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LXXII. On ripples

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mineral (206·06 found ; 206·2 theoretical value). As more evidence accumulates, the ratios for other geological periods will be similarly tested. At present it would seem that the age of the older intrusive rocks of Ceylon, as deduced from Pb/U ratios in thorianite and thorite, is probably too high. These ratios are nearly always greater than 0·2, giving an age exceeding 1600 million years. Zircon from the same pegmatites, however, gives a ratio of 0·164 (1370 million years), which seems to be more probable, in the light of atomic weight estimations. Zircon is much less likely to contain original lead than thorite or thorianite, and apart from the difficulty of estimating the very small quantities of lead which have accumulated, zircon represents one of the most valuable minerals for age determinations.

Finally, in addition to our previous acknowledgments to Professors Strutt and Mache, we wish to express our thanks to Professor Stefan Meyer, for his kindly interest and encouragement during the progress of this work.

LXXII. *On Ripples.* *By* J. R. WILTON, *M.A.*, *D.Sc.*, *Assistant Lecturer in Mathematics at the University of Sheffield* *.

IN carrying the approximation to the form of a wave to such an extent as is done in my paper on "On Deep Water Waves" (Phil. Mag. Feb. 1914, pp. 385–394), which will here be referred to as "Waves," it is important to make certain that the sense of accuracy thus obtained is not illusory. The present paper therefore takes up the consideration of the corrections which have to be applied. We shall, however, still suppose that the wave (or ripple) is formed under ideal conditions,—that there is no wind, no secondary disturbance of any kind,—that the "ocean" is "deep" (a depth of ten centimetres will be ample for the ripples we shall actually consider) and of unlimited extent. With this understanding there are three things for which we have to make allowance, namely :—

(1) Surface Tension,
(2) The formation of waves in the air,
(3) Viscosity.

Now the first order approximation to the velocity when (1) and (2) are taken into account is known to be given by

$$c^2 = \frac{g\lambda}{2\pi}\frac{\rho-\rho'}{\rho+\rho'} + \frac{2\pi}{\lambda}\frac{T}{\rho+\rho'},$$

* Communicated by the Author.

in which ρ is the density of water, ρ' of the air, T is the surface tension, and the rest of the notation is that of "Waves."

It is assumed in obtaining the above value of c^2 that the air is incompressible, but the removal of this restriction will rather lessen than increase the effect of air waves.

Since ρ'/ρ is small the effect of ρ' is to multiply one term in c^2 by $12-\rho'/\rho$, the other by $1-\rho'/\rho$. Now

$$\rho'/\rho = \cdot 0013$$

nearly. Hence the correction is of the order 1/400 of the uncorrected value. We shall consider this as negligible. In fact, in such a (relatively) high wave as that of "Waves," fig. 1, the ratio of the last term retained to the first is of the order

$$A_{12}/A_1 = 1/150.$$

If, then, we omit ρ', we have as a first approximation

$$\frac{2\pi c^2}{g\lambda} = 1 + \left(\frac{2\pi}{\lambda}\right)^2 \frac{T}{g\rho},$$

which we shall, for brevity, write in the form

$$\mu = 1 + \kappa.$$

But

$$T/g\rho = 74/981 = \cdot 075,$$

so that κ, the correction due to surface tension, is appreciable for waves of length less than about 25 cm., and for very short ripples it may become very large.

Finally, we have to take account of viscosity. It is shown in Lamb's 'Hydrodynamics,' § 332, p. 566 (Third Edition), that, if $\lambda/\cdot 0048$ cm. may be considered large (say 10 or more), the effect of viscosity is to introduce a time factor $e^{-2\nu(2\pi/\lambda)^2 t}$ which does not affect the *form* of the wave, and to introduce a correcting factor to the form of the wave of the order of magnitude

$$\tau = \frac{2\nu(2\pi/\lambda)^2}{\sqrt{\dfrac{g\lambda}{2\pi} + \dfrac{2\pi}{\lambda}\dfrac{T}{\rho}}} = \frac{2\nu g^{-\frac{1}{2}}(2\pi/\lambda)^{\frac{5}{2}}}{\sqrt{1+\kappa}},$$

ν being the coefficient of dynamical viscosity.

The following table will show that surface tension is always of considerably greater effect than viscosity *.

Wave length, λ.	Suface Tension Correction, κ.	Viscosity Correction, τ.	Ratio, κ/τ.
9·4 cm.	·033	·0004	80
6·3 „	·075	·0011	68
3·1 „	·30	·006	50
1·6 „	1·2	·03	40
·8 „	4·8	·11	44
·4 „	19	·36	53

Thus κ/τ is never less than 40, so that we may neglect viscosity even for small ripples without risk of serious error, provided always that the condition that $\lambda/{\cdot}0048$ is to be "large" is not forgotten†. But for longer waves the correction due to the formation of air waves is of the same order of magnitude as κ: thus, when $\lambda=35$ cm.,

$$\kappa=1/400,$$

nearly. Hence, if we include T but omit the other two corrections, we must apply our results only to ripples and waves of from, say, 1 mm. to 20 cm. in length, so that the form of waves which ordinarily occur in the open sea will not be affected by any of these considerations. We shall, actually, apply our formulæ only to ripples of from 5 mm. to 25 mm. in length. With this understanding we proceed to determine the form of a wave when surface tension is taken into account.

Let R be the radius of curvature of the wave: R will be reckoned positive when the concavity is upwards. Then, if Π is the atmospheric pressure, the pressure along the free surface, within the water, is

$$p=\Pi-\mathrm{T}/\mathrm{R},$$

* It will be observed that for ripples, down to half a centimetre in length, we may still speak of the correction due to viscosity, but it is absurd to speak of the "correction" due to surface tension, for the latter is the predominating influence in determining the form of ripples of 2 cm. length, or less.

† The viscous *time* factor is, however, far from being negligible for short ripples.

and the surface condition becomes *

$$\dot{\phi}+\tfrac{1}{2}q^2=gy+\mathrm{T}/\rho\mathrm{R}-\tfrac{1}{2}\mathrm{C}'. \quad . \quad . \quad . \quad . \quad (1)$$

To determine the form of the term $\mathrm{T}/\rho\mathrm{R}$ we expand $1/\mathrm{R}$ in a series of cosines of multiples of ξ by means of the equations for x and y on p. 392 of "Waves." Since on the free surface $\xi'=\xi$, we have

$$\frac{1}{\mathrm{R}}=\frac{2\pi\iota}{\lambda}\left(\frac{d\eta}{d\xi}\frac{d^2\eta'}{d\xi^2}-\frac{d\eta'}{d\xi}\frac{d^2\eta}{d\xi^2}\right)\Big/\left(\frac{d\eta}{d\xi}\frac{d\eta'}{d\xi}\right)^{3/2}.$$

The sign is determined by the fact that R is positive at the trough of the wave where $\xi=0$. A_1, $=-a$, is negative, and therefore the predominant term in $1/\mathrm{R}$ is $2\pi a/\lambda$, which is positive.

On substituting in (1) we find, as in "Waves," p. 387,

$$0=\tfrac{1}{2}\mu+\left\{\mathrm{C}+\sum_1^\infty \mathrm{A}_n\cos n\xi\right\}\left\{(1+\sum_1^\infty n\mathrm{A}_n\cos n\xi)^2+(\sum_1^\infty n\mathrm{A}_n\sin n\xi)^2\right\}$$
$$+\kappa\left\{\sum_1^\infty n^2\mathrm{A}_n\cos n\xi+\sum_1^\infty n^3\mathrm{A}_n^2\right.$$
$$\left.+\sum_{m=1}^\infty\sum_{n=m+1}^\infty mn(m+n)\mathrm{A}_m\mathrm{A}_n\cos(m-n)\xi\right\}$$
$$\left\{(1+\sum_1^\infty n\mathrm{A}_n\cos n\xi)^2+(\sum_1^\infty n\mathrm{A}_n\sin n\xi)^2\right\}^{-\frac{1}{2}}, \quad . \quad (2)$$

where, as before,

$$\mu=\frac{2\pi c^2}{g\lambda}, \quad \kappa=\left(\frac{2\pi}{\lambda}\right)^2\frac{\mathrm{T}}{g\rho}.$$

As a first approximation we see that

$$\mathrm{C}+\tfrac{1}{2}\mu=0,$$
$$\mathrm{A}_1(1+2\mathrm{C}+\kappa)=0,$$

so that A_1 is arbitrary,

$$\mathrm{A}_1=-a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$c^2=\frac{g\lambda}{2\pi}(1+\kappa),$$

as it should.

* The explanation of any notation not given in this paper will be found in "Waves," pp. 385–6.

To obtain closer approximations we must expand the right hand side of (2) and re-arrange in cosines of multiples of ξ, The coefficient of $\cos n\xi$ is then to be equated to zero. It would be possible to obtain the general form of the equation thus derived, but it would be extremely complicated and there would be no advantage in doing this. We shall therefore write down the resulting equations only in so far as they are necessary to obtain the approximation we desire. We shall determine each approximation accurately so far as T is concerned.

The equations derived from (2) are, if we retain terms of the fifth order and reject those of higher orders,

$$\mu+2C(1+A_1^2+4A_2^2)+2A_1^2+4A_2^2+4A_1^2A_2$$
$$+\kappa(A_1^2+\frac{1}{8}A_1^4-2A_1^2A_2+8A_2^2)=0, \quad . \quad . \quad . \quad (4)$$

$$2C(A_1+2A_1A_2+6A_2A_3)+A_1+A_1^3+3A_1A_2+5A_2A_3+6A_1A_2^2+3A_1^2A_3$$
$$+\kappa(A_1-\frac{3}{8}A_1^3+3A_1A_2-\frac{5}{64}A_1^5+\frac{5}{4}A_1^3A_2-\frac{15}{8}A_1^2A_3-5A_1A_2^2+15A_2A_3)$$
$$=0, \quad . \quad (5)$$

$$2C(2A_2+3A_1A_3)+A_2+A_1^2+3A_1^2A_2+4A_1A_3$$
$$+\kappa(4A_2-\frac{1}{2}A_1^2+\frac{1}{4}A_1^4-2A_1^2A_2+6A_1A_3)=0, \quad . \quad . \quad (6)$$

$$2C(3A_3+4A_1A_4)+A_3+3A_1A_2+5A_1A_4+4A_1^2A_3+2A_1A_2^2$$
$$+\kappa(9A_3+\frac{3}{8}A_1^3-3A_1A_2-\frac{25}{128}A_1^5+\frac{15}{8}A_1^3A_2-\frac{15}{4}A_1^2A_3$$
$$-\frac{5}{2}A_1A_2^2+10A_1A_4)=0, \quad . \quad (7)$$

$$8CA_4+A_4+4A_1A_3+2A_2^2$$
$$+\kappa(16A_4-\frac{5}{16}A_1^4-4A_2^2+3A_1^2A_2-6A_1A_3)=0, \quad . \quad (8)$$

$$10CA_5+A_5+5A_1A_4+5A_2A_3$$
$$+\kappa(25A_5+\frac{35}{128}A_1^5-\frac{25}{8}A_1^3A_2+\frac{45}{8}A_1^2A_3+\frac{15}{2}A_1A_2^2-10A_1A_4-15A_2A_3)$$
$$=0. \quad . \quad (9)$$

From these equations, remembering (3), I find by successive approximation :—

$$A_1=-a,$$

$$A_2=\frac{1}{2}\frac{\kappa-2}{2\kappa-1}a^2-\frac{1}{16}\frac{30\kappa^3-71\kappa^2+17\kappa-8}{(2\kappa-1)^3(3\kappa-1)}a^4, \quad . \quad . \quad . \quad (10)$$

$$A_3=-\frac{3}{16}\frac{2\kappa^2-11\kappa+8}{(2\kappa-1)(3\kappa-1)}a^3$$
$$+\frac{13248\kappa^5-53640\kappa^4+63260\kappa^3-29010\kappa^2+7971\kappa-1216}{768(2\kappa-1)^3(3\kappa-1)^2(4\kappa-1)}a^5, \quad . \quad (11)$$

$$A_4=\frac{18\kappa^3-183\kappa^2+361\kappa-128}{48(2\kappa-1)(3\kappa-1)(4\kappa-1)}a^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$A_5=-\frac{5}{1536}\frac{288\kappa^5-4680\kappa^4+18980\kappa^3-24786\kappa^2+11091\kappa-1600}{(2\kappa-1)^2(3\kappa-1)(4\kappa-1)(5\kappa-1)}a^5, \quad . \quad (13)$$

$$C=-\frac{1}{2}-\frac{1}{2}\kappa+\frac{1}{16}\frac{2\kappa^2-15\kappa+16}{2\kappa-1}a^2$$
$$-\frac{1}{256}\frac{24\kappa^5+220\kappa^4-2422\kappa^3+4701\kappa^2-2858\kappa+704}{(2\kappa-1)^3(3\kappa-1)}a^4, \quad . \quad . \quad (14)$$

$$\mu=1+\kappa-\frac{1}{8}\frac{2\kappa^2+\kappa+8}{2\kappa-1}a^2$$
$$+\frac{1}{128}\frac{24\kappa^5-164\kappa^4-566\kappa^3+1821\kappa^2-1322\kappa+448}{(2\kappa-1)^3(3\kappa-1)}a^4. \quad . \quad . \quad . \quad (15)$$

I have also calculated, independently, the values of these constants in the particular cases $\kappa=1$ and $\kappa=2$.

When $\kappa=1$, $A_1=-a$,

$$\left.\begin{array}{l} A_2=-\frac{1}{2}a^2+a^4, \quad A_3=\frac{3}{32}a^3+\frac{613}{9216}a^5, \\ A_4=\frac{17}{72}a^4, \quad A_5=\frac{3535}{36864}a^5, \\ C=-1+\frac{3}{16}a^2-\frac{369}{512}a^4, \quad \mu=2-\frac{11}{8}a^2+\frac{241}{256}a^4. \end{array}\right\} \quad . \quad (16)$$

When $\kappa=2$, $A_1=-a$,

$$\left.\begin{array}{l} A_2=\frac{1}{120}a^4, \quad A_3=\frac{3}{40}a^3-\frac{547}{67200}a^5, \\ A_4=\frac{1}{840}a^4, \quad A_5=-\frac{47}{5376}a^5, \\ C=-\frac{3}{2}-\frac{1}{8}a^2+\frac{3}{80}a^4, \quad \mu=3-\frac{3}{4}a^2-\frac{3}{40}a^4. \end{array}\right\} \quad . \quad (17)$$

It will be found that equations (10), ... (15) agree with (16) and (17), and that they also agree with the known result when $\kappa=0$. Hence it is improbable that there is any undetected error in calculation.

The most interesting thing about the constants whose values are given in equations (10), ... (15) is the unexpected form of the denominators. It is easy to see that this form is general; for the coefficient of A_n in the equation which determines it is

$$2nC+1+n^2\kappa = n^2\kappa+1-n(1+\kappa)$$
$$= (n-1)(n\kappa-1),$$

to a first approximation. Hence the denominator of A_n contains as a factor $(n-1)!\prod_{r=1}^{n}(r\kappa-1)$. Now we have seen that we are justified in neglecting the effect of air waves if κ is greater than about ·01; so that for a considerable range of values of n we have to consider the possibility of values of κ of the form $\kappa=1/n$, where n is a positive integer. To the consideration of these values of κ we shall return later. On the other hand, since we are not justified in neglecting the effect of air-waves if κ is less than about ·01, we cannot, as might appear at first sight from equations (10), ... (15), conclude that however small κ is, so long as it is finite, the value of one of the coefficients A_n becomes illusory, and therefore that the ordinary theory, in which $\kappa=0$, is incorrect.

Let us take first the case of a ripple whose form is largely determined by surface tension—say that for which $\kappa=10$, and therefore $\lambda=\cdot 54$ cm. From equations (10), ... (15) I find, for this ripple,

$$A_1=-a, \quad A_2=\cdot 21a^2-\cdot 0072a^4,$$
$$A_3=-\cdot 033a^3+\cdot 0049a^5, \quad A_4=\cdot 0031a^4, \quad A_5=\cdot 00023a^5,$$
$$\mu=11-1\cdot 43a^2+\cdot 014a^4.$$

The largest value of a which we may safely insert in these equations is * $a=1\cdot 5$. We then find

$$A_1=-1\cdot 5, \quad A_2=\cdot 44, \quad A_3=-\cdot 075,$$
$$A_4=\cdot 016, \quad A_5=\cdot 0017,$$
$$\mu=7\cdot 85, \text{ i. e., } c=25\cdot 7 \text{ cm./sec.},$$
$$\lambda=\cdot 54 \text{ cm.}$$

* It may be verified that R is positive for this value of a when $\xi=0$.

The corresponding values of x and y are ("Waves," p. 392) given by

$$ct-x=\cdot086(\xi-1\cdot5\sin\xi+\cdot44\sin 2\xi-\cdot075\sin 3\xi$$
$$+\cdot016\sin 4\xi+\cdot0017\sin 5\xi),$$
$$y=\cdot086(1\cdot5\cos\xi-\cdot44\cos 2\xi+\cdot075\cos 3\xi$$
$$-\cdot016\cos 4\xi-\cdot0017\cos 5\xi).$$

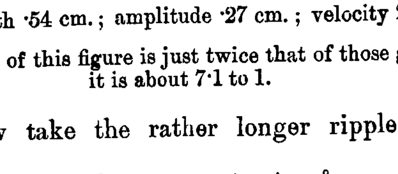
Corresponding values of x and y are given in Table I., and the ripple is drawn in fig. 1.

TABLE I.

Ripple of wave-length ·54 cm.

ξ.	$ct-x$.	y.
0	0	·095
$\pi/6$	·008	·094
$\pi/3$	·009	·077
$\pi/2$	·013	·037
$2\pi/3$	·037	−·039
$5\pi/6$	·128	−·129
π	·272	−·175

Fig. 1.



Ripple of length ·54 cm.; amplitude ·27 cm.; velocity 25·7 cm./sec.

N.B.—The scale of this figure is just twice that of those given below: it is about 7·1 to 1.

Let us now take the rather longer ripples $\kappa=2$ and $\kappa=1$.

First, when $\kappa=2$ the constants A_1, &c. are given by equations (17). In the particular case when $a=1$, which is a relatively high ripple, we have

$$\kappa=2, \quad \lambda=1{\cdot}22 \text{ cm.}, \quad \lambda/2\pi=\cdot194,$$
$$A_1=-1, \quad A_2=\cdot0083, \quad A_3=\cdot067, \quad A_4=\cdot0012,$$
$$A_5=-\cdot0087,$$
$$\mu=2{\cdot}18, \quad c=20{\cdot}3 \text{ cm./sec.}$$

Corresponding values of x and y are given in Table II., and the ripple is shown in fig. 2.

TABLE II.

Ripple of wave-length 1·22 cm.

ξ.	$ct-x$.	y.
0	0	·181
$\pi/6$	·018	·165
$\pi/3$	·038	·111
$\pi/2$	·096	·001
$2\pi/3$	·239	−·110
$5\pi/6$	·421	−·167
π	·611	−·185

Fig. 2.

Ripple of length 1·22 cm.; amplitude ·37 cm.; velocity 20·3 cm./sec.

Again, when $\kappa=1$ we find A_1, &c. from equations (16). Taking in this case $a=1/3$, we find

$$\kappa=1, \quad \lambda=1{\cdot}31 \text{ cm.}, \quad \lambda/2\pi=\cdot208,$$
$$A_1=-\cdot333, \quad A_2=-\cdot043, \quad A_3=\cdot0037, \quad A_4=\cdot0029,$$
$$A_5=\cdot0004,$$
$$\mu=1{\cdot}85, \quad c=19{\cdot}4 \text{ cm./sec.}$$

Corresponding values of x and y are given in Table III. and the ripple is drawn in fig. 3.

TABLE III.

Ripple of wave-length 1·31 cm.

ξ.	$ct-x$.	y.
0	0	·077
$\pi/6$	·068	·068
$\pi/3$	·144	·031
$\pi/2$	·258	−·009
$2\pi/3$	·389	−·039
$5\pi/6$	·522	−·054
π	·653	−·060

Fig. 3.

Ripple of length 1·31 cm.; amplitude ·14 cm.; velocity 19·4 cm./sec.

We come now to the most interesting portion of our inquiry,—the consideration of the form of those waves for which κ is the reciprocal of a positive integer n other than unity. When κ is not actually equal to $1/n$, it is always possible to choose a sufficiently small to insure the convergence of the series for A_2, A_3, ... For it is easy to satisfy oneself that the index of the power of $n\kappa-1$ in the denominator of any coefficient is less than the index of the power of a which it multiplies. Hence if we put $a=(n\kappa-1)b$, some power of $n\kappa-1$ will divide every coefficient A_m, and it is now manifestly possible to choose a value of b which secures convergence of the series. If $n\kappa-1$ is small a will be small, *i. e.* the amplitude of the wave will be small; but as A_n and the succeeding coefficients become relatively important the form of the wave may be very different from that of a simple cosine curve.

When $\kappa=1/n$ the case is different. The ordinary method of approximation breaks down altogether, and we have to start again *ab initio*. We shall consider in particular the case $\kappa=\frac{1}{2}$.

When $\kappa=\frac{1}{2}$, to a first approximation $C=-\frac{3}{4}$, and therefore equation (6) leads to

$$\tfrac{3}{4}A_1^2+\text{terms of order higher than the second} = 0.$$

Hence A_1 cannot be of the first order unless A_2 is of the same order. But if A_2 is of the first order, the second approximation to (5) is

$$(2C+\tfrac{3}{2})A_1+\tfrac{3}{2}A_1A_2 = 0,$$

i. e.,

$$C = -\tfrac{3}{4}-\tfrac{3}{4}A_2;$$

and substituting in (6) we have, to the second order,

$$-3A_2^2+\tfrac{3}{4}A_1^2 = 0,$$

i. e.,

$$A_2 = \pm\tfrac{1}{2}A_1.$$

From equations (7), (8), and (9) we see that, if A_1 and A_2 are both of the first order, A_3 and A_4 are both of the second order, A_5 and A_6 are both of the third order, and so on.

If $\kappa=1/3$, it is easy to see that we may take the orders of the successive coefficients to be

$$1, 2;\quad 1, 2, 3;\quad 2, 3, 4;\quad 3, 4, 5;\ \ldots.$$

While in the general case, when $\kappa=1/n$, the orders are

$$1, 2, 3, \ldots n-1;\quad n-2, n-1, n \ldots 2n-3;$$
$$2n-4, 2n-2, \ldots 3n-5;\quad 3n-6, \ldots$$

It is only in the particular case of $\tfrac{1}{2}\kappa=$, *i. e.* $n=2$, that there is any ambiguity in the form of A_n.

Let us return now to the consideration of the case $\kappa=\tfrac{1}{2}$. To the first order we have

$$A_1=-a,\qquad A_2=\pm\tfrac{1}{2}a.$$

And, on substituting these values in the other equations, we find as a first approximation,

$$A_1=-a,\quad A_2=\pm\tfrac{1}{2}a,\quad A_3=\pm\tfrac{3}{4}a^2,\quad A_4=0,\quad A_5=0,$$
$$C=-\tfrac{3}{4}\mp\tfrac{3}{8}a,\qquad \mu=\tfrac{3}{2}\pm\tfrac{3}{4}a.$$

As a second approximation I find, after rather long analysis,

$$A_1=-a,\quad A_2=\pm\tfrac{1}{2}a-\tfrac{1}{8}a^2,\quad A_3=\pm\tfrac{3}{4}a^2+\tfrac{15}{8}a^3,$$
$$A_4=\mp\tfrac{1}{2}a^2,\quad A_5=0,\quad A_6=\mp\tfrac{3}{80}a^3,$$
$$C=-\tfrac{3}{4}\mp\tfrac{3}{8}a+\tfrac{9}{32}a^2,\qquad \mu=\tfrac{3}{2}\pm\tfrac{3}{4}a-\tfrac{25}{16}a^2.$$

In particular, when $a=\cdot2$ the two sets of values are :—

(1) $A_1=-\cdot2,\quad A_2=\cdot095,\quad A_3=\cdot045,\quad A_4=-\cdot004,$

$$A_5=0,\qquad A_6=-\cdot0003,$$
$$\mu=1\cdot59,\quad c=24\cdot6 \text{ cm./sec.}$$

(2) $A_1=-\cdot2,\quad A_2=-\cdot105,\quad A_3=-\cdot015,\quad A_4=\cdot004,$

$$A_5=0,\quad A_6=\cdot0003,$$
$$\mu=1\cdot29,\quad c=22\cdot2 \text{ cm./sec.}$$

The two corresponding sets of values of x and y are given in Tables IV. *a* and IV. *b*, and the ripples* are shown in figs. 4 *a* and 4 *b*, respectively.

TABLE IV.

Ripple of length 2·44 cm.

a.

ξ.	$ct-x$.	y.
0	0	·025
$\pi/6$	·21	·050
$\pi/3$	·37	·087
$\pi/2$	·52	·038
$2\pi/3$	·71	−·039
$5\pi/6$	·96	−·086
π	1·22	−·095

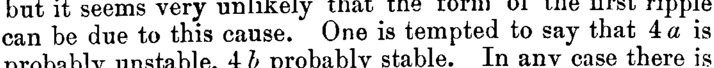
b.

ξ.	$ct-x$.	y.
0	0	·123
$\pi/6$	·12	·088
$\pi/3$	·30	·014
$\pi/2$	·54	−·042
$2\pi/3$	·78	−·052
$5\pi/6$	1·00	−·046
π	1·22	−·044

Fig. 4 *a*.

Ripple of length 2·44 cm.; amplitude ·182 cm.; velocity 24·6 cm./sec.

Fig. 4 *b*.



Ripple of length 2·44 cm.; amplitude ·175 cm.; velocity 22·2 cm./sec.

It is possible that the dimple at the crest of the second ripple is due only to the neglect of terms of higher order; but it seems very unlikely that the form of the first ripple can be due to this cause. One is tempted to say that 4 *a* is probably unstable, 4 *b* probably stable. In any case there is room for experimental investigation of the forms of ripples of this particular length, and also of the form of "high" ripples of very short wave-length, such as that shown in fig. 1. It would also be interesting to obtain experimentally the forms of the ripples whose lengths are given by $\kappa=1/3$, $\kappa=1/4$, &c.

* I have not strictly adhered to the customary distinction between ripples and waves.

It must, however, be remembered that although viscosity does not, to any appreciable extent, affect the forms of these ripples, it does very rapidly damp them out. Thus the amplitude of the ripple in fig. 1 is halved in less than one fifth of a second, so that it must be sought within a very few centimetres of the generating source. But, if the ripples of fig. 4 could be produced, they might be expected to travel some twenty or thirty centimetres without any serious diminution of amplitude.

LXXIII. *On the Operator ∇ in Combination with Homogeneous Functions.* By FRANK L. HITCHCOCK, *Ph.D.**

1. AMONG the uses of the Hamiltonian operator ∇ there are three which are particularly remarkable. First is the use of ∇ to distinguish the character of fields of force, fluid motion, and other vector fields. Second is its use to express integral relations having to do with space-integration over surfaces and volumes. Third, when ∇ is combined with functions which are homogeneous in the point-vector ρ, many new results are obtained.

To recall the leading facts under the first category:—If a vector function F of the point-vector ρ satisfies the relation $V\nabla F=0$, its rotation vector or "curl" is zero, and its distribution is lamellar. If $S\nabla F=0$, the "divergence" is zero, and the distribution solenoidal. If both these relations hold, so that $\nabla F=0$, the distribution is Laplacean. If F is everywhere at right angles to its own curl, we have $SF\nabla F=0$; as I am not aware of any name for such a distribution, I shall venture to call it ORTHOGYRAL †. The most significant property of an orthogyral vector is that it becomes lamellar when multiplied by a suitably chosen variable scalar ‡.

Under the second category fall the quaternionic forms of the theorems of Gauss and of Stokes on multiple integrals, which have been greatly extended by the late Profs. Tait and C. J. Joly and by Dr. Alex. McAulay.

My present object is to develop somewhat further the uses

* Communicated by the Author.

† Pronounced ortho jī′ral.

‡ Such a characterization of vector fields by means of differential operators may be greatly extended. Thus the four fields to which names are above given are characterized by the linear operators $V\nabla$, $S\nabla$, ∇, and $SF\nabla$, special cases of the general linear quaternion function of ∇, which in these combinations is, analytically, both vector and differentiator. I have considered the general question in a former paper ("The Double Nature of Nabla," Phil. Mag. Jan. 1909).