XX. *On Induced Stability.* By ANDREW STEPHENSON.*

THE conditions under which an imposed periodic change in the spring of an oscillation exerts a cumulative effect in magnifying the motion, have already been investigated †. We shall now examine the influence of such a variation on instability of equilibrium and, in certain cases, of steady motion.

The equation of motion about statically unstable equilibrium under variable spring is

$$\ddot{x}-(\mu^2-2\alpha n^2\cos nt)x=0,$$

and the complete solution is given by

$$x=\sum_{-\infty}^{\infty}\mathrm{A}_r\sin\{(c-rn)t+\epsilon\},$$

where

$$-\mathrm{A}_r\{\mu^2+(c-rn)^2\}+\alpha n^2(\mathrm{A}_{r-1}+\mathrm{A}_{r+1})=0. \quad . \quad . \quad (r)$$

In the limit when k, taken positive, is infinite

$$\mathrm{A}_{\pm k}=\frac{\alpha}{k^2}\mathrm{A}_{\pm(k-1)},$$

so that the series is convergent in both directions. If the eliminant of the equations (r) gives a real value for c, the elementary oscillations are of constant amplitude and the equilibrium is stable.

An approximation ‡ for c is readily obtained when α is small, for in that case, r being positive,

$$\mathrm{A}_{\pm r}=\frac{\alpha n^2}{\mu^2+(c\mp rn)^2}\mathrm{A}_{\pm(r-1)}$$

approximately, and on substituting for A_1 and A_{-1} in (0) we find that for a real c when α is small n must be large compared with μ: then

$$c^2=2\alpha^2n^2-\mu^2,$$

and for stability n must be greater than $\mu/\alpha\sqrt{2}$.

Thus the system can always be maintained about a position of otherwise unstable equilibrium by a periodic variation in spring of sufficiently large frequency. The inverted pendulum, for example, is rendered stable by rapid vertical

* Communicated by the Author.

† "On a class of forced oscillations," Quarterly Journal of Mathematics, no. 168, 1906. "On the forcing of oscillations by disturbances of different frequencies," Phil. Mag. July 1907.

‡ Similar to that employed in the former of the two papers referred to above.

vibration of the point of support. The above analysis shows that quite apart from gravity, an imposed motion of small amplitude and high frequency produces a comparatively slow simple oscillation about its own direction ; the impressed action must therefore exert a restoring moment on the body proportional to the displacement of the mass centre from the line of the applied motion, and if this moment numerically exceeds the outward moment due to gravity stability is ensured *.

If the pivot is given a simple vibration of amplitude a with frequency n per 2π seconds, the equation of motion about the unstable position is

$$\ddot{x} - \frac{h}{k^2}(g - an^2 \cos nt)\, x = 0.$$

For a simple pendulum 1 metre long $\mu^2 = 10$ approximately, and if $a = 10$ cm. for stability $n > 44{\cdot}7$; thus a frequency of 7·2 per second is sufficient to maintain relative equilibrium. If the pendulum is 20 cm. in length $\mu^2 = 50$, and if $a = 10$ the condition would give $n > 20$, a frequency of over 3·2 per second. In this case, however, μ/n is not sufficiently small for the approximate formula to be applicable.

There is no difficulty in verifying the stability under the imposed motion by experiment. It is found furthermore that the pendulum may be rendered approximately steady in a position sensibly oblique by a comparatively small inclination of the direction of vibration, and it is of interest to enquire under what circumstances this occurs.

If the path of the pivot makes a small angle β with the vertical the equation of motion is

$$\ddot{x} - \frac{h}{k^2}(g - an^2 \cos nt)\, x = \beta \frac{h}{k^2} an^2 \cos nt.$$

For the particular solution giving the forced motion due to the disturbance represented by the term on the right, we have

$$x = \sum_{0}^{\infty} \mathrm{B}_r \cos rnt,$$

where

$$-\mathrm{B}_r\left(\frac{gh}{k^2} + r^2 n^2\right) + \frac{1}{2}\frac{ha}{k^2} n^2 (\mathrm{B}_{r-1} + \mathrm{B}_{r+1}) = 0, \quad . \quad . \quad (r)'$$

* The stability of this particular system is worked out from first principles for a special case in a paper "On a new type of dynamical stability," read before the Manchester Literary and Philosophical Society in January 1908.

except for $r=0$ and 1, for which

$$-B_0\frac{gh}{k^2}+\frac{ha}{k^2}n^2 B_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad (0)'$$

and

$$-B_1\left(\frac{gh}{k^2}+n^2\right)+\frac{1}{2}\frac{ha}{k^2}n^2(2B_0+B_2) = \beta\frac{ha}{k^2}n^2. \quad . \quad . \quad (1)'$$

The series is evidently convergent. Since $\frac{gh}{k^2}=\mu^2$ is small compared with n^2,

$$B_r = \frac{1}{r^2}\cdot\frac{ah}{k^2}B_{r-1}$$

approximately if $r>1$.

Also $B_1 = \frac{2g}{an^2}B_0$, and therefore $B_2 = \frac{1}{2}\frac{gh}{k^2n^2}B_0$.

B_2 is therefore negligible in (1)′, and finally

$$B_0 = \frac{\beta}{1-\frac{2gk^2}{n^2a^2h}}.$$

As before for the stability of the "free" motion n^2 must exceed $\mu^2/2\alpha$, which is equal to $2gk^2/a^2h$: if, therefore, n is in the vicinity of this limit B_0, the mean inclination of the pendulum, is large compared with β, the inclination of the path of the pivot. Since $2g/an^2$ is small, being less than ah/k^2, it is clear that B_0 is large compared with $\sum_1^\infty |B_r|$; *i. e.*, the forced oscillation about the inclined position is comparatively small. This fact is very evident experimentally.

As n increases B_0 decreases to the limit β, and the rod approaches the line of vibration.

2. A case of steady motion for which the stability equation is of the type

$$\ddot{x}+\lambda x = 0$$

is furnished by a solid of revolution rolling on a horizontal plane. Consider the rectilinear motion of such a body symmetrical about the middle plane normal to its axis. The small oscillations are determined by

$$(A+Mb^2)\ddot{x}+\{Cp^2(C+Mb^2)/A-(b-\rho)Mg\}x=0,$$

where p is the angular velocity, b the radius of the mid-section, ρ the other principal radius at any point on it, and the other constants are in the usual notation.

When, therefore, $p^2<g\frac{(b-\rho)MA}{C(C+Mb^2)}$ the motion is unstable.

If now the plane is given a simple oscillation vertically, g must be replaced in the above equation by $g+an^2\cos nt$, and the motion is rendered stable if

$$2\left\{\frac{1}{2}\frac{(b-\rho)\mathrm{M}}{\mathrm{A}+\mathrm{M}b^2}a\right\}^2 n^2 > \frac{\mathrm{C}(\mathrm{C}+\mathrm{M}b^2)}{\mathrm{A}(\mathrm{A}+\mathrm{M}b^2)}\left\{\frac{g(b-\rho)\mathrm{MA}}{\mathrm{C}(\mathrm{C}+\mathrm{M}b^2)}-p^2\right\},$$

where, as before, in making the approximation it is assumed that $\frac{(b-\rho\mathrm{M})}{\mathrm{A}+\mathrm{M}b^2}a$ is small. The range of action is limited by the conditions of the problem which require that an^2 must not exceed g.

3. In general the single equation determining the oscillations of a system about steady motion is of order higher than the second, as for example in the case of the spinning top, which we now examine in the present connexion. The unstable position of equilibrium of a symmetrical top may be rendered stable either by an axial spin or by an imposed vertical vibration of the point of support. The two actions together might therefore be expected to reinforce one another, if either singly is not of sufficient intensity.

For the small oscillations about the position of equilibrium

$$\ddot{x}-p\dot{y}-(\mu^2-2\alpha n^2\cos nt)x=0$$
$$\ddot{y}+p\dot{x}-(\mu^2-2\alpha n^2\cos nt)y=0,$$

and a solution is

$$x=\sum_{-\infty}^{\infty}\mathrm{A}_r\sin\{(c-rn)t+\epsilon\},$$

$$y=\sum_{-\infty}^{\infty}\mathrm{A}_r\cos\{(c-rn)t+\epsilon\},$$

where

$$-\{\mu^2+(c-rn)^2-p(c-rn)\}\mathrm{A}_r+\alpha n^2(\mathrm{A}_{r-1}+\mathrm{A}_{r+1})=0,$$

This set of conditional equations is similar to the system (r) in § 1, and admits of corresponding approximate treatment. If α is small and n is large compared with μ and p, we find

$$\mu^2+c^2-pc-2\alpha^2n^2=0,$$

and therefore for stability in this case

$$2\alpha^2n^2>\mu^2-\tfrac{1}{4}p^2.$$

Thus when the imposed motion is comparatively rapid the two actions are simply cumulative.

Manchester, October 1907.