$$(\alpha c)^2 = \frac{3}{4}\frac{h^2(\lambda+2\mu)}{\lambda+\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

whence

$$V^2 = \tfrac{4}{3}(\alpha c)^2\frac{\mu(\lambda+\mu)}{\rho(\lambda+2\mu)}, \quad . \quad . \quad . \qquad (10)$$

and

$$p^2 = \alpha^2V^2 = \alpha^4c^2\frac{4\mu(\lambda+\mu)}{3\rho(\lambda+2\mu)}. \quad . \quad . \quad . \qquad (11)$$

* In these calculations, σ has been taken to be 0·25.

† A. E. H. Love, *op. cit.* § 214.

Replacing λ by the quantity

$$\lambda' = \frac{2\lambda\mu}{\lambda+2\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

we should obtain the corresponding approximate solution for the case of plane stress, and it is easily verified that this is equivalent to the solution of equation (1).

Proceeding to a further approximation in the case of plane strain, we observe that the quantity d in (6) is small in comparison with a and b, and that the solution of this equation can therefore be written as follows:

$$(\alpha c)^2 = -\frac{d}{b}\left(1+\frac{a}{b}\cdot\frac{d}{b}\right). \quad . \quad . \quad . \quad . \quad (13)$$

The second term in the bracket on the right-hand side is a small correction. In calculating it, we may take for $\frac{d}{b}$ the first approximation (8), and in the expression for a/b we may retain the terms of the order h^2 and f^2 only. We then have

$$\frac{a}{b}\cdot\frac{d}{b} = \tfrac{4}{5}(\alpha c)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The first term on the right of (13) must be calculated more exactly. Retaining in the expression for b terms of the order f^4 and h^4, and substituting for V the first approximation (10), we have

$$-\frac{d}{b} = \tfrac{3}{4}h^2\frac{\lambda+2\mu}{\lambda+\mu}\left[\frac{1}{1-\frac{1}{3}(\alpha c)^2\frac{3\lambda+2\mu}{\lambda+2\mu}}\right] \quad . \quad . \quad (15)$$

Substituting from (14) and (15) in (13), and neglecting small quantities of higher order, we obtain

$$(\alpha c)^2 = \tfrac{3}{4}h^2\frac{\lambda+2\mu}{\lambda+\mu}\left[1+\tfrac{1}{3}(\alpha c)^2\left(\frac{3\lambda+2\mu}{\lambda+2\mu}+\frac{12}{5}\right)\right],$$

or

$$p^2 = \frac{\alpha^4c^2}{3}\,\frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}\left[1-\tfrac{1}{3}(\alpha c)^2\left(\frac{3\lambda+2\mu}{\lambda+2\mu}+\frac{12}{5}\right)\right]. \quad . \quad . \quad . \quad (16)$$

§ 4. The square brackets in (16) contain the required corrections to (11). These correspond principally to the effects of "rotatory inertia" and of shearing force, and could have been obtained with sufficient accuracy from the equation for transverse vibrations of rods, if this is supplemented in the manner explained in my previous paper, where the correction to the frequency p was given in the form of a multiplying term of amount *

$$\left[1-\frac{1}{2}\frac{\pi^2 k^2}{L^2}\left(1+\frac{E}{\lambda C}\right)\right]. \quad . \quad . \quad . \quad . \quad (17)$$

In this expression, π/L is equivalent to the α of the present paper, and $k^2=c^2/3$. λ is a constant relating the shearing force with the angle of shear at the section considered: its value for a rectangular section (if we make use of the experimental results of L. N. G. Filon †) is 8/9. The ratio E/C must be replaced, for comparison with our present problem of plane strain, by the quantity $\frac{4(\lambda+\mu)}{\lambda+2\mu}$ (in the notation of the present paper). We then have from (17), in our present notation, as the correcting factor required in the expression for the *square of the frequency*,

$$\left[1-\tfrac{1}{3}(\alpha c)^2\left(1+\frac{3}{2}\frac{\lambda+\mu}{\lambda+2\mu}\right)\right].$$

With $\sigma=0{\cdot}25$, or $\lambda=\mu$, the correction to p^2, given by (17), is thus

$$\left[1-\tfrac{4}{3}(\alpha c)^2\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

whilst (16) gives

$$\left[1-\tfrac{61}{45}(\alpha c)^2\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

§ 5. This close agreement gives us some confidence in applying the approximate solution of my earlier paper to other shapes of cross-section. We take, for example, the case of a circular cross-section ‡: the exact solution may be written in the form

$$V=\frac{\pi c}{l}\sqrt{\frac{E}{\rho}}\left[1-\frac{\pi^2 c^2}{3l^2}\left(\frac{7}{2}+\frac{E}{\mu}-\frac{\mu}{E}\right)\right], \quad . \quad . \quad (20)$$

where c denotes the radius of the cross-section; whilst the methods of my earlier paper would give

$$V=\frac{\pi c}{l}\sqrt{\frac{E}{\rho}}\left[1-\frac{\pi^2 c^2}{2l^2}\left(1+\frac{E}{\kappa\mu}\right)\right], \quad . \quad . \quad . \quad (21)$$

where κ is the "shear constant" previously denoted by λ.

* *Cf.* equation (13) of the paper referred to.

† A. E. H. Love, *op. cit.* § 245.

‡ See Pochhammer, *Journal f. d. reine u. angw. Math.* Bd. lxxxi. p. 335 (1876).

If we put

$$\lambda = \mu\,; \quad 1/\kappa = \tfrac{3}{4} \times 1{\cdot}40 = 1{\cdot}05,$$

the correcting factor found from (20) will be $1{\cdot}87\,\dfrac{\pi^2 c^2}{l^2}$, and from (21) will be $1{\cdot}81\,\dfrac{\pi^2 c^2}{l^2}$. Thus we have again obtained very satisfactory results from the approximate formula (17).

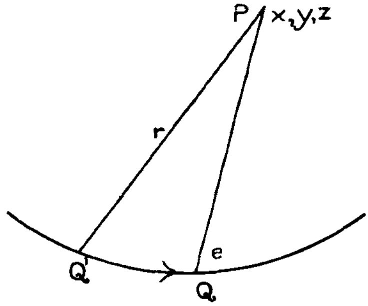
§ 6. In this investigation of transverse vibrations we have taken the expressions (3) for Δ and ϖ. By taking the even function of y for Δ and the uneven function for ϖ, we may obtain the solution of the problem of longitudinal vibrations.

Zagreb,
January 26, 1921.

XI. *On Scalar and Vector Potentials due to Moving Electric Charges.* *By* Prof. A. Anderson *.

In the Philosophical Magazine (March, 1921) Prof. A. Liénard criticises my paper on "A method of finding Scalar and Vector Potentials due to Motion of Electric Charges" (Phil. Mag. August 1920). The criticism, though just, touches only a slight error in notation; the results of the paper are not affected thereby. The last part of my paper is very much condensed and perhaps, on that account, unintelligible, and the object of the present communication is to make the argument fuller, and, I hope, clearer.

Fig. 1.



In fig. 1, Q is the position at time t of a moving charge e, and Q′ is a point moving in the path of Q—the companion

* Communicated by the Author.