It appears that if we take A = B so that half the electrons are in crystals and half distributed at random, the calculated values of I_θ/I_{90} agree fairly well with those found by Crowther.

The excess of scattered radiation in the emergence direction seems therefore to be easily explicable on the theory that Röntgen rays are very short electromagnetic waves or pulses. The only property of Röntgen rays which is not easily explicable on this theory seems to be that of causing the emission of high-velocity electrons. The most reasonable way of explaining the emission of these electrons seems to be that proposed by Planck *, according to which matter absorbs radiant energy continuously but only emits it in definite amounts inversely proportional to the wave-length of the radiation. This view enables the ordinary electromagnetic wave theory of light and Röntgen rays to be retained.

XLI. *On Deep Water Waves.* By J. R. WILTON, *M.A., B.Sc., Assistant Lecturer in Mathematics in the University of Sheffield* †.

SOME time ago, when studying Stokes's papers on Oscillatory Waves (Collected Works, vol. i. and vol. v.), I noticed that the work of determining the coefficients in the expansions of the coordinates might be very considerably reduced, and that the order of magnitude of any coefficient was determinate. These expansions may, without great labour, be carried to a high order of approximation, and the form of the wave may be drawn for a value of the amplitude not far short of that for the highest wave.

By a change in the notation we are enabled to treat the progressive wave directly, and at the same time to retain all the advantages of the customary method of reducing the problem first to one of steady motion. The advantage of the change will be more apparent to anyone who attempts to discuss the stability of the wave.

We shall use throughout the abbreviations :

$$z = x + \iota y, \qquad z' = x - \iota y,$$

$$w = \phi + \iota\psi, \qquad w' = \phi - \iota\psi,$$

$$\eta = \frac{2\pi}{\lambda}(ct - z), \qquad \eta' = \frac{2\pi}{\lambda}(ct - z'),$$

$$\xi = \frac{2\pi}{\lambda}\frac{w}{c} + \eta, \qquad \xi' = \frac{2\pi}{\lambda}\frac{w'}{c} + \eta',$$

* *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften,* April 3, 1913.

† Communicated by the Author.

where λ, c, ϕ, and ψ are the wave-length, wave velocity, velocity potential, and current function, respectively.

We assume that η is a function of ξ, say,

$$\eta = \mathrm{F}(\xi). \qquad (1)$$

The conditions which have to be satisfied are $w=0$ when $y=\infty$; and, on the free surface, $y=f(ct-x)$,

$$\frac{\partial f}{\partial x}(c-u)=v,$$

$$\dot{\phi}+\tfrac{1}{2}q^2=gy-\tfrac{1}{2}\mathrm{C}',$$

where u and v are, respectively, the horizontal and vertical velocities, and q is the resultant velocity. The axis of y is vertically downwards, that of x horizontal. We shall also take the origin at the trough of a wave when $t=0$.

The first of the two surface conditions is satisfied whatever the form of the function F, provided that ξ is real on the surface, *i. e.* if

$$\psi = cy$$

on the surface.

Let

$$\eta' = \mathrm{F}'(\xi') \qquad (2)$$

be the equation obtained from (1) by changing the sign of ι throughout. We then have

$$u-\iota v=\frac{dw}{dz},$$

$$\frac{d\eta}{d\xi}\left(\frac{dw}{dz}-c\right)=-c,$$

or

$$c-u+\iota v=c\Big/\frac{d\eta}{d\xi}$$

and

$$c-u-\iota v=c\Big/\frac{d\eta'}{d\xi'}.$$

Also

$$c=\frac{\lambda}{2\pi}\frac{\partial\eta}{\partial t}=\frac{d\eta}{d\xi}\left(\frac{\dot{\phi}+\iota\dot{\psi}}{c}+c\right),$$

i. e.

$$\frac{\dot{\phi}+\iota\dot{\psi}}{c}=-c+c\Big/\frac{d\eta}{d\xi}=-(u-\iota v),$$

$$\frac{\dot{\phi}-\iota\dot{\psi}}{c}=-(u+\iota v);$$

therefore

$$\dot{\phi}=-uc,$$

so that the condition of constant pressure on the free surface becomes

$$(c-u)^2+v^2=c^2-\mathrm{C}'+2gy,$$

i. e.

$$\frac{1}{\frac{d\eta}{d\xi}\frac{d\eta'}{d\xi'}} = \frac{g\lambda}{2\pi c^2}[\iota(\eta-\eta')-2\mathrm{C}]. \quad . \quad . \quad . \quad (3)$$

We must, further, have $w=0$, *i. e.* $\eta=\xi$, when $y=\infty$, or $\xi=-\iota\infty$. This condition is satisfied if we assume the expansion

$$\eta=\xi+\iota\mathrm{A}_1e^{-\iota\xi}+\iota\mathrm{A}_2e^{-2\iota\xi}+\ldots$$

as the form of η *. Making the assumption, and substituting in equation (3), we find, as the equation to determine the coefficients A_n,

$$0=\frac{\pi c^2}{g\lambda}+\left[\mathrm{C}+\sum_{n=1}^{\infty}\mathrm{A}_n\cos n\xi\right]\left[\left(1+\sum_{n=1}^{\infty}n\mathrm{A}_n\cos n\xi\right)^2+\left(\sum_{n=1}^{\infty}n\mathrm{A}_n\sin n\xi\right)^2\right].$$

The terms of this equation must be re-arranged in cosines of multiples of ξ, and then the coefficient of each cosine is to be equated to zero. The general form of the equations so

* I find it difficult to persuade myself that this is the *only* form of F. It is possible, if we do not make this assumption, to satisfy equation (3), together with the condition of rest at the bottom of the liquid, in an infinite number of ways, and in certain cases the exact expression for the function F may be obtained. In fact if we put $\eta=\eta_1+\iota\eta_2$, where η_1 and η_2 are real when ξ is real, equation (3) is satisfied provided that

$$\eta_1=\int\sqrt{-\frac{\pi c^2}{g\lambda(\mathrm{C}+\eta_2)}-\eta_2'^2}\,d\xi;$$

and the only remaining condition is that $\eta=\xi$, when $\xi=-\iota\infty$. A function satisfying this condition is that given by the equation

$$\eta_2e^{\frac{g\lambda}{\pi c^2}\eta_2}=\mathrm{A}\cos\frac{g\lambda}{2\pi c^2}\xi,$$

which presents some remarkable points of resemblance to Stokes's wave. Such "waves" are, however, not in general such that the crests of all the stream-lines are vertically under the crests of the free surface. Stokes's solution is the only one which satisfies this condition.

The wave considered in the paper "On the Highest Wave in Deep Water" (Phil. Mag. Dec. 1913, pp. 1053–8) is of the type considered in this note. Another example, perhaps even more curious, is that of the steady motion represented by the equations

$$\frac{gz}{c^2}=\theta+\iota-\iota e^{\iota\theta},$$

$$\frac{gw}{2c^3}=\theta+\sin\theta,$$

in which the complete cycloid obtained by making θ a real quantity is a free surface, for which $\psi=0$. The fluid is, moreover, at rest at the bottom, where $y=-\infty$, $\theta=-\iota\infty$. In this case there is no need, as in the paper referred to, to "fit on" various distinct arcs of the complete curve.

obtained is as follows :—

$$0=\frac{\pi c^2}{g\lambda}+C\left(1+\sum_{n=1}^{\infty}n^2A_n^{\,2}\right)+\sum_{n=1}^{\infty}nA_n^{\,2}+\sum_{n=1}^{\infty}A_n\sum_{m=1}^{\infty}m(m+n)A_mA_{m+n},$$

$$0=C\left[2rA_r+2\sum_{m=1}^{\infty}m(m+r)A_mA_{m+r}\right]+A_r+\sum_{n=1}^{\infty}(2n+r)A_nA_{n+r}$$
$$+\tfrac{1}{2}\sum_{n=1}^{r-1}rA_nA_{r-n}+\sum_{n=1}^{\infty}A_n\sum_{m=1}^{\infty}m(m+n+r)A_mA_{m+n+r}$$
$$+\sum_{n=1}^{\infty}A_n\sum_{m=1}^{\infty}m(m+n-r)A_mA_{m+n-r},$$

for all values of r from 1 to infinity. It is understood that in the last summation $m+n-r$ is positive.

These equations must be solved by a process of successive approximation. The equations, to any desired order, may be written down from the above general expression, or by means of a rule derived from it. The rule, which is somewhat complicated, is as follows :—

The equation obtained by equating the coefficient of $\cos p\xi$ to zero is made up of terms :

(1) C multiplied by

$$2pA_p+2(p+1)A_1A_{p+1}+4(p+2)A_2A_{p+2}+\text{\&c.}$$

(2) A_p.

(3) All terms made up of the product of two coefficients the sum or difference of whose subscripts is p ; and the numerical multiplier of any term is the sum of the subscripts unless the term is a square, in which case it must be halved.

(4) All terms made up of the product of three coefficients whose subscripts are such that the sum of two of them minus the third is $\pm p$; and the numerical multiplier of any term is the product of the two numbers whose difference is p and whose sum is the sum of the subscripts of the coefficients forming the term. If, however, one of the terms contains the square of a coefficient, and is such that twice the subscript of this term minus the subscript of the remaining coefficient is $\pm p$, the term is to be halved, but not otherwise.

As an example of the use of the rule, the equation for which $p=2$ is here written down to the twelfth order. It is

$$0=C(4A_2+6A_1A_3+16A_2A_4+30A_3A_5+48A_4A_6+70A_5A_7)$$
$$+A_2+A_1^2+3A_1^2A_2+4A_1A_3+8A_1A_2A_3+4A_1^2A_4+4A_2^3$$
$$+6A_2A_4+15A_1A_3A_4+15A_1A_2A_5+15A_2A_3^2+8A_3A_5$$
$$+24A_1A_4A_5+24A_1A_3A_6+24A_2A_4^2+12A_2^2A_6+12A_3^2A_4$$
$$+10A_4A_6+35A_1A_5A_6+35A_1A_4A_7+35A_2A_5^2$$
$$+35A_2A_3A_7+35A_3A_4A_5+12A_5A_7+6A_6^2.$$

The leading term in each coefficient A_n may be written down. In fact,

$$A_n = (-)^n \frac{n^{n-2}a^n}{(n-1)\,!} + \text{higher powers of } a,$$

where $A_1 = -a$.

For, if we retain only the first approximation to each coefficient, we shall have, in order to determine A_n from the values of A_1, $A_2 \ldots A_{n-1}$, supposed known, the equation

$$(n-1)A_n = n(A_1A_{n-1} + A_2A_{n-2} + A_3A_{n-3} + \ldots),$$

where the last term on the right-hand side is $\frac{1}{2}A^2_{\frac{1}{2}n}$ if n is even, and $A_{\frac{n-1}{2}}\,A_{\frac{n+1}{2}}$ if n is odd. We have also put $C = -\frac{1}{2}$, its approximate value.

The solution of this system of equations is easily seen to be, assuming $A_1 = -a$,

$$A_n = (-)^n \frac{n^{n-2}a^n}{(n-1)\,!}.$$

The most direct proof is obtained by expanding x and x^2 in powers of xe^x by Burmann's theorem, squaring the first result, and comparing with the second.

I have not been able to find a general formula for even the second order approximation to the value of A_n. It is, however, easy to derive a sequence formula by means of which the second order approximation to any given coefficient may be calculated. In fact, if we retain terms of order $r+2$ in the equation for A_r, we obtain

$$C[2rA_r + 2(r+1)A_1A_{r+1}] + A_r + \tfrac{1}{2}r(A_1A_{r-1} + A_2A_{r-2} + \ldots A_{r-1}A_1) + A_1[rA_1A_r + (r-1)A_2A_{r-1} + \ldots A_rA_1] = 0,$$

i. e.

$$(r-1)A_r - \tfrac{1}{2}r(A_1A_{r-1} + A_2A_{r-2} + \ldots A_{r-1}A_1) + 2ra^2A_r - (r+1)A_1A_{r+1} = 0.$$

If we put

$$(-)^rA_r = \frac{r^{r-2}a^r}{(r-1)\,!} + B_ra^{r+2},$$

we obtain the following sequence equation for B_r :—

$$(r-1)B_r - r\left[B_{r-1} + B_{r-2} + \frac{3}{2}B_{r-3} + \frac{8}{3}B_{r-4} + \ldots + \frac{(r-2)^{r-4}}{(r-3)\,!}B_2\right] = \frac{1}{r\,!}\left[(r+1)^r - 2r^r\right].$$

In the same way, if we put

$$(-)^rA_r = \frac{r^{r-2}a^r}{(r-1)\,!} + B_ra^{r+2} + C_ra^{r+4},$$

we shall find the following sequence equation for C_r :—

$$(r-1)C_r - r\left[C_{r-1} + C_{r-2} + \frac{3}{2}C_{r-3} + \frac{8}{3}C_{r-4} + \ldots\right]$$

$$= (r+1)B_{r+1} + \frac{1}{(r+1)\,!}(r+2)^{r+1} + r\left[B_2B_{r-2} + B_3B_{r-3} + \ldots\right.$$

$$\left. + \frac{1}{2}B^2_{\frac{1}{2}r}\left(\text{or } B_{\frac{r-1}{2}}B_{\frac{r+1}{2}}\right)\right] - 2rB_r - \frac{11}{2}\frac{r^{r-1}}{(r-1)\,!}.$$

The corresponding sequence equations for the higher approximations become exceedingly complicated in form.

I find to the eighth order the following values of the coefficients :—

$$A_1 = -a,$$

$$A_2 = a^2 + \frac{1}{2}a^4 + \frac{29}{12}a^6 + \frac{1123}{72}a^8,$$

$$A_3 = -\left(\frac{3}{2}a^3 + \frac{19}{12}a^5 + \frac{1183}{144}a^7\right),$$

$$A_4 = \frac{8}{3}a^4 + \frac{313}{72}a^6 + \frac{103727}{4320}a^8,$$

$$A_5 = -\left(\frac{125}{24}a^5 + \frac{16603}{1440}a^7\right),$$

$$A_6 = \frac{54}{5}a^6 + \frac{54473}{1800}a^8,$$

$$A_7 = -\frac{7^5}{6\,!}a^7,$$

$$A_8 = \frac{8^6}{7\,!}a^8,$$

$$C = -\left(\frac{1}{2} + a^2 + \frac{11}{4}a^4 + \frac{167}{12}a^6 + \frac{29893}{288}a^8\right),$$

$$\frac{2\pi c^2}{g\lambda} = 1 + a^2 + \frac{7}{2}a^4 + \frac{229}{12}a^6 + \frac{7427}{48}a^8,$$

and to the tenth order—

$$-A_1 = a,$$

$$A_2 = a^2 + \cdot 5a^4 + 2\cdot 417a^6 + 15\cdot 597a^8 + 64\cdot 08a^{10},$$

$$-A_3 = 1\cdot 5a^3 + 1\cdot 583a^5 + 8\cdot 215a^7 + 55\cdot 01a^9,$$

$$A_4 = 2\cdot 667a^4 + 4\cdot 347a^6 + 24\cdot 01a^8 + 166\cdot 2a^{10},$$

$$-A_5 = 5\cdot 208a^5 + 11\cdot 53a^7 + 67\cdot 40a^9,$$

$$A_6 = 10\cdot 8a^6 + 30\cdot 26a^8 + 186\cdot 5a^{10},$$

$$-A_7 = 23{\cdot}34a^7 + 79{\cdot}20a^9 + 498{\cdot}3a^{11},$$

$$A_8 = 52{\cdot}01a^8 + 207{\cdot}4a^{10} + 1390a^{12},$$

$$-A_9 = 118{\cdot}6a^9 + 543{\cdot}4a^{11},$$

$$A_{10} = 275{\cdot}6a^{10} + 1426a^{12},$$

$$-A_{11} = 649{\cdot}8a^{11},$$

$$A_{12} = 1551a^{12},$$

$$-C = {\cdot}5 + a^2 + 2{\cdot}75a^4 + 13{\cdot}92a^6 + 103{\cdot}8a^8 + 823{\cdot}8a^{10},$$

$$\frac{2\pi c^2}{g\lambda} = 1 + a^2 + 3{\cdot}5a^4 + 19{\cdot}08a^6 + 154{\cdot}7a^8 + 1297a^{10}.$$

The series for η certainly becomes divergent in the neighbourhood of the crest of the wave when a is greater than $1/e$, where e is the base of natural logarithms, and the differential coefficients which occur in equation (3) become divergent when $a = 1/e$. For the series formed by the leading terms of the various coefficients is

$$\sum_{m=1}^{\infty} \frac{m^{m-2}a^m}{(m-1)!},$$

and this is divergent when a is greater than $1/e$, though it converges when $a = 1/e$. For when m is large the mth term of this series is

$$m^{m-2}a^m e^{m-1} \Big/ \left\{\sqrt{2\pi}(m-1)^{m-\frac{1}{2}}\right\}$$

$$= a^m (m-1)^{\frac{1}{2}} e^m \Big/ \left\{\sqrt{2\pi} e \left(1 - \frac{1}{m}\right)^m m^2\right\}$$

$$= (ae)^m m^{-\frac{3}{2}} \Big/ \sqrt{2\pi},$$

which proves the desired result. Further, the corresponding term of the series for $\dfrac{d\eta}{d\xi}$ is of order $(ae)^m m^{-\frac{1}{2}}$, so that this series diverges when $a = 1/e$. Both series are, however, convergent when $a < 1/e$.

Moreover, the terms of any one coefficient are all of the same sign, so that the divergence of the series when a is greater than $1/e$ is increased by the presence of these terms, and its convergence when a is less than $1/e$ is rendered doubtful. It is impossible, with the numbers given above, to say with any certainty whether the series for any given coefficient is convergent or not, but the general impression is that all become divergent when $a = 1/3$ or thereabouts.

For $a=1/\sqrt{10}$, I find :—

$$-A_1=\cdot316,\quad -A_5=\cdot026,\quad -A_9=\cdot006,$$
$$A_2=\cdot112,\quad A_6=\cdot019,\quad A_{10}=\cdot004,$$
$$-A_3=\cdot060,\quad -A_7=\cdot013,\quad -A_{11}=\cdot003,$$
$$A_4=\cdot037,\quad A_8=\cdot010,\quad A_{12}=\cdot002,$$
$$\frac{2\pi A}{\lambda}=\cdot86,\quad \lambda/A=7\cdot3,\quad \frac{2\pi c^2}{g\lambda}=1\cdot2,$$

where A is the amplitude of the wave. Comparison of these figures with those obtained by Michell * for the corresponding quantities in the case of the highest wave show that the wave for which $a=1/\sqrt{10}$ is not far short of the highest. Michell's figures are

$$\lambda/A=7\cdot04,\qquad \frac{2\pi c^2}{g\lambda}=1\cdot20.$$

From the above values of the coefficients I find the following table of values of $ct-x$ and y on the free surface, whose equation, given by putting $\psi=cy$ in the expression for η as a function of ξ and then equating real and imaginary parts, is found to be that resulting from the elimination of ξ

$$\left(\xi=\frac{2\pi}{\lambda c}\{\phi+c(ct-x)\}\right)$$

between

$$\frac{2\pi}{\lambda}(ct-x)=\xi+A_1\sin\xi+A_2\sin 2\xi+\ldots$$

and

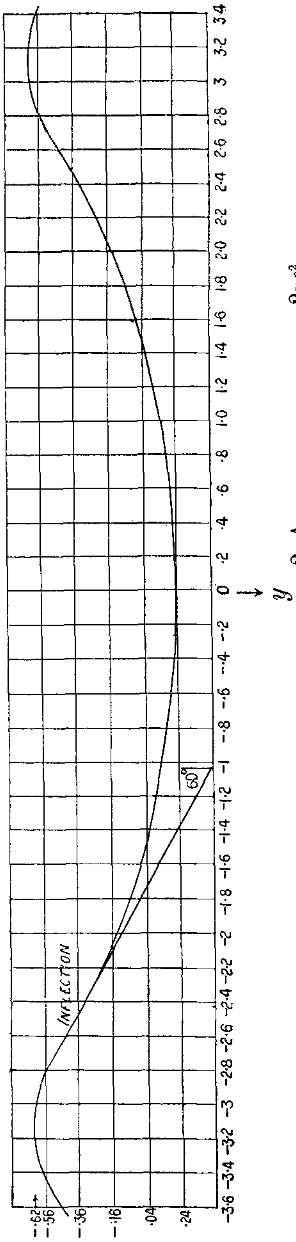
$$-\frac{2\pi y}{\lambda}=\quad A_1\cos\xi+A_2\cos 2\xi+\ldots$$

ξ.	$\frac{2\pi}{\lambda}(ct-x)$.	$\frac{2\pi}{\lambda}y$.	ξ.
0°	0	·24	360°
45°	·64	·20	315°
90°	1·30	·09	270°
135°	2·01	−·13	225°
150°	2·28	−·26	210°
165°	2·60	−·42	195°
170°	2·75	−·53	190°
175°	2·93	−·59	185°
180°	3·14	−·62	180°

* Phil. Mag. November 1893.

The curve obtained from these numbers is shown in fig. 1. It will be seen that the point of inflexion is very much nearer to the crest of the wave than to the trough, and that the greatest slope of the wave is, as nearly as can be measured, 30°. This is just what it should be at the crest of the highest wave.

Fig. 1.



Profile of Deep Water Wave when $a = \cdot 316$, $\dfrac{2\pi A}{\lambda} = \cdot 86$, $\lambda/A = 7\cdot 3$, $\dfrac{2\pi c^2}{g\lambda} = 1\cdot 2$.

There can be no doubt that Stokes's series become, as he supposed, divergent in the neighbourhood of the crest of the highest wave, but they evidently hold right up to this point.

I have also attempted to determine whether the wave-profile becomes unstable for a ratio of amplitude to wave-length less than that for the highest wave, but as the work is laborious, and the conclusion arrived at is the merely negative one that, so far as it is possible to tell from the somewhat imperfect analysis, it does not become exponentially unstable until a exceeds the value corresponding to the highest wave for which Stokes's series converge, I have not included it.

In attempting to determine the stability of the wave, we are compelled to consider only a small disturbance of the progressive wave profile, depending on a time-factor of the form e^{-kt}, where k may be complex. We are also compelled, in order to prevent the analysis from becoming unmanageable, to assume that a^4 is negligible. The first supposition is the one which is always made, although it may on occasion lead to difficulty, as, for example, in the case of the flow of viscous fluid between two parallel planes which, as is well known, is mathematically stable for small disturbances, but is experimentally unstable if the velocity exceeds a certain value. If the real part of k had been negative, the wave would necessarily have been unstable ; but, although it is actually zero, it is not therefore absolutely certain that the wave is stable.

The second supposition, that a^4 is negligible, may be justified by the considerations that a^2 does not exceed one-tenth, and that in the final equation which is found for k the coefficient of a^2 (a does not occur to any odd power in this equation) is of the same order of magnitude as the term independent of a.

XLII. *On the Number of Ions produced by the Gamma Radiation from Radium.* By A. S. Eve, *D.Sc., Macdonald Professor of Physics, McGill University, Montreal**.

IF q ions are produced, directly or indirectly, in a cubic centimetre of air, at standard temperature and pressure, at a distance of r cm. from a source of Q gm. of radium, then

$$q = \mathrm{KQ}/r^2 e^{\mu r},$$

where K is a constant, and μ is the coefficient of absorption of the γ rays in air.

Also the total number of ions produced in air by the γ rays from Q gm. will be found by integration to be

$$\mathrm{N} = 4\pi \mathrm{KQ}/\mu.$$

The first determination† of K was made in 1906 with an aluminium vessel, 0·4 mm. thick, and for radium bromide, assuming $e = 3{\cdot}4 \times 10^{-10}$, the value was $3{\cdot}1 \times 10^9$. This is equivalent to $\mathrm{K} = 3{\cdot}8 \times 10^9$ for a gramme of radium with $e = 4{\cdot}7 \times 10^{-10}$.

The second and third determinations‡, made in 1911, with very thin-walled testing vessels, gave values $\mathrm{K} = 3{\cdot}74 \times 10^9$ and $3{\cdot}81 \times 10^{-9}$ respectively. Accepting Chadwick's value for μ as ·000060 the corresponding value of N is $7{\cdot}8 \times 10^{14}$.

In Rutherford's 'Radioactive Substances' (1913), p. 295, it is stated that Moseley and Robinson have found $\mathrm{N} = 13 \times 10^{14}$. This corresponds to $\mathrm{K} = 6{\cdot}2 \times 10^9$, a value

* Communicated by the Author.
† Phil. Mag. September 1906.
‡ Phil. Mag. October 1911.