LXVI. *On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars.* By Prof. S. P. Timoshenko *.

IN studying the transverse vibrations of prismatic bars, we usually start from the differential equation

$$EI\frac{\partial^4 y}{\partial x^4}+\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

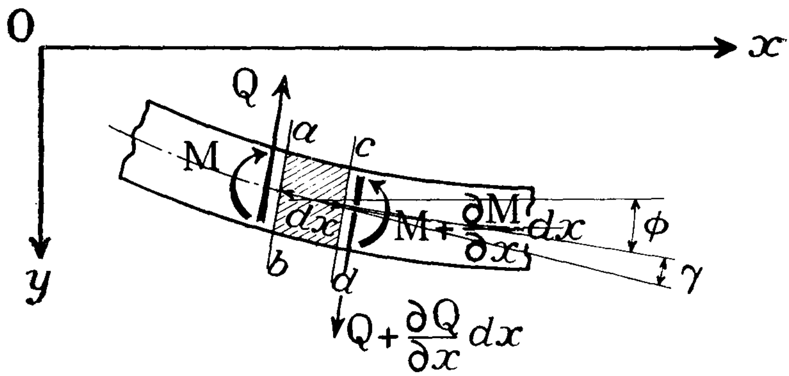
in which EI denotes the flexural rigidity of the bar,
Ω the area of the cross-section,
and $\frac{\rho}{g}$ the density of the material.

When the "rotatory inertia" is taken into consideration, the equation takes the form

$$EI\frac{\partial^4 y}{\partial x^4}-\frac{I\rho}{g}\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}=0. \quad . \quad . \quad . \quad (2)$$

I now propose to show how the effect of the shear may be taken into account in investigating transverse vibrations, and I shall deduce the general equation of vibration, from which equations (1) and (2) may be obtained as special cases.

Fig. 1.



Let $a\ b\ c\ d$ (fig. 1) be an element bounded by two adjacent cross-sections of a prismatic bar. M and Q denote respectively the bending moment and the shearing force.

* Communicated by Mr. R. V. Southwell, M.A. Translated from the Russian by Prof. M. G. Yatsevitch.

The position of the element during vibration will be determined by the displacement of its centre of gravity and by the angular rotation ϕ in the (x, y) plane: the axis Ox may be taken as coinciding with the initial position of the axis of the bar.

The angle at which the tangent to the curve into which the axis of the bar is bent (the curve of deflexion) is inclined to the axis Ox will differ from the angle ϕ by the angle of shear γ. Hence, for very small deflexions, we may write

$$\frac{\partial y}{\partial x}=\phi+\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For determining M and Q we have the familiar expressions

$$\mathrm{M}=-\mathrm{EI}\frac{\partial \phi}{\partial x}, \quad \mathrm{Q}=\lambda \mathrm{C}\Omega\gamma=\lambda\mathrm{C}\Omega\left(\frac{\partial y}{\partial x}-\phi\right), \quad . \quad . \quad (4)$$

where C denotes the modulus of rigidity, for the material of the bar, and λ is a constant which depends upon the shape of the cross-section.

The equations of motion will now be :—

for the rotation—

$$-\frac{\partial \mathrm{M}}{\partial x}dx+\mathrm{Q}dx=\frac{\rho\mathrm{I}}{g}\frac{\partial^2\phi}{\partial t^2}dx,$$

or
$$\mathrm{EI}\frac{\partial^2\phi}{\partial x^2}+\lambda\mathrm{C}\Omega\left(\frac{\partial y}{\partial x}-\phi\right)-\frac{\rho\mathrm{I}}{g}\frac{\partial^2\phi}{\partial t^2}=0, \quad . \quad . \quad . \quad (5)$$

if we substitute from equations (4);

for translation in the direction of Oy—

$$\frac{\partial \mathrm{Q}}{\partial x}dx=\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}dx,$$

or
$$\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}-\lambda\mathrm{C}\Omega\left(\frac{\partial^2 y}{\partial x^2}-\frac{\partial\phi}{\partial x}\right)=0. \quad . \quad . \quad . \quad . \quad (6)$$

Eliminating ϕ from (5) and (6), we obtain the required equation in the form

$$\mathrm{EI}\frac{\partial^4 y}{\partial x^4}+\frac{\rho\Omega}{g}\frac{\partial^2 y}{\partial t^2}-\frac{\rho\mathrm{I}}{g}\left(1+\frac{\mathrm{E}}{\lambda\mathrm{C}}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{\rho^2\mathrm{I}}{g^2\lambda\mathrm{C}}\frac{\partial^4 y}{\partial t^4}=0. \quad (7)$$

Introducing the notation

$$\frac{\mathrm{EI}g}{\rho\Omega}=\alpha^2, \quad \frac{\mathrm{I}}{\Omega}=k^2,$$

we may write equation (7) in the form

$$\alpha^2\frac{\partial^4 y}{\partial x^4}+\frac{\partial^2 y}{\partial t^2}-k^2\left(1+\frac{\mathrm{E}}{\lambda\mathrm{C}}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\frac{k^2\rho}{g\lambda\mathrm{C}}\frac{\partial^4 y}{\partial t^4}=0. \quad (8)$$

In order to estimate the influence of the shear upon the frequency of the vibrations, let us consider the case of a prismatic bar with supported ends. The type of the vibrations may be assumed to be given by

$$y = \mathrm{Y} \sin \frac{m\pi x}{l} \cos p_m t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where l represents the length of the bar, and p_m is the required frequency. By substitution from (9) in equation (8), we obtain the following equation for the frequency:

$$\alpha^2 \frac{m^4\pi^4}{l^4} - p_m{}^2 - \frac{m^2\pi^2k^2}{l^2}\left(1 + \frac{\mathrm{E}}{\lambda \mathrm{C}}\right) p_m{}^2 + \frac{k^2\rho}{g\lambda \mathrm{C}} p_m{}^4 = 0. \quad (10)$$

If only the first two terms on the left side of this equation are retained (this will correspond to the equation (1)), we have

$$p_m = \alpha \frac{m^2\pi^2}{l^2} = \frac{\alpha\pi^2}{\mathrm{L}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $\mathrm{L} = \frac{l}{m}$ represents the length of a wave.

By retaining the first three terms of equation (10) (*i.e.* by neglecting the terms which involve λ), we find

$$p_m = \frac{\alpha\pi^2}{\mathrm{L}^2}\left(1 - \frac{1}{2}\frac{\pi^2k^2}{\mathrm{L}^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

approximately: this result corresponds to equation (2), where the rotatory inertia is taken into consideration.

By using the complete equation (10), and neglecting small quantities of the second order, we find

$$p_m = \frac{\alpha\pi^2}{\mathrm{L}^2}\left[1 - \frac{1}{2}\frac{\pi^2k^2}{\mathrm{L}^2}\left(1 + \frac{\mathrm{E}}{\lambda \mathrm{C}}\right)\right] \quad . \quad . \quad . \quad (13)$$

approximately.

Assuming the values

$$\lambda = \tfrac{2}{3}, \qquad \mathrm{E} = \tfrac{8}{3}\mathrm{C},$$

we have

$$\frac{\mathrm{E}}{\lambda \mathrm{C}} = 4,$$

and hence we see that the correction for shear is four times greater than the correction for rotatory inertia. The value of the correction of course increases with a decrease in the wave-length L, *i. e.*, with an increase in m.

Yougoslavia, Videm.
Summer 1920.