

in outline Table XIII., of which it may be said to be a continuation. The values of Σc_a calculated by means of the values of H_{c_a} , omitting those referring to water, methyl alcohol, and propyl alcohol, agree approximately with the values contained in the third column which were derived directly from experimental data. The agreement is, however, not so good as that exhibited in Table XIII. From the way equation (14) was obtained it is obvious that this must be due to deviations from the relation expressed by equation (13).

Cambridge, June 24, 1909.

LVII. *The Principle of Relativity, and Non-Newtonian Mechanics.* By GILBERT N. LEWIS and RICHARD C. TOLMAN*.

UNTIL a few years ago every known fact about light, electricity, and magnetism was in agreement with the theory of a stationary medium or æther, pervading all space, but offering no resistance to the motion of ponderable matter. This theory of a stagnant æther led to the belief that the absolute velocity of the earth through this medium could be determined by optical and electrical measurements. Thus it was predicted that the time required for a beam of light to pass over a given distance, from a fixed point to a mirror and back, should be different in a path lying in the direction of the earth's motion and in a path lying at right angles to this line of motion. This prediction was tested in the crucial experiment of Michelson and Morley †, who found, in spite of the extreme precision of their method, not the slightest difference in the different paths.

It was also predicted from the æther theory that a charged condenser suspended by a wire would be subject to a torsional effect due to the earth's motion. But the absence of this effect was proved experimentally by Trouton and Noble ‡.

The skill with which these experiments were designed and executed permits no serious doubt as to the accuracy of their results, and we are therefore forced to adopt certain new views of far-reaching importance.

It is true that the results of Michelson and Morley might be simply explained by assuming that the velocity of light depends upon the velocity of its source. Perhaps this assumption has formerly been dismissed without sufficient

* Communicated by the Authors.

† Amer. Jour. Sci. xxxiv. p. 333 (1887).

‡ Phil. Trans. Roy. Soc. (A) ccii. p. 165 (1904).

reason, but recent experimental evidence, to which we shall revert, seems to prove it untenable.

This possibility being excluded, the only satisfactory explanation of the Michelson-Morley experiment which has been offered is due to Lorentz*, who assumed that all bodies in motion are shortened in the line of their motion by an amount which is a simple function of the velocity. This shortening would produce a compensation just sufficient to offset the predicted positive effect in the Michelson-Morley experiment, and would also account for the result obtained by Trouton and Noble. It would not, however, prevent the determination of absolute motion by other analogous experiments which have not yet been tried.

Einstein† has gone one step farther. Because of the experiments that we have cited, and because of the failure of every other attempt that has ever been made to determine absolute velocity through space, he concludes that further similar attempts will also fail. In fact he states as a law of nature that absolute uniform translatory motion can be neither measured nor detected.

The second fundamental generalization made by Einstein he calls "the law of the constancy of light velocity." It states that the velocity of light in free space appears the same to all observers, regardless of the motion of the source of light or of the observer.

These two laws taken together constitute *the principle of relativity*. They generalize a number of experimental facts, and are inconsistent with none. In so far as these generalizations go beyond existing facts they require further verification. To such verification, however, we may look forward with reasonable confidence, for Einstein has deduced from the principle of relativity, together with the electromagnetic theory, a number of striking consequences which are remarkably self-consistent. Moreover the system of mechanics which he obtains is identical with the non-Newtonian mechanics developed from entirely different premises by one of the present authors ‡. Finally, one of the most important equations of this non-Newtonian mechanics has within the past year been quantitatively verified by the experiments of

* *Abhandlungen über theoretische Physik*, Leipzig, 1907, p. 443.

† An excellent summary of the conclusions drawn from the principle of relativity, by Einstein, Planck, and others is given by Einstein in the *Jahrbuch der Radioaktivität*, iv. p. 411 (1907). An interesting treatment of certain phases of this problem is given by Bumstead, *Amer. Jour. Sci.* xxvi. p. 493 (1908).

‡ Lewis, *Phil. Mag.* xvi. p. 705 (1908).

Bucherer * on the mass of a β particle, to which we shall refer later.

Therefore, in as far as present knowledge goes, we may consider the principle of relativity established on a pretty firm basis of experimental fact. Accepting this principle, we shall accept the consequences to which it leads, however extraordinary they may be, provided that they are not inconsistent with one another, nor with known experimental facts.

The consequences which one of us has obtained from a simple assumption as to the *mass of a beam of light*, and the fundamental *conservation laws* of mass, energy, and momentum, Einstein has derived from the *principle of relativity* and the *electromagnetic theory*. We propose in this paper to show that these consequences may also be obtained merely from the *conservation laws* and the *principle of relativity*, without any reference to electromagnetics.

In dealing with such fundamental questions as we meet here it seems especially desirable to avoid as far as possible all technicalities. We have endeavoured to find for each of the following theorems the simplest and most obvious proof, and have used no mathematics beyond the elements of algebra and geometry.

The Units of Space and Time.

The following development will be based solely upon the conservation laws, and the two postulates of the principle of relativity.

The first of these postulates is that there can be no method of detecting absolute translatory motion through space, or through any kind of æther which may be assumed to pervade space. The only motion which has physical significance is the motion of one system relative to another. Hence two similar bodies having relative motion in parallel paths form a perfectly symmetrical arrangement. If we are justified in considering the first at rest and the second in motion, we are equally justified in considering the second at rest and the first in motion.

The second postulate is that the velocity of light as measured by any observer is independent of relative motion between the observer and the source of light †. This idea, that the velocity of light will seem the same to two different observers, even though one may be moving towards and the

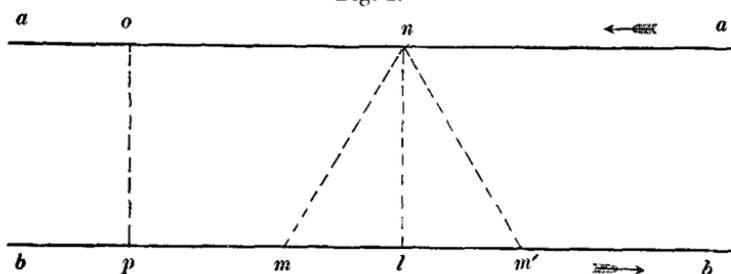
* *Ber. Phys. Ges.* vi. p. 688 (1908); *Ann. Physik*, xxviii. p. 513 (1909).

† We will imagine that the observer measures the velocity of light by means of two clocks placed at the ends of a metre stick which is situated lengthwise in the path of the light.

other away from the source of light, constitutes the really remarkable feature of the principle of relativity, and forces us to the strange conclusions which we are about to deduce.

Let us consider two systems, moving past one another, with a constant relative velocity, provided with plane mirrors aa and bb parallel to one another and to the line of motion (fig. 1). An observer A on the first system sends a beam of light across to the opposite mirror, which is reflected back to the starting-point. He measures the time taken by the light in transit.

Fig. 1.



A, assuming that his system is at rest (and the other in motion), considers that the light passes over the path opo , but he believes that if a similar experiment is conducted by an observer B in the moving system, the light must pass over the longer path $mm'm'$, in order to return to the starting-point. For the point m moves to the position m' while the light is passing; he therefore predicts that the time required for the return of the reflected beam will be longer than in his own experiment. A, however, having established communication with B, learns that the time measured is the same as in his own experiment*.

The only explanation which A can offer for this surprising state of affairs is that the clock used by B for his measurement does not keep time with his own, but runs at a rate which is to the rate of his own clock, as the lengths of the paths, opo to $mm'm'$.

B, however, is equally justified in considering his system at rest, and A's in motion, and by identical reasoning has come to the conclusion that A's clock is not keeping time.

* This is evidently required by the principle of relativity, for contrary to A's supposition the two systems are in fact entirely symmetrical. Any difference in the observations of A and B would be due to a difference in the absolute velocity of the two systems, and would thus offer a means of determining absolute velocity.

Thus to each observer it seems that the other's clock is running too slowly.

This divergence of opinion evidently depends not so much on the fact that the two systems are in relative motion, but on the fact that each observer arbitrarily assumes that his own system is at rest. If, however, they both decide to call A's system at rest, then both will agree that in the two experiments the light passes over the paths *opo* and *mnm'* respectively, and that B's clock runs more slowly than A's. In general, whatever point may be arbitrarily chosen as a point of rest, it will be concluded that any clock in motion relative to this point runs too slowly.

Consider fig. 1 again, assuming system *a* at rest. We have shown that it is necessary to assume that B's clock runs more slowly than A's in the ratio of the lengths of the path *opo* to the path *mnm'*; in other words, the second of B's clock is longer than the second of A's, in the ratio *mnm'* to *opo*. This ratio between the two paths will evidently depend on the relative velocity of the two systems, *v*, and on the velocity of light, *c*.

Obviously from the figure,

$$(op)^2 = (ln)^2 = (mn)^2 - (ml)^2.$$

Dividing by $(mn)^2$,

$$\frac{(op)^2}{(mn)^2} = 1 - \frac{(ml)^2}{(mn)^2}.$$

But the distance *ml* is to the distance *mn* as *v* is to *c*.

Hence

$$\frac{mn}{op} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

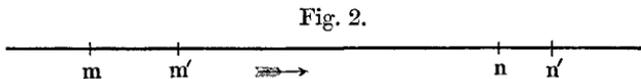
Denoting the important ratio $\frac{v}{c}$ by the letter β , we see that in general a second measured by a moving clock bears to a second measured by a stationary clock the ratio

$$\frac{1}{\sqrt{1 - \beta^2}}.$$

Whatever assumption the observers A and B may make as to their motion, it is obvious that their measurements of length, at least in a direction perpendicular to their line of relative motion, will lead to no disagreement. For evidently, if each observer with a measuring-rod determines the distance from his system to the other, the two determinations must

agree. Otherwise the condition of symmetry required by the principle of relativity would not be fulfilled.

But let us now consider distances *parallel* to the line of relative motion.



A system (fig. 2) has a source of light at m and a reflecting mirror at n . If we consider the whole system to be at absolute rest, it is evident that a light-signal sent from m to the mirror, and reflected back, passes over the path mmn . If, however, the entire system is considered to be in absolute motion with a velocity v , the light must pass over a different path $mn'm'$ where nn' is the distance through which the mirror moves before the light reaches it, and mm' is the distance traversed by the source before the light returns to it.

Obviously then,

$$\frac{nn'}{mn'} = \frac{v}{c}, \quad \text{and}$$

$$\frac{mm'}{mn'm'} = \frac{v}{c}.$$

Also from the figure,

$$mn' = mn + nn',$$

$$mn'm' = mnm + 2nn' - mm'.$$

Combining we have,

$$\frac{mn'm'}{mnm} = \frac{1}{1 - \frac{v^2}{c^2}} = \frac{1}{1 - \beta^2}.$$

Hence if we call the system in motion, instead of at rest, the calculated path of the light is greater in the ratio $\frac{1}{1 - \beta^2}$.

Now the velocity of light must seem the same to the observer, whether he is at rest or in motion. His measurements of velocity depend upon his units of length and time. We have already seen that a second on a moving clock is

lengthened in the ratio $\frac{1}{\sqrt{1 - \beta^2}}$, and therefore if the path

of the beam of light were also greater in this same ratio, we should expect that the moving observer would find no discrepancy in his determination of the velocity of light. From

the point of view of a person considered at rest, however, we have just seen that the path is increased by the larger ratio $\frac{1}{1-\beta^2}$. In order to account for this larger difference, we must assume that the unit of length in the moving system has been *shortened* in the ratio $\frac{\sqrt{1-\beta^2}}{1}$.

We thus see that a metre-stick, which, when held perpendicular to its line of motion, has the same length as a metre-stick at rest, will be shortened when turned parallel to the line of motion in the ratio $\frac{\sqrt{1-\beta^2}}{1}$, and indeed any moving body must be shortened in the direction of its motion in the same ratio*.

Let us emphasize once more, that these changes in the units of time and length, as well as the changes in the units of mass, force, and energy which we are about to discuss, possess in a certain sense a purely factitious significance;

* Certain of Einstein's other deductions from the principle of relativity will not be needed in the development of this paper, but may be directly obtained by the methods here employed. For example, the principle of relativity leads to certain curious conclusions as to the comparative readings of clocks in a system assumed to be in motion. Consider two systems in relative motion. An observer on system *a* places two carefully compared clocks, unit distance apart, in the line of motion, and has the time on each clock read when a given point on the other system passes it. An observer on system *b* performs a similar experiment. The difference between the readings of the two clocks in one system must be the same as the difference in the other system, for by the principle of relativity, the relative velocity *v* of the systems must appear the same to an observer in either. However, the observer A, considering himself at rest, and familiar with the change in the units of length and time in the moving system which we have already deduced, expects that the velocity determined by B will be greater than that which he himself observes in the ratio $\frac{1}{1-\beta^2}$, since he has concluded that B's unit of time is longer, and his unit of length in this direction is shorter, each by a factor involving $\sqrt{1-\beta^2}$. The only possible way in which A can explain this discrepancy is to assume that the clocks which B claims to have set together are not so in reality. In other words he has to conclude that clocks which in a moving system appear to be set together really read differently at any instant (in stationary time), and that a given clock is "slower" than one immediately to the rear of it by an amount proportional to the distance. From what has preceded it can be readily shown that if in a moving system two clocks are situated, one in front of the other by a distance *l*, in units of this system, the difference in setting will be $\frac{lv}{c^2}$. From this point Einstein's equations concerning the addition of velocities also follow directly.

although, as we shall show, this is equally true of other universally accepted physical conceptions. We are only justified in speaking of a body in motion when we have in mind some definite, though arbitrarily chosen, point as a point of rest. The distortion of a moving body is not a physical change in the body itself, but is a scientific fiction.

When Lorentz first advanced the idea that an electron or in fact any moving body is shortened in the line of its motion, he pictured a real distortion of the body in consequence of a real motion through a stationary æther, and his theory has aroused considerable discussion as to the nature of the forces which would be necessary to produce such a deformation. The point of view first advanced by Einstein, which we have here adopted, is radically different. Absolute motion has no significance. Imagine an electron and a number of observers moving in different directions with respect to it. To each observer, naïvely considering himself to be at rest, the electron will appear shortened in a different direction and by a different amount; but the physical condition of the electron obviously does not depend upon the state of mind of the observers.

Although these changes in the units of space and time appear in a certain sense psychological, we adopt them rather than abandon completely the fundamental conceptions of space, time, and velocity, upon which the science of physics now rests. At present there appears no other alternative.

Non-Newtonian Mechanics.

Having obtained these relations for the units of space and time, we may turn to some of the other important quantities used in mechanics.

Let us again consider two systems, a and b , in relative motion with the velocity v . An experimenter A on the first system constructs a ball of some rigid elastic material, with a volume of one cubic centimetre, and sets it in motion, with a velocity of one centimetre per second, towards the system b (in a direction perpendicular to the line of relative motion of the two systems). On the other system, an experimenter b constructs of the same material a similar ball with a volume of one cubic centimetre in his units, and imparts to it, also in his units, a velocity of one centimetre per second towards a . The experiment is so planned that the balls will collide and rebound over their original paths. Since the two systems are entirely symmetrical, it is evident by the principle of relativity, that the (algebraic) change in velocity of the first

ball, as measured by A, is the same as the change in velocity of the other ball, as measured by B. This being the case, the observer A, considering himself at rest, concludes that the real change in velocity of the ball *b* is different from that of his own, for he remembers that while the unit of length is the same in this transverse direction in both systems, the unit of time is longer in the moving system.

Velocity is measured in centimetres per second, and since the second is longer in the moving system, while the centimetre in the direction which we are considering is the same in both systems, the observer A, always using the units of his own system, concludes that the change in velocity of the ball *b* is smaller in the ratio $\frac{\sqrt{1-\beta^2}}{1}$ than the change in velocity of the ball *a*. The change in velocity of each ball multiplied by its mass gives its change in momentum. Now, from the law of conservation of momentum, A assumes that each ball experiences the same change in momentum, and therefore since he has already decided that the ball *b* has experienced a smaller change of velocity in the ratio $\frac{\sqrt{1-\beta^2}}{1}$, he must conclude that the mass of the ball in system *b* is greater than that of his own in the ratio $\frac{1}{\sqrt{1-\beta^2}}$. In general, therefore, we must assume that the mass of a body increases with its velocity. We must bear in mind, however, as in all other cases, that the motion is determined with respect to some point *arbitrarily* chosen as a point of rest.

If *m* is the mass of a body in motion and *m*₀ its mass at rest, we have *

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}} \dots \dots \dots (1)$$

The only opportunity of testing experimentally the change of a body's mass with its velocity has been afforded by the experiments on the mass of a moving electron or β particle. The actual measurements were indeed not of the mass of the electron, but of the ratio of charge to mass $\left(\frac{e}{m}\right)$.

* This equation and others developed in this section are identical with those obtained through an entirely different course of reasoning by Lewis (Phil. Mag. xvi. p. 705, 1908). The equations were there obtained for systems in motion with respect to a point at absolute rest. We shall show here, however, that they are true, whatever arbitrary point is selected as a point of rest.

It has, however, been universally considered that the charge e is constant. In other words, that the force acting upon the electron in a uniform electrostatic field is independent of its velocity relative to the field. Hence the observed change in $\frac{e}{m}$ is attributed solely to the change in mass. It might be well to subject this view to a more careful analysis than has hitherto been done. At present, however, we will adopt it without further scrutiny.

The original experiments of Kaufmann* showed only a qualitative agreement with equation (1). Recently, however, Bucherer † by a method of exceptional ingenuity, has made further determinations of the mass of electrons moving with varying velocities, and his results are in remarkable accord with this equation obtained from the principle of relativity.

This very satisfactory corroboration of the fundamental equation of non-Newtonian mechanics, must in future be regarded as a very important part of the experimental material which justifies the principle of relativity. By a slight extrapolation we may find with accuracy from the results of Bucherer that limiting velocity at which the mass becomes infinite, in other words, a numerical value of c which in no way depends upon the properties of light. Indeed merely from the first postulate of relativity and these experiments of Bucherer we may deduce the second postulate and all the further conclusions obtained in this paper. This fact can hardly be emphasized too strongly.

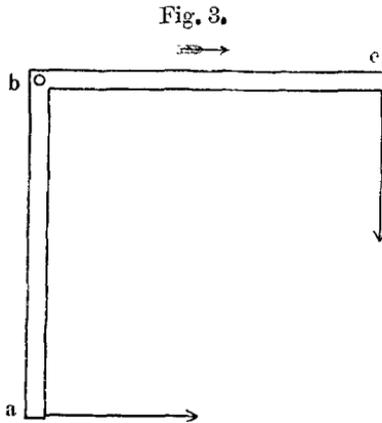
Leaving now the subject of mass, let us consider whether the unit of force depends upon our choice of a point of rest. An observer in a given system allows such a force to act upon unit mass as to give it an acceleration of one $\frac{\text{cm.}}{\text{sec.}^2}$, and calls this force the dyne. If now we assume that the system is in motion, with a velocity v , in a direction perpendicular to the line of application of the force, we conclude that the acceleration is really less than unity, since in a moving system the second is longer in the ratio $\frac{1}{\sqrt{1-\beta^2}}$ and the centimetre in this transverse direction is the same as at rest. On the other hand, the mass is increased owing to the motion of the system by the factor $\frac{1}{\sqrt{1-\beta^2}}$. Since the time enters to the second power, the product of mass and acceleration

* See Lewis, *loc. cit.*

† Bucherer, *loc. cit.*

is smaller by the ratio $\frac{\sqrt{1-\beta^2}}{1}$ than it would be if the system were at rest. And we conclude, therefore, that the unit of force or the dyne in a direction transverse to the line of motion is smaller in a moving system than in one at rest by this same ratio.

In order now to obtain a value for the force in a longitudinal direction in the moving system, let us consider (fig. 3) a rigid lever abc , whose arms are equal and perpendicular, and equal forces applied at a and c in directions parallel to bc and ba . The system is thus in equilibrium.



Now let us assume that the whole system is in motion with velocity v in the direction bc . Obviously, merely by making such an assumption we cannot cause the lever to turn, nevertheless we must now regard the length bc as shortened in

the ratio $\frac{\sqrt{1-\beta^2}}{1}$, while ab has the same length as at rest.

We must therefore conclude that to maintain equilibrium the force at a must be less than the force at c in the same ratio.

We thus see that in a moving system unit force in the longitudinal direction is smaller than unit transverse force in the

ratio $\frac{\sqrt{1-\beta^2}}{1}$, and therefore, by the preceding paragraph, smaller than unit force at rest in the ratio $\frac{1-\beta^2}{1}$. It is

interesting to point out, as Bumstead* has already done, that the repulsion between two like electrons, as calculated from

* Bumstead, *loc. cit.*

the electromagnetic theory, is diminished in the ratio $\frac{\sqrt{1-\beta^2}}{1}$ if they are moving perpendicular to the line joining them; and in the ratio $\frac{1-\beta^2}{1}$ if moving parallel to the line joining them.

From the standpoint of the principle of relativity, one of the most interesting quantities in mechanics is the so-called kinetic energy, which is the increase in energy attributed to a body when it is set in motion with respect to an arbitrarily chosen point of rest. Knowing the change of the mass with velocity as given by equation (1), the general equation for kinetic energy*, E' , may readily be shown to be,

$$E' = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right). \quad \dots \quad (2)$$

From equations 1 and 2 we may derive one of the most interesting consequences of the principle of relativity. If E is the total energy (including internal energy) of a body in motion, and E_0 is its energy at rest, the kinetic energy E' is equal to $E - E_0$ and equation (2) may be written,

$$E - E_0 = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right). \quad \dots \quad (3)$$

Moreover, we may write equation (1) in the form

$$m - m_0 = m_0 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right), \quad \dots \quad (4)$$

and dividing (3) by (4),

$$\frac{E - E_0}{m - m_0} = c^2. \quad \dots \quad (5)$$

In other words when a body is in motion its energy and mass are both increased, and the increase in energy is equal to the increase in mass multiplied by the square of the velocity of light. From the conservation laws we know that when a body is set in motion and thus acquires mass and energy, these must come from the environment. So also when a moving body is brought to rest it must give up mass as well as energy to the environment. The mass thus acquired by the environment is independent of the particular form

* Consider a body moving with the velocity v subjected to a force f in the line of its motion. Its momentum M and its kinetic energy E' will be changed by the amounts $dM = f dt$, $dE' = f v dt = f v dt$. Hence $dE' = v dM$ or substituting mv for M , $dE' = m v dv + v^2 dm$. Eliminating m between this equation and equation (1), and integrating, gives at once the above equation (2).

which the energy may assume and we are thus forced to the important conclusion that *when a system acquires energy in any form it acquires mass in proportion*, the ratio of the energy to the mass being equal to the square of the velocity of light. We might go further and assume that if a system should lose all its energy it would lose all its mass. If we admit this plausible although unproved assumption, then we may regard the mass of every body as a measure of its total energy according to the equation,

$$m = \frac{E}{c^2} \dots \dots \dots (6)$$

For a body at rest

$$m_0 = \frac{E_0}{c^2}.$$

Combining this equation with (3) gives

$$\frac{E}{E_0} = \frac{1}{\sqrt{1-\beta^2}}.$$

We thus see that energy changes with the velocity in the same way that mass does, and that the so-called kinetic energy is a "second-order effect" of the same character as the change of length and the change of mass. The only reason that this effect is easily measured, and has become a familiar conception in mechanics, while the others are obtainable only by the most precise measurements, is that we are in the habit of measuring quantities of energy which are extremely minute in comparison with the total energy of the systems investigated.

Conclusion.

We have shown how observers stationed on systems in motion relative to one another have been able to preserve their fundamental principles of mechanics only by adopting certain novel conclusions. These conclusions are self-consistent ; in the one case where they have been tested they are in accord with experiment ; and they enable us to save all the fundamental physical concepts which have been found useful in the past. We have, however, considered primarily only systems which are initially in uniform relative motion. Whether our conclusions can be retained when we consider processes in which the relative motion is being established, in other words, processes in which acceleration takes place, it is not our present purpose to determine.

The ideas here presented appear somewhat artificial in character, and we cannot but suspect that this is due to the

arbitrary way in which we have assumed this point or that point to be at rest, while at the same time we have asserted that a condition of rest in the absolute sense possesses no significance.

If our ideas possess a certain degree of artificiality, this is also true of others which have long since been adopted into mechanics. The apparent change in rate of a moving clock, and the apparent change in length and mass of a moving body, are completely analogous to that apparent change in energy of a body in motion which we have long been accustomed to call its kinetic energy. We may with equal reason speak of the kinetic mass found by Kaufmann and Bucherer, or the kinetic length assumed by Lorentz. We say that the heat evolved when a moving body is brought to rest comes from the kinetic energy which it possessed. We thus preserve the law of conservation of energy. It is in order to maintain such fundamental conservation laws, and to reconcile them with the Principle of Relativity, which rests on the experiments of Michelson and Morley and of Bucherer, that we have adopted the principles of non-Newtonian Mechanics.

These principles, bizarre as they may appear, offer the only method of preserving the science of mechanics substantially in its present form. If later, when more complex systems are considered, and especially when we deal with acceleration, these views prove untenable, it will then be necessary to revolutionize the whole of mechanics.

Research Laboratory of Physical Chemistry,
Mass. Inst. of Technology, Boston,
May 11th, 1909.

LVIII. *On the Moving Force of Terrestrial and Celestial Bodies in Relation to the Attraction of Gravitation.* By HENRY WILDE, D.Sc., D.C.L., F.R.S.*

1. **I**N the course of a lecture which I delivered before the Society in 1902, "On the Evolution of the Mental Faculties in relation to some Fundamental Principles of Motion," prominence was given to the historic controversy respecting the measure of moving force of terrestrial bodies which has exercised the minds of distinguished men of science and learning for more than two centuries.

* Communicated by the Author. Reprinted from the Memoirs and Proceedings of the Manchester Literary and Philosophical Society, vol. liii. pt. ii. (1909).