VIII. *A* New Series *for the* Rectification *of the* Ellipsis; *together with some* Observations *on the* Evolution *of the* Formula $(a^2 + b^2 - 2ab \operatorname{cos}\varphi)^n$. *By James Ivory*, A. M. *Communicated by John Playfair, Professor of Mathematics in the University of Edinburgh.*

[*Read Nov.* 7. 1796.]

Dear Sir,

HAVING, as you know, bestowed a good deal of time and attention on the study of that part of physical astronomy which relates to the mutual disturbances of the planets, I have, naturally, been led to consider the various methods of resolving the formula $(a^2 + b^2 - 2ab \operatorname{cos}\varphi)^n$ into infinite series of the form $A + B \operatorname{cos}\varphi + C \operatorname{cos} 2\varphi +$ &c. In the course of these investigations, a series for the rectification of the ellipsis occurred to me, remarkable for its simplicity, as well as its rapid convergency. As I believe it to be new, I send it you, inclosed, together with some remarks on the evolution of the formula just mentioned, which, if you think proper, you may submit to the consideration of the Royal Society.

I am, Dear Sir,
Your's, &c.

Douglastown, near Forfar,
20th October 1796.

James Ivory.

To Mr John Playfair, Professor of Mathematics, &c.

Let

Let ϵ denote the excentricity of an ellipſe, of which the ſemi-tranſverſe axis is unity, and ϖ the length of the ſemicircle, radius being unity: Then,

if we put $e = \frac{1 - \sqrt{1 - \epsilon^2}}{1 + \sqrt{1 - \epsilon^2}}$,

half the periphery of the ellipſis will be

$$= \frac{\varpi}{1+e}\left(1 + \frac{1^2}{2^2}\cdot e + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} e^4 + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} e^6 + \frac{1^2 \cdot 1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} e^8 + \&c.\right),$$

the coefficients being the ſquares of the coefficients of the radical $\sqrt{1 - \epsilon^2}$.

The common ſeries is,

$$\varpi \times \left(1 - \frac{1}{2} \cdot \frac{1}{2} \epsilon^2 - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1 \cdot 3}{2 \cdot 4} \epsilon^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \epsilon^6 - \&c.\right).$$

The firſt of theſe ſeries converges faſter than the other on two accounts: firſt, becauſe the coefficients decreaſe more rapidly; and, next, becauſe e is very ſmall in compariſon of ϵ, even when ϵ is great: Thus, if ϵ be $\frac{4}{5}$, e will be $\frac{1}{4}$, and $e^2 = \frac{1}{16}$.

In order to point out the way in which the preceding ſeries was diſcovered, let us ſuppoſe $(a^2 + b^2 - 2ab \operatorname{cos}\varphi)^n = A + B \operatorname{cos}\varphi + C \operatorname{cos} 2\varphi + \&c.$; and to determine the coefficients, A, B, C, &c. let us, with M. de la Grange, conſider the quantity $(a^2 + b^2 - 2ab \operatorname{cos}\varphi)$ as the product of the two imaginary expreſſions $\left(a - bc^{\varphi\sqrt{-1}}\right)$, and $\left(a - bc^{-\varphi\sqrt{-1}}\right)$, where c denotes the number whoſe hyperbolic logarithm is unity. Then, by expanding the powers $\left(a - bc^{\varphi\sqrt{-1}}\right)^n$, and $\left(a - bc^{-\varphi\sqrt{-1}}\right)^n$ into the ſeries $a^n\left(1 - \alpha \cdot \frac{b}{a} c^{\varphi\sqrt{-1}} + \beta c^{2\varphi\sqrt{-1}} - \gamma c^{3\varphi\sqrt{-1}} + \&c.\right)$

and

and $a^n\left(1 - \alpha.\frac{b}{a}c^{-\varphi\sqrt{-1}} + \beta c^{-2\varphi\sqrt{-1}} - \gamma c^{-3\varphi\sqrt{-1}} + \text{\&c.}\right.$

we have $\alpha = n$, $\beta = \frac{n.\overline{n+1}}{1.2}$, $\gamma = \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}$ &c.

THEN multiplying these two series together, and putting $2\cos m\varphi$ for its imaginary value $c^{+m\varphi\sqrt{-1}} + c^{-m\varphi\sqrt{-1}}$, we shall find, on equating the terms,

$$A = a^{2n} \times \left(1 + \alpha^2.\frac{b^2}{a^2} + \beta^2.\frac{b^4}{a^4} + \gamma^2.\frac{b^6}{a^6} + \text{\&c.}\right),$$

$$B = -2a^{2n} \times \left(\alpha\cdot\frac{b}{a} + \alpha\beta\cdot\frac{b^3}{a^3} + \beta\gamma\cdot\frac{b^5}{a^5} + \text{\&c.}\right),$$

and so on.

OF the several series for A, B, C, &c. the first deserves particular attention, on account of the simplicity of the law of its terms. It deserves the more attention, too, that the whole fluent $\int\dot{\varphi}\,(a^2 + b^2 - 2ab\cos\varphi)^n$, generated while φ from o becomes $= \pi$, half the circumference of the circle, is $= A + \pi$: all the other terms of the fluent then vanishing.

SUPPOSE now, in an ellipsis, the semi-transverse $= 1$, the excentricity $= \epsilon$, and φ an arch of the circumscribing circle, reckoned from the extremity of the transverse: then the fluxion of the correspondent arch of the ellipsis, cut off by the same ordinate, will be $= \dot{\varphi}\sqrt{1 - \epsilon^2\cos^2\varphi}$.

IN this expression, I write $\frac{1}{2} + \frac{1}{2}\cos 2\varphi$, for $\cos^2\varphi$: and put the result, $\dot{\varphi}\sqrt{1 - \frac{\epsilon^2}{2} - \frac{\epsilon^2}{2}\cos 2\varphi} = \dot{\varphi}\sqrt{a^2 + b^2 - 2ab\cos 2\varphi}$, a and b being indeterminate quantities.

To determine a and b, we have $a^2 + b^2 = 1 - \frac{\epsilon^2}{2}$, and $2ab = \frac{\epsilon^2}{2}$: whence $a + b = 1$, and $a - b = \sqrt{1 - \epsilon^2}$ so that $a = \frac{1 + \sqrt{1 - \epsilon^2}}{2}$ and $b = \frac{1 - \sqrt{1 - \epsilon^2}}{2}$.

I THUS obtain $\varphi\sqrt{1-\varepsilon^2\cos^2\varphi} = \dot{\varphi}\sqrt{a^2+b^2-2ab\cos 2\varphi}$: and, taking the whole fluent, while φ from o becomes $=\varpi$, it is manifest, from what has been premised, that the semiperiphery of the ellipsis is $=$

$$\varpi\times a\times\left(1+\frac{1}{2^2}\cdot\frac{b^2}{a^2}+\frac{1^2.1^2}{2^2.4^2}\cdot\frac{b^4}{a^4}+\frac{1^2.1^2.3^2}{2^2.4^2.6^2}\cdot\frac{b^6}{a^6}+\&c.\right),$$

or putting $\frac{b}{a}=e=\frac{1-\sqrt{1-\varepsilon^2}}{1+\sqrt{1-\varepsilon^2}}$, and $a=\frac{a}{a+b}=\frac{1}{1+\frac{b}{a}}=\frac{1}{1+e}$,

the semiperiphery of the ellipsis $=\frac{\varpi}{1+e}\times$

$$\left(1+\frac{1^2}{2^2}e^2+\frac{1^2.1^2}{2^2.4^2}e^4+\frac{1^2.1^2.3^2}{2^2.4^2.6^2}e^6+\&c,\right)$$

IN this series, as was before observed, e is a small fraction even when ε is very considerable, and the coefficients are more simple in the law of progression, and converge faster, (especially in the first terms), than in the common series.

IF we suppose the ellipsis to be infinitely flattened, in which case $\varepsilon=1$, and $e=1$, and the semiperiphery $=2$, this series gives $2=\frac{\varpi}{2}\times\left(1+\frac{1^2}{2^2}+\frac{1^2.1^2}{2^2.4^2}+\frac{1^2.1^2.3^2}{2^2.4^2.6^2}+\&c.\right)$, and so

$$\frac{4}{\varpi}=1+\frac{1^2}{2^2}+\frac{1^2.1^2}{2^2.4^2}+\frac{1^2.1^2.3^2}{2^2.4^2.6^2}+\frac{1^2.1^2.3^2.5^2}{2^2.4^2.6^2.8^2}+\&c.$$

BUT, we may remark, that as we have here obtained the sum of the squares of the coefficients of the binomial when the exponent is $\frac{1}{2}$; so, from the same source, we may determine the sum of the squares of the coefficients corresponding to any other exponent, at least by a fluent.

FOR taking the whole fluent when $\varphi=\varpi$, we have

$$\int(a^2+b^2-2ab\cos\varphi)^n\dot{\varphi}=a^{2n}\varpi\left(1+\alpha^2.\frac{b^2}{a^2}+\beta^2.\frac{b^4}{a^4}+\gamma^2.\frac{b^6}{a^6}+\&c.\right).$$

and so when $a=1$, and $b=1$,

$$\int\left(\frac{a^2+b^2-2ab\cos\varphi}{\varpi}\right)^n\dot{\varphi}=1+\alpha^2+\beta^2+\gamma^2+\&c.$$

NOW,

Now, when $a = 1$, and $b = 1$, $\int \dot{\varphi} (a^2 + b^2 - 2ab \cos \varphi)^n = 2^{2n} \times \int \dot{\varphi} \left(\sin \frac{\varphi}{2}\right)^{2n}$ because $2 \left(\sin \frac{\varphi}{2}\right)^2 = 1 - \cos \varphi$: we thus obtain

$$\frac{2^{2n} \times \int \dot{\varphi} \left(\sin \frac{\varphi}{2}\right)^{2n}}{\varpi} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c.$$

the whole fluent to be taken when $\varphi = \varpi$, or $\frac{\varphi}{2} = \frac{\varpi}{2}$.

If we put $x = \sin \frac{\varphi}{2}$, we shall have

$$\frac{2^{2n} \times \int \frac{x^{2n} \dot{x}}{\sqrt{1 - x^2}}}{\frac{1}{2}\varpi} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c,$$

the whole fluent to be taken when $x = 1$; and in this formula n is any number fractional or integral, positive or negative; and α, β, γ, &c. the coefficients of the binomial raised to a power of which the exponent is n.

When n is a whole positive number,

$\int \frac{x^{2n} \dot{x}}{\sqrt{1 - x^2}} = \frac{1 . 3 . 5 \ldots . (2n - 1)}{2 . 4 . 6, \ldots \quad 2n} . \frac{\varpi}{2}$, in the case when $x = 1$:

And so, $2^{2n} \times \frac{1 . 3 . 5 \ldots . (2n - 1)}{2 . 4 . 6 \ldots . \quad 2n} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \&c.$

Now, $2^{2n} \times \frac{1 . 3 . 5 \ldots (2n - 1)}{2 . 4 . 6 \ldots . \quad 2n}$ is no other than the coefficient of the middle term of a binomial, raised to the power expressed by $2n$: Hence we have a very curious property of those numbers: viz. *that the sum of the squares of the coefficients of a binomial, the exponent being n, is equal to the coefficient of the middle term of a binomial, of which the exponent is 2n.*

Another remark, which I have to offer on this subject, may be considered not only as curious, in an analytical point of view, but as, in some measure, accomplishing an object that has much engaged the attention of mathematicians.

In the computation of the planetary disturbances, it becomes necessary to evolve the fraction $(a^2 + b^2 - 2ab \cos \varphi)^{-\frac{1}{2}}$ into a series

ſeries of this form, A + B coſφ + C coſ 2φ + &c. The quantities a and b repreſent the diſtances of the diſturbing planets from the ſun; and when theſe bear ſo great a proportion to one another, (as in the caſe of Jupiter and Saturn, or Venus and the Earth), that the fraction $\frac{b}{a}$ is large, it becomes extremely difficult to compute the coefficients A, B, &c. by ſeries, on account of the great number of terms that muſt be taken in. This matter not a little perplexed the firſt geometers who conſidered this ſubject, and they were obliged to approximate to the quantities ſought by the method of quadratures, and by other artifices.

Two things are to be attended to with regard to the quantities A, B, C, &c. The firſt is, That it is not neceſſary to compute all of them ſeparately by ſeries, or by other methods: They form a recurring ſeries; and the two firſt being ſo computed, all the reſt may be derived from them. The ſecond thing is, That the quantities A and B having been computed for any exponent n, the correſpondent quantities are thence derived, by eaſy formulæ, for the exponents $n+1$, $n+2$; $n-1$, $n-2$; and in general for the exponent $n+m$, m being any integer number, poſitive or negative.

From theſe remarks, it follows, that the whole difficulty lies in the computation of the two firſt quantities, A and B; and that we are not confined to a given exponent n, but may chooſe any one in the ſeries, $n+1$, $n+2$, &c.; $n-1$, $n-2$, &c.; that will render the computation moſt eaſy and expeditious.

Thus, in order to compute the quantities A and B, for the exponent $-\frac{3}{2}$, M. de la Grange makes choice of the exponent $+\frac{1}{2}$, which, in the whole ſeries of exponents $+\frac{3}{2}$, $+\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$, &c. is the moſt favourable for computation, on account of the convergency of the coefficients of the ſeries for A and B.

In considering these subjects, however, I have fallen upon a method of computing the quantities A and B for the exponent $-\frac{1}{2}$ by series that converge so fast, that, even taking the most unfavourable case that occurs in the theory of the planets, two or three terms give the values required with a sufficient degree of exactness. This is what I am now to communicate.

We are then to consider the expression $(a^2+b^2-2ab\,\text{cos}\,\varphi)^{-\frac{1}{2}}$ $= \frac{1}{\sqrt{a^2+b^2-2ab\,\text{cos}\,\varphi}}$: for the sake of simplicity in calculation, I write $\frac{b}{a} = c$, throwing out a altogether; and I suppose

$$\frac{1}{\sqrt{(1+c^2-2c\,\text{cos}\,\varphi)}} = A + B\,\text{cos}\,\varphi + C\,\text{cos}\,2\varphi + \&c.$$

Let ψ be an angle, so related to φ, that $\text{sin}\,(\psi-\varphi) = c\,\text{sin}\,\psi$: It is obvious, from this formula, that $\psi = \varphi$ when $\text{sin}\,\psi = 0$, that is, when ψ is equal to 0, or to ϖ, 2ϖ, &c.

We have then, $\text{cos}\,(\psi-\varphi) = \sqrt{1-c^2\,\text{sin}^2\psi}$: and taking the fluxions, $\dot{\psi}-\dot{\varphi} = \frac{c\,\text{cos}\,\psi\times\dot{\psi}}{\text{cos}\,(\psi-\varphi)} = \frac{c\,\text{cos}\,\psi\times\dot{\psi}}{\sqrt{(1-c^2\,\text{sin}^2\psi)}}$:

whence $\dot{\varphi} = \dot{\psi}\times\frac{\sqrt{1-c^2\,\text{sin}^2\psi}-c\,\text{cos}\,\psi}{\sqrt{1-c^2\,\text{sin}^2\psi}}$.

But $(\sqrt{1-c^2\,\text{sin}^2\psi}-c\,\text{cos}\,\psi)^2 = 1-c^2\,\text{sin}^2\psi+c^2\,\text{cos}^2\psi$ $-2c\,\text{cos}\,\psi\sqrt{1-c^2\,\text{sin}^2\psi} = 1+c^2-2c^2\,\text{sin}^2\psi-2c\,\text{cos}\,\psi$ $\sqrt{1-c^2\,\text{sin}\,2\psi}$, (because $c^2\,\text{cos}^2\psi = c^2-c^2\,\text{sin}^2\psi$) $= 1+c^2$ $-2c\times(c\,\text{sin}\,\psi\times\text{sin}\,\psi+\text{cos}\,\psi\sqrt{1-c^2\,\text{sin}^2\psi})$. Now, if we write for $c\,\text{sin}\,\psi$ its equal, $\text{sin}\,(\psi-\varphi)$, and for $\sqrt{1-c^2\,\text{sin}^2\psi}$ its equal, $\text{cos}\,(\psi-\varphi)$, we shall have $c\,\text{sin}\,\psi\times\text{sin}\,\psi+\text{cos}\,\psi\times$ $\sqrt{1-c^2\,\text{sin}^2\psi} = \text{sin}\,(\psi-\varphi)\times\text{sin}\,\psi+\text{cos}\,\psi\times\text{cos}\,(\psi-\varphi)$ $= \text{cos}\,\varphi$: which being substituted, there comes out $(\sqrt{1-c^2\,\text{sin}^2\psi}-c\,\text{cos}\,\psi)^2 = 1+c^2-2c\,\text{cos}\,\varphi$.

Our

Our fluxional formula thus becomes $\dot{\phi} = \dot{\psi}\,\frac{\sqrt{1 + c^2 - 2c\cos\phi}}{\sqrt{1 - c^2 \sin^2\psi}}$: whence $\frac{\dot{\phi}}{\sqrt{1 + c^2 - 2c\cos\phi}} = \frac{\dot{\psi}}{\sqrt{1 - c^2 \sin^2\psi}}$.

I next transform the quantity $\sqrt{1 - c^2 \sin^2\psi}$ as in the investigation for the elliptic series, and putting $c' = \frac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}}$, I find $\sqrt{1 - c^2 \sin^2\psi} = \frac{\sqrt{1 + c'^2 + 2c'\cos 2\psi}}{1 + c'}$, and so $\frac{\dot{\phi}}{\sqrt{1 + c^2 - 2c\cos\phi}} = \frac{(1 + c')\,\dot{\psi}}{\sqrt{1 + c'^2 + 2c'\cos 2\psi}}$.

Now, taking the fluents when $\phi = \varpi$, and $\psi = \varpi$, we shall have $\int \frac{\dot{\phi}}{\sqrt{1 + c^2 - 2c\cos\phi}} = A \times \varpi$: And according to the method of M. de la Grange, $\int \frac{\dot{\psi}}{\sqrt{1 + c'^2 + 2c'\cos 2\psi}} = \varpi \times \left(1 + \frac{1^2}{2^2} c'^2 + \frac{1^2 . 3^2}{2^2 . 4^2} c'^4 + \&c.\right)$: Hence $A = (1 + c') \times \left(1 + \frac{1^2}{2^2} c'^2 + \frac{1^2 . 3^2}{2^2 . 4^2} c'^4 + \&c.\right)$. And in this value of A, c' will be a small fraction, even though c be large; and the series will therefore converge very fast.

But, taking the value of A directly in a series, we have $A = 1 + \frac{1^2}{2^2} c^2 + \frac{1^2 . 3^2}{2^2 . 4^2} c^4 + \&c.$ And so $1 + \frac{1^2}{2^2} c^2 + \frac{1^2 . 3^2}{2^2 . 4^2} c^4 + \&c. = (1 + c') \times \left(1 + \frac{1^2}{2^2} c'^2 + \frac{1^2 . 3^2}{2^2 . 4^2} c'^4 + \&c.\right)$. Now, the two series being exactly alike, it is evident that we may transform the one, as we have transformed the other, and that, if we put $c'' = \frac{1 - \sqrt{1 - c'^2}}{1 + \sqrt{1 - c'^2}}$ we shall have $1 + \frac{1^2}{2^2} c'^2 + \frac{1^2 . 3^2}{2^2 . 4^2} c'^4 = (1 + c'') \times \left(1 + \frac{1^2}{2^2} c''^2 + \frac{1^2 . 3^2}{2^2 . 4^2} c''^4 + \&c.\right)$: whence $A = (1 + c')(1 + c'')\left(1 + \frac{1^2}{2^2} c''^2 + \frac{1^2 . 3^2}{2^2 . 4^2} c''^4 + \&c.\right)$.

It

It is manifest we may proceed in this manner as far as we please, and that, if we put $c''' = \frac{1-\sqrt{1-c''^2}}{1+\sqrt{1-c''^2}}$; $c'''' = \frac{1-\sqrt{1-c'''^2}}{1+\sqrt{1-c'''^2}}$ and so on, we shall have the value of A in an infinite product, $A = (1+c') \times (1+c'') \times (1+c''') (1+c'''') \times$ &c, the quantities c', c'', c''', c'''', &c. converging very rapidly.

Nothing more seems to be wished for, with regard to the computation of the quantity A: since we can, by methods sufficiently simple, exhibit the value of it in series that shall converge as fast as we please. By a similar mode of reasoning, I find the series $1 - \frac{1}{2^2}\gamma^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\gamma^4 - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}\gamma^6 +$ &c. (which occurs in determining the time of a body's descent in the arch of a circle), $= (1-c) \times (1 + \frac{1^2}{2^2}c^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}c^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}c^6 +$ &c.) where $c = \frac{\sqrt{1+\gamma^2}-1}{\sqrt{1+\gamma^2}+1}$: so that the summation of this series also is accomplished by the method above.

I have now only to explain the method of computing B. For this purpose I resume,

$$\frac{1}{\sqrt{1+c^2-2c\cos\varphi}} = A + B\cos\varphi + C\cos 2\varphi + \&c.$$

Multiply by $2\cos\varphi$, and there results

$$\frac{2\cos\varphi}{\sqrt{1+c^2-2c\cos\varphi}} = B + (2A + C)\cos\varphi + \&c.$$

whence it is manifest that the whole fluent

$$\int \frac{2\cos\varphi \times \dot{\varphi}}{\sqrt{1+c^2-2c\cos\varphi}}, \text{ when } \varphi = \pi, \text{ is equal to } B \times \pi.$$

From the preceding investigation we have $\frac{\dot{\varphi}}{\sqrt{1+c^2-2c\cos\varphi}} = \frac{\dot{\psi}}{\sqrt{1-c^2\sin^2\psi}}$, and $\cos\varphi = c\sin^2\psi + \cos\psi\sqrt{1-c^2\sin^2\psi}$,

 whence

whence $\frac{2\dot{\varphi}\cos\varphi}{\sqrt{1+c^2-2c\cos\varphi}} = \frac{2c\dot{\psi}\sin^2\psi}{\sqrt{1-c^2\sin^2\psi}} + 2\dot{\psi}\cos\psi$. Again, $2\sin^2\psi = 1 - \cos 2\psi$, and $\frac{1}{\sqrt{1-c^2\sin^2\psi}} = \frac{(1+c')}{\sqrt{1+c'^2+2c'\cos 2\psi}}$, c' being $= \frac{1-\sqrt{1-c^2}}{1+\sqrt{1-c^2}}$: these substitutions being made, we get

$$\frac{2\dot{\varphi}\cos\varphi}{\sqrt{1+c^2-2c\cos\varphi}} = c \times \frac{(1+c')\dot{\psi}}{\sqrt{1+c'^2+2c'\cos 2\psi}} - c \times \frac{(1+c')\dot{\psi}\cos 2\psi}{\sqrt{1+c'^2+2c'\cos 2\psi}}$$

$$+ 2\dot{\psi}\cos\psi.$$

Suppose now, $\frac{1}{\sqrt{1+c'^2+2c'\cos 2\psi}} = A' - B'\cos 2\psi + c'\cos 4\psi - \&c.$ it is evident, from what goes before, that, taking the fluents of the above fluxions, when φ and $\psi = \varpi$, we shall have $B \times \varpi = c \times (1+c') \times (A' + \frac{B'}{2}) \times \varpi$, and so $B = c \times (1+c') \times (A' + \frac{B'}{2})$.

The values of A' and B', in series according to the method of M. de la Grange, are

$$A' = 1 + \frac{1^2}{2^2}c'^2 + \frac{1^2.3^2}{2^2.4^2}c'^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}c'^6 + \&c.$$

$$\frac{1}{2}B' = \left(\frac{1}{2}c' + \frac{1}{2}\cdot\frac{1.3}{2.4}c'^3 + \frac{1.3.1.3.5}{2.4.2.4.6}c'^5 + \&c.\right)$$

which series converge very fast, on account of the smallness of c' in respect of c.

If, however, it be required to find the value of B by series still more converging, we may easily do so: For it is manifest that B and B′ are similar functions of c and c': and that if we make $c'' = \frac{1-\sqrt{1-c'^2}}{1+\sqrt{1-c'^2}}$, $c''' = \frac{1-\sqrt{1-c''^2}}{1+\sqrt{1-c''^2}}$, and so on, and put

— A″,

A″, A‴, &c.; B″, B‴, &c. to denote the correſponding values of A′ and B′, we ſhall have

$$B = c.(1 + c')(A + \frac{B'}{2})$$

$$B' = c'.(1 + c'')(A'' + \frac{B''}{2})$$

$$B'' = c''.(1 + c''')(A''' + \frac{B'''}{2}) \text{ \&c.}:$$

Now, remarking that $A' = (1 + c'')A''$; $A'' = (1 + c''')A'''$, &c. we have the following values of B:

$$B = c \times (1 + \frac{c'}{2}).(1 + c').(1 + c'')A'' + \frac{c}{2}.\frac{c'}{2}(1 + c')(1 + c'')B''.$$

$$B = c \times (1 + \frac{c'}{2} + \frac{c'}{2}.\frac{c''}{2})(1 + c')(1 + c'')(1 + c''')A''' + \frac{c}{2}.\frac{c'}{2}.\frac{c''}{2}(1 + c')$$
$$(1 + c'')(1 + c''')B'''.$$

And we may proceed in this manner to find the value of B in ſeries that ſhall converge as faſt as we pleaſe.

As the quantities c', c'', c''', &c. diminiſh very faſt, the ſeries A′, A″, A will approach rapidly to unity, and B′, B″, B‴ will decreaſe rapidly to nothing: Hence we have ultimately,

$$B = c \times (1 + \frac{c'}{2} + \frac{c'}{2}.\frac{c''}{2} + \frac{c'}{2}.\frac{c''}{2}.\frac{c'''}{2} + \text{\&c.}) \times (1 + c')(1 + c'')(1 + c''')$$
&c.

or, ſince $A = (1 + c')(1 + c'')(1 + c''')$ &c.:

$$B = c \times (1 + \frac{c'}{2} + \frac{c'}{2}.\frac{c''}{2} + \frac{c'}{2}.\frac{c''}{2}.\frac{c'''}{2} + \text{\&c.}) \times A.$$

We ſhall beſt ſee the degree of convergency of the quantities c, c', c'', &c. if we take the infinite ſeries by which they are derived one from another. Now, if $y = \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}}$, then alſo $y = \frac{x^2}{4} + \frac{x^4}{8} + \frac{5x^6}{64} + \frac{7x^8}{128} +$ &c.: whence it is obvious, that in the ſeries of quantities c, c', c'', &c. the fourth part of the ſquare of

 any

any term is nearly equal to the following term, and the rapidity with which the ſeries decreaſes is therefore very great.

THE method, then, that reſults from the preceding inveſtigations for computing A and B, is ſhortly this:

Put $c' = \frac{1 - \sqrt{1 - c^2}}{1 + \sqrt{1 - c^2}}$: and compute

$$1 + \frac{1^2}{2^2} c'^2 + \frac{1^2 . 3^2}{2^2 . 4^2} c'^4 + \frac{1^2 . 3^2 . 5^2}{2^2 . 5^2 . 6^2} c'^6 + \&c. = M,$$

$$\text{and } \frac{1}{2} c' + \frac{1}{2} . \frac{1 . 3}{2 . 4} c'^3 + \frac{1 . 3}{2 . 4} . \frac{1 . 3 . 5}{2 . 4 . 6} c'^5 + \&c. = N.$$

$$\text{Then } A = (1 + c') \times M, \text{ and}$$

$$B = c \times (1 + c') \times (M + N).$$

THE ſeries M and N will converge ſo faſt, even in the moſt unfavourable caſe that occurs in the theory of the planets, that the firſt three terms will give the ſums ſufficiently exact; and it will therefore not be neceſſary to have recourſe to the more converging ſeries A″ and B″.

SUCH is the method that I had firſt imagined, for facilitating theſe ſort of computations. I have ſince found, however, that by means of the common tables of ſines and tangents, the quantities A and B may be computed in a ſtill eaſier way from the expreſſions,

$$A = (1 + c') (1 + c'') (1 + c''') \&c.$$

$$B = c \times \left(1 + \frac{c'}{2} + \frac{c'}{2} . \frac{c''}{2} + \frac{c'}{2} . \frac{c''}{2} . \frac{c'''}{2} + \&c.\right) \times A.$$

FOR if $c = \text{ſin}\, m$, then $\sqrt{1 - c^2} = \text{coſ}\, m$ and $c' = \frac{1 - \text{coſ}\, m}{1 + \text{coſ}\, m}$ $= \tan^2 \frac{m}{2}$: conſequently $1 + c' = \text{ſec}^2 \frac{m}{2}$. In like manner, if $c' = \text{ſin}\, m'$, $c'' = \text{ſin}\, m''$, &c. we ſhall have $\text{ſin}\, m' = \tan^2 \frac{m}{2}$;

ſin m''

ſin $m''=\tan^2\frac{m'}{2}$, and ſo on: And $1+c''=$ ſec$^2\frac{m'}{2}$; $1+c'''=$ ſec$^2\frac{m''}{2}$, and ſo on. Thus:

$$A=\text{ſec}^2\frac{m}{2}\times\text{ſec}^2\frac{m'}{2}\times\text{ſec}^2\frac{m''}{2}\times\text{\&c.}$$

To find the logarithm of A, we have then only to add together the logarithm ſecants of the angles $\frac{m}{2}$, $\frac{m'}{2}$, $\frac{m''}{2}$, &c. to diminiſh the ſum by as many times the radius as there are ſecants, and to take twice the remainder. As the angles m, m', m'', &c. decreaſe very faſt, it will ſeldom be neceſſary to compute more than two or three of them.

The ſeries $\left(1+\frac{c'}{2}+\frac{c'}{2}\cdot\frac{c''}{2}+\frac{c'}{2}\cdot\frac{c''}{2}\cdot\frac{c'''}{2}+\text{\&c.}\right)$ is alſo readily computed from the tables; becauſe the logarithms of c', c'', c''', &c. being the ſines of the angles m', m'', &c. are all found in the tables.

As an example, let $c=0.72333$: which is the fraction that ariſes from dividing the mean diſtance of Venus from the ſun, by the mean diſtance of the Earth; and this is the moſt unfavourable caſe that occurs in the theory of the planets: Then to compute A, I find, in the table of natural ſines, that 0.72333 correſponds to $46°\ 19'\ 48\frac{1}{3}''$: we have therefore

$m=46°\ 19'\ 48\frac{1}{3}''$	
L. tan $\frac{m}{2}=$ l. tan $23°\ 9'\ 54\frac{1}{6}''=9.6313206$	L. ſec $\frac{m}{2}=10.0365070$
2	
L. ſin $m'=$ 9.2626412	
L. $m'=10°\ 32'\ 57''$	
L. tan $\frac{m'}{2}=$ l. tan $5°\ 16'\ 28\frac{1}{2}''=8.9652949$	L. ſec $\frac{m'}{2}=10.0018429$
2	
L. ſin $m''=$ 7.9305898	
L. $m''=0°\ 29'\ 18''$	

L. tan

$\text{L. tan}\,\frac{m''}{2} = \text{l. tan } 0^\circ\ 14'\ 39'' = 7.6295664$

$ 2$

$\text{L. sin}\,m''' = 5.2591328$

As m''' will only be a few seconds, it may be neglected.

Hence $A = 1.19318$

$\text{L. sec}\,\frac{m''}{2} = 10.0000039$

0.0383538

2

$\text{L. A} = 0.0767076$

To compute B, let $S = 1 + \frac{c'}{2} + \frac{c'}{2}\cdot\frac{c''}{2} + \&c.$

$\text{L. } c' = \text{l. sin } m' = 9.2626412$;

$\text{L. } c'' = \text{l. sin } m'' = 7.9305898$

$\text{l. } c'.\ c'' = 7.1932310$

1	$= 1.000000$
$\frac{c'}{2}$	$= 0.091540$
$\frac{c'}{2}\cdot\frac{c''}{2}$	$= 0.000390$
S	$= 1.091830$

$B = c \times S \times A.$

$\text{L. } c = \bar{1}.8593365$

$\text{L. S} = 0.0381948$

$\text{L. A} = 0.0767076$

$\text{L. B} = \bar{1}.9742389$, and $B = 0.942408$