Cycle Index of Uncertain Random Graph

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Abstract. With the increasing of the complexity of a system, there is a variety of indeterminacy in the practical applications of graph theory. We focus on uncertain random graph, in which some edges exist with degrees in probability measure and others exist with degrees in uncertain measure. In this paper, the chance theory is applied to construct the cycle index of an uncertain random graph. Then a method to calculate the cycle index of an uncertain random graph is presented. We also discuss some properties of the cycle index.

Keywords: Cycle index, Uncertain random graph, Chance theory, Uncertainty theory

1. Introduction

In real life, graph theory is applied to many problems, such as vertex covering problem (Chen et al. [13]), traveling salesman problem (Ma et al. [36]), vertex coloring problem (Chen et al. [14]), and so on. In classical graph theory, these problems are often considered in a determinacy environment, in which all the edges and the vertices can be completely determined. For more research on the theoretical problems and applications of classical graph theory, we may consult Wallis [42].

As the system becomes more complex, different types of vagueness are frequently encountered in practical application of graph theory. Sometimes, it cannot be completely determined whether two vertices are joined by an edge or not. Then, a problem is arise: how to deal with these vagueness phenomena? On one hand, many researchers regarded that vagueness phenomena belonged to the fuzziness phenomena, and they introduced fuzzy set theory into the graph theory. Rosenfeld [38] introduced a fuzzy graph with fuzzy vertex set and fuzzy edge set. The membership function of the fuzzy edge set is employed by Rosenfeld [38] to describe whether two vertices were joined by an edge or not. There are some new concepts related to fuzzy phenomena in graph theory, such as: bipolar fuzzy graphs [2], intuitionistic fuzzy hypergraphs [6], strong intuitionistic fuzzy graphs [5], $\mathcal{N}$-hypergraphs [4], fuzzy soft graphs [8], bipolar fuzzy digraphs [3], regular interval-valued fuzzy graphs [41], generalized fuzzy hypergraphs [17], domination in m-polar fuzzy graphs [9], and neutrosophic graphs [10]. Furthermore, the fuzzy phenomena in graph coloring problem has been investigated by Rosyida et al. [39] which developed a method to determine a fuzzy chromatic number of fuzzy graph.

On the other hand, some researchers handled the indeterminacy phenomena with randomness phenomena, and they used probability theory into the graph theory. The concept of random graph was first investigated by Erdős and Rényi [16], also by Gilbert [27] at nearly the same time. They thought that whether

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two vertices were joined by an edge or not can be described as a random variable. After that, many researchers, such as Alon et al. [11], Gamarnik and Sviridenko [18], and Gilmer and Kopparty [28] have done a lot of works in this field.

Although randomness has been used to characterize the indeterminacy phenomena, some imprecise quantities, such as information and knowledge represented by human language, do not behave like randomness (Akram and Dudek [6], Akram and Nawaz [8]). In order to model these imprecise quantities, Liu [29] founded the uncertainty theory. In the area of graph theory, some researchers have employed the uncertainty theory to deal with imprecise quantities in graphs. The concept of uncertain graph was given by Gao and Gao [24]. In their paper, a connectedness index was introduced to measure the degree that an uncertain graph was connected. After that, Zhang and Peng [43] discussed the Euler tour in an uncertain graph. In addition, Gao investigated the cycle index [21] and the regularity index [22], Gao [23] proposed a method to calculate the tree index, path index, and star index, and Gao et al. [26] discussed the diameter for an uncertain graph. Recently, Gao and Qin [25] concerned with how to calculate the uncertain measure that an uncertain graph was edge-connected. Different from the coloring problem of classical graph, Rosyida et al. [40] introduced a new approach to color an uncertain graph based on an α-cut of the uncertain graph. Chen et al. [14] investigated uncertain vertex coloring problem based on maximum uncertain independent vertex set.

In order to propose the usefulness of uncertain graph, we explain the difference between fuzzy graph and uncertain graph. The fuzzy graph $\widetilde{G}(V,\tilde{E})$ is a generalization of classical graph based on fuzzy set theory. It contains a fuzzy edge set $\tilde{E}$ with a membership function $\mu_{\tilde{E}}$ which has the value in $[0,1]$. Whether two vertices u and v connected by an edge or not is represented by the membership function $\mu_{\tilde{E}}(uv)$. If $\mu_{\tilde{E}}(uv) = 0$, then the vertices u and v are not connected by an edge. Different from this point of view, the uncertain graph $\breve{G} = (V,E,\xi)$ is a generalization of classical graph based on uncertainty theory. It is a triple consisting of a vertex set V, an edge set E, and an indicator function ξ. The indicator function ξ is a Boolean uncertain variable which is used to indicate the existence of an edge between two vertices with a belief degree in uncertain measure. If two vertices v_i and v_j are connected by an edge, then $\xi(\{v_iv_j\}) = 1$ with the belief degree $\mathcal{M}\{\xi_{ij} = 1\} = \alpha_{ij}$ where $\alpha_{ij} \in [0,1]$. Otherwise, if v_i and v_j are not connected by an edge, then $\xi(\{v_iv_j\}) = 0$ with the belief degree $\mathcal{M}\{\xi_{ij} = 0\} = 1 - \alpha_{ij}$.

As the system becomes more complex, the uncertainty and randomness are required to be considered simultaneously in a system. Chance theory, which was pioneered by Liu [33] via giving the concepts of uncertain random variable and chance measure, can be employed to deal with such complex system. Some basic concepts of chance theory such as chance distribution, expected value, and variance were proposed by Liu [33]. As an important contribution to chance theory, Liu [33] presented an operational law of uncertain random variables. After that, chance theory was developed steadily and applied widely. It follows from the development of chance theory, Liu [32] initialized the concept of uncertain random graph. In an uncertain random graph, some edges exist with degrees in probability measure and others exist with degrees in uncertain measure. In the paper of Liu [32], the connectivity index of an uncertain random graph was investigated. Recently, Zhang et al. [44] considered the Euler tour in an uncertain random graph. They put forward a formula to calculate the Euler index of the uncertain random graph, and discussed some properties of the Euler index. Zhang et al. [45] investigated the matching index of uncertain random graph and gave an algorithm to compute the matching index and perfect matching index for uncertain random graph.

Cycle is one of the basic concepts of graph theory (Akram et al. [7]). A tour in traveling salesman problem is one of the representation of cycle in classical graph. The traveling salesman problem has been constructed in an uncertain environment [36]. Meanwhile, Gao investigated the cycle index [21] of uncertain graph. In order to develop some applications of cycle in uncertain random environment, we need to know whether an uncertain random graph is a cycle or not. What is the truth value that an uncertain random graph being a cycle? In order to answer this problem, we need to investigate the cycle index problem of uncertain random graph. To the best of our knowledge, no prior work has investigated the cycle problem of an uncertain random graph. In view of this fact, this paper focuses on studying the cycle index of an uncertain random graph. In this paper, the concept of the cycle index of an uncertain random graph is firstly proposed. Then a method to calculate the cycle index is presented and discussed within the framework of chance theory. We also present an algorithm to calculate the cycle index.

The remainder of this paper is organized as follows. Section 2 presents some basic concepts and theorems selected from uncertainty theory to chance theory and from classical graph to uncertain random graph. In Section 3, we discuss the results on the cycle index of an uncertain random graph, and give some examples to illustrate how to obtain the cycle index. At the end of the paper, a brief summary is presented.

2. Preliminary

At the beginning of this section, we would like to introduce some basic concepts and results about the uncertainty theory and the chance theory, respectively. The former is a branch of axiomatic mathematics for dealing with belief degrees, while the latter is a mathematical methodology for dealing with complex systems with both uncertainty and randomness. We also recall some fundamental definitions and results of classical graph and uncertain random graph in Subsections 2.3 and 2.4, respectively.

2.1. Uncertainty theory

Uncertainty theory is a new mathematical tool to deal with indeterminacy phenomena. Nowadays, uncertainty theory has been applied to network optimization ([15]), supply chain management ([35]), and supplier selection ([37]). In this subsection, we briefly state some basic concepts and results about uncertainty theory, which are introduced by Liu [29,30,31] and will be used throughout the paper.

Given a nonempty set Γ and let $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. An uncertain measure $\mathcal{M}$: $\mathcal{L} \to [0,1]$, is a set function, which satisfies the following three axioms (Liu [29]):

Axiom I. (**Normality**) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom II. (**Duality**) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom III. (**Subadditivity**) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leqslant \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be an uncertainty space. In order to define the product uncertain measure, Liu [30] gave the following product axiom of uncertain measure.

Axiom IV. (**Product**) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

As a useful tool for describing uncertain quantity, uncertain variable is defined as a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\}$ is an event (Liu [29]). In order to characterize an uncertain variable in practice, the uncertainty distribution Φ: $\Re \to [0,1]$ of an uncertain variable ξ was defined by Liu [29] as follows $\Phi(x) = \mathcal{M}\{\xi \leqslant x\}, \forall x \in \Re$.

In order to measure the size of an uncertain variable ξ, the expected value of ξ was defined by Liu [29] as the following form,

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geqslant x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leqslant x\}\mathrm{d}x$$

provided that at least one of the two integrals was finite. In addition, for independent uncertain variables ξ and η with finite expected values, Liu [31] proved the linearity property of expected value, i.e.,

$$E[a_1\xi + a_2\eta] = a_1E[\xi] + a_2E[\eta]$$

for any real numbers a_1 and a_2. Assumed that uncertain variable ξ has a finite expected value, the variance of ξ was defined by [29] as

$$V[\xi] = E[(\xi - E[\xi])^2].$$

Liu [30] introduced the concept of independent uncertain variables. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are called independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets B_1, B_2,. . ., B_n of real numbers.

When the uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are represented by uncertainty distributions, the operational law was given by Liu [31] as follows. Assume that

$\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots,$$
$$\Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

2.2. Chance theory

In many cases, uncertainty and randomness appear simultaneously in a complex system. In order to describe this phenomenon, Liu [33] first proposed the chance theory, which is a mathematical methodology for modeling complex systems with both uncertainty and randomness. Regarding the theoretical aspect, Ahmadzade et al. [1] studied some properties of uncertain random sequences. As an application of chance theory, Liu [34] proposed the uncertain random programming as a branch of mathematical programming involving uncertain random variables. Besides, chance theory was applied into many fields, such as uncertain random optimization (Gao et al. [19]) and uncertain random system (Gao and Yao [20]).

Assume that $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space, and $(\Omega, \mathcal{A}, \Pr)$ is a probability space. The product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is called a chance space. Each element $\Theta \in \mathcal{L} \times \mathcal{A}$ is called an event in the chance space. The chance measure of event Θ was defined by Liu [33] as

$$\mathrm{Ch}\{\Theta\} =$$
$$\int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq x\} \mathrm{d}x.$$

Based on chance space, an uncertain random variable was introduced by Liu [33] as follows. The uncertain random variable ξ is a function from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B.

In order to describe uncertain random variables in practice, a concept of chance distribution of an uncertain random variable ξ was defined as $\Phi(x) = \mathrm{Ch}\{\xi \leq x\}$ for any real number $x \in \Re$ [33]. As an important contribution to the chance theory, Liu [34] provided the following operation law to determine the chance distribution of uncertain random variable. Assume that $\eta_1, \eta_2, \cdots, \eta_m$ are independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and $\tau_1, \tau_2, \cdots, \tau_n$ are uncertain variables, then the chance distribution of the uncertain random variable $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ can be presented as

$$\Phi(x) =$$
$$\int_{\Re^m} F(x; y_1, y_2, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m),$$

where $F(x; y_1, y_2, \cdots, y_m)$ is the uncertainty distribution of $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$ for any real numbers $y_1, y_2, \cdots, y_m$.

A function is said to be a Boolean function if it is a mapping from $\{0,1\}^n$ to the set $\{0,1\}$. A random variable (with a probability measure) is said to be Boolean if it takes values either 0 or 1 and an uncertain variable (with an uncertain measure) is said to be Boolean if it takes values either 0 or 1.

Theorem 2.1 *(Liu [34]) Assume that $\eta_1, \eta_2, \cdots, \eta_m$ are independent Boolean random variables, i.e.,*

$$\eta_i = \begin{cases} \text{1 with probability measure } a_i \\ \text{0 with probability measure } 1 - a_i \end{cases}$$

for $i = 1, 2, \cdots, m$ and the variables $\tau_1, \tau_2, \cdots, \tau_n$ are independent Boolean uncertain variables, i.e.,

$$\tau_j = \begin{cases} \text{1 with uncertain measure } b_j \\ \text{0 with uncertain measure } 1 - b_j \end{cases}$$

for $j = 1, 2, \cdots, n$. If f is a Boolean function, then

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$$

is a Boolean uncertain random variable such that

$$\mathrm{Ch}\{\xi = 1\} =$$
$$\sum_{(x_1, \cdots, x_m) \in \{0,1\}^m} \left(\prod_{i=1}^m \mu_i(x_i)\right) f^*(x_1, \cdots, x_m),$$

where

$$f^*(x_1, \cdots, x_m) =$$

$$\begin{cases} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \textit{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=0} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \textit{if} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) \ge 0.5, \end{cases}$$

$$\mu_i(x_i) = \begin{cases} a_i, & \textit{if } x_i = 1 \\ 1 - a_i, & \textit{if } x_i = 0 \end{cases} \quad (i = 1, 2, \cdots, m),$$

$$\nu_j(y_j) = \begin{cases} b_j, & \textit{if } y_j = 1 \\ 1 - b_j, & \textit{if } y_j = 0 \end{cases} \quad (j = 1, 2, \cdots, n).$$

2.3. Classical graph

In this subsection, a formal definition of graph and some basic terminologies of graph theory are offered, which are excerpted from Bondy and Murty [12].

Definition 2.2 *(Bondy and Murty [12]) A graph G is an order triple $(V(G), E(G), \psi_G)$ consisting of a nonempty vertex set $V(G)$, a set $E(G)$ of edges, and an incidence function ψ_G that associates with each edge an unordered pair of vertices of G.*

A loop is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints. A simple graph is a graph having no loops or multiple edges. The number of vertices in G is often called the order of G, while the number of edges is called its size. Every graph in this paper is simple graph with finite order. Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$. The adjacency matrix of G is the $n \times n$ matrix

$$\mathrm{D} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix},$$

where

$$d_{ij} = \begin{cases} 1, \text{ if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, \text{ otherwise.} \end{cases}$$

A graph is called disconnected if its vertex set can be partitioned into two subsets, V_1 and V_2, that have no common element, in such a way that there is no edge with one endpoint in V_1 and the other one in V_2. If a graph is not disconnected, then it is called connected. The degree of a vertex v in a graph G, denoted by deg v, is the number of edges that are incident with v. A graph G is called regular if the vertices of G have the same degree. If deg $v = k$ for every vertex v of G, where $0 \le k \le n-1$, then G is k-regular.

A graph G of order n is a cycle if and only if it is connected and 2-regular. This conclusion is very useful in the classical graph theory, which is a necessary and sufficient condition to determine whether a graph is a cycle or not.

2.4. Uncertain random graph

Recently, Liu [32] presented a concept of uncertain random graph. In uncertain random graph, all edges are independent and some edges exist with degrees in probability measure and others exist with degrees in uncertain measure. We next introduce some denotations in uncertain random graph, which are stemmed from Liu [32].

Let $\mathcal{V}$ be a set of n vertices, that is,

$$\mathcal{V} = \{v_1, v_2, \cdots, v_n\}.$$

Let us define two collections of edges:

$$\mathcal{U} = \{(v_i, v_j) \mid (v_i, v_j) \text{ are uncertain edges}\},$$

$$\mathcal{R} = \{(v_i, v_j) \mid (v_i, v_j) \text{ are random edges}\}.$$

for $1 \le i < j \le n$.

Then $\mathcal{V} \cup \mathcal{R} = \{(v_i, v_j) \mid 1 \le i < j \le n\}$. We call the matrix

$$\mathcal{T} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

an uncertain random adjacency matrix if α_{ij} represent the truth values in uncertain measure or probability measure that the edges between vertices v_i and v_j exist, $i, j = 1, 2, \cdots, n$, respectively. Note that $\alpha_{ii} = 0$ and $\alpha_{ij} = \alpha_{ji}$ for $i, j = 1, 2, \cdots, n$, respectively, that is, $\mathcal{T}$ is a symmetric matrix.

Definition 2.3 *(Liu [32]) Assume that $\mathcal{V}$ is the collection of vertices, $\mathcal{U}$ is the collection of uncertain edges, $\mathcal{R}$ is the collection of random edges, and $\mathcal{T}$ is the uncertain random adjacency matrix. Then the quartette $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ is said to be an uncertain random graph.*

From the definition of uncertain random graph, the edge set of uncertain random graph $\mathbb{G}$ is a set of Boolean variable

$$E(\mathbb{G}) = \{\xi_{12}, \cdots, \xi_{1n}, \cdots, \xi_{(n-1)n}\},$$

where $\text{Ch}\{\xi_{ij} = 1\} = \alpha_{ij}$ and $\text{Ch}\{\xi_{ij} = 0\} = 1 - \alpha_{ij}$ for any $1 \le i < j \le n$.

A realization of the uncertain graph $\mathbb{G}$ is a deterministic graph $G(V, E)$ where $V = \mathcal{V}, E \subset \mathcal{U} \cup \mathcal{R}$. Let $\varepsilon = \{(i,j) | 1 \le i < j \le n$ and $0 < \alpha_{ij} < 1\}$ and $|\varepsilon| = m$. It follows from Gao et al. [26] that $\mathbb{G}$ has 2^m realizations. We can call a realization as model.

Example 1. *Consider an uncertain random graph $\mathbb{G}$ in Figure 1.*

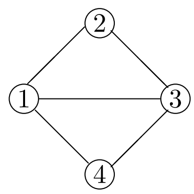

Figure 1. Uncertain random graph $\mathbb{G}$

The edges of uncertain random graph $\mathbb{G}$ is

$$\mathcal{U} = \{(v_1, v_2), (v_1, v_4)\},$$

$$\mathcal{R} = \{(v_2, v_3), (v_1, v_3), (v_3, v_4)\}.$$

The uncertain random graph $\mathbb{G}$ has uncertain random adjacency matrix as follows:

$$\mathcal{T} = \begin{pmatrix} 0 & 0.2 & 1 & 0.6 \\ 0.2 & 0 & 0.5 & 0 \\ 1 & 0.5 & 0 & 0.8 \\ 0.6 & 0 & 0.8 & 0 \end{pmatrix}.$$

The uncertain random graph $\mathbb{G}$ has 2^4 models. Some models of $\mathbb{G}$ are presented in Figure 2.

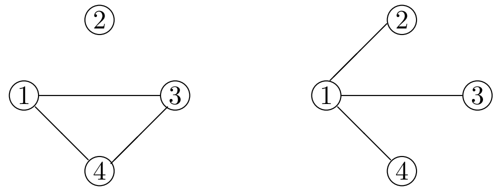

Figure 2. Some realizations of the uncertain random graph $\mathbb{G}$ in Figure 1

Similar to Liu [32], we first assume that

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \tag{1}$$

and

$$\mathbb{X} = \left\{ X \left| \begin{array}{l} x_{ij} = 0 \text{ or } 1, \text{ if } (v_i, v_j) \in \mathcal{R} \\ x_{ij} = 0, \text{ if } (v_i, v_j) \in \mathcal{U} \\ x_{ij} = x_{ji}, \ i, j = 1, 2, \cdots, n \\ x_{ii} = 0, \ i = 1, 2, \cdots, n \end{array} \right. \right\}. \tag{2}$$

Next, given a matrix

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}, \tag{3}$$

we define the extension class of Y as

$$Y^* = \left\{ X \left| \begin{array}{l} x_{ij} = y_{ij}, \text{ if } (v_i, v_j) \in \mathcal{R} \\ x_{ij} = 0 \text{ or } 1, \text{ if } (v_i, v_j) \in \mathcal{U} \\ x_{ij} = x_{ji}, \ i, j = 1, 2, \cdots, n \\ x_{ii} = 0, \ i = 1, 2, \cdots, n \end{array} \right. \right\}. \tag{4}$$

In order to show how likely an uncertain random graph is connected, a connectivity index was defined by Liu [32] as follows.

Definition 2.4 *(Liu [32]) The connectivity index of an uncertain random graph is the chance measure that the uncertain random graph is connected.*

For a given uncertain random graph, Liu [32] proposed a method to obtain the connectivity index of the graph. The main results can be summarized as follows.

Theorem 2.5 *(Liu [32]) Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. If all edges are independent, then the connectivity index of $\mathbb{G}$ is*

$$\rho(\mathbb{G}) = \sum_{Y \in \mathbb{X}} \left(\prod_{(v_i,v_j)\in\mathcal{R}} \nu_{ij}(Y) \right) f^*(Y),$$

where

$$F = \sup_{X\in Y^*,\, f(X)=1} \min_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X),$$

$$f^*(Y) = \begin{cases} \sup\limits_{X\in Y^*,\, f(X)=1} \min\limits_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X), & \text{if } F < 0.5 \\ 1 - \sup\limits_{X\in Y^*,\, f(X)=0} \min\limits_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X), & \text{if } F \geq 0.5, \end{cases}$$

$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1-\alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases} \quad (v_i, v_j) \in \mathcal{U},$$

$$f(X) = \begin{cases} 1, \text{ if } & I + X + X^2 + \cdots + X^{n-1} > 0 \\ 0, \text{ otherwise.} \end{cases}$$

3. Cycle Index of Uncertain Random Graph

The object of this section is to investigate how likely an uncertain random graph being a cycle. For this purpose, we first give the definition of cycle function for an uncertain random graph as follows.

Definition 3.1 *Let $\mathbb{G}$ be an uncertain random graph with edge set $E(\mathbb{G}) = \{\xi_1, \xi_2, \cdots, \xi_m\}$. The cycle function of $\mathbb{G}$ is denoted as:*

$$\mathcal{C}(E(\mathbb{G})) = \begin{cases} 1, \text{ if graph } \mathbb{G} \text{ is cycle} \\ 0, \text{ otherwise.} \end{cases}$$

Obviously, $\mathcal{C}(E(\mathbb{G}))$ is a Boolean function. In order to show how likely an uncertain random graph is cycle, a cycle index is given in Definition 3.2.

Definition 3.2 *A cycle index of an uncertain random graph $\mathbb{G}$ is the chance measure that the uncertain random graph is cycle, i.e.,*

$$\mu(\mathbb{G}) = \text{Ch}\{\mathcal{C}(E(\mathbb{G})) = 1\}.$$

A basic problem for us is how to calculate the cycle index when an uncertain random graph is given. The following theorem can be completely solve this problem for an uncertain random graph.

Theorem 3.3 *Let $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. If all edges are independent, then the cycle index of $\mathbb{G}$ is*

$$\mu(\mathbb{G}) = \sum_{Y \in \mathbb{X}} \left(\prod_{(v_i,v_j)\in\mathcal{R}} \nu_{ij}(Y) \right) \mathcal{C}^*(Y),$$

where

$$H = \sup_{X\in Y^*,\, \mathcal{C}(X)=1} \min_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X),$$

$$\mathcal{C}^*(Y) = \begin{cases} \sup\limits_{X\in Y^*,\, \mathcal{C}(X)=1} \min\limits_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X), & \text{if } H < 0.5 \\ 1 - \sup\limits_{X\in Y^*,\, \mathcal{C}(X)=0} \min\limits_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X), & \text{if } H \geq 0.5, \end{cases}$$

$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1-\alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases} \quad (v_i, v_j) \in \mathcal{U},$$

$$\nu_{ij}(Y) = \begin{cases} \alpha_{ij}, & \text{if } y_{ij} = 1 \\ 1-\alpha_{ij}, & \text{if } y_{ij} = 0 \end{cases} \quad (v_i, v_j) \in \mathcal{R},$$

$$\mathcal{C}(X) = \begin{cases} 1, \text{ if } \begin{cases} I + X + X^2 + \cdots + X^{n-1} > 0 \\ \sum\limits_{j=1}^{n} x_{ij} = 2 \text{ for } i = 1, 2, \cdots, n \end{cases} \\ 0, \text{ otherwise.} \end{cases}$$

Proof. Note that all random edges are independent Boolean random variables, being represented by

$$\eta_{ij} = \begin{cases} 1 \text{ with probability measure } \alpha_{ij} \\ 0 \text{ with probability measure } 1 - \alpha_{ij} \end{cases}$$

for $(v_i, v_j) \in \mathcal{R}$.

Also, all uncertain edges are independent Boolean uncertain variables, being represented by

$$\tau_{ij} = \begin{cases} 1 \text{ with uncertain measure } \alpha_{ij} \\ 0 \text{ with uncertain measure } 1-\alpha_{ij} \end{cases}$$

for $(v_i, v_j) \in \mathcal{U}$.

Let G be a classical graph of order n with adjacency matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix},$$

where $x_{ij} \in \{0,1\}$, and X is a symmetric matrix with $x_{ii} = 0, i = 1,2,\cdots,n$. Then the graph $\mathbb{G}$ is connected if and only if

$$I + X + X^2 + \cdots + X^{n-1} > 0.$$

The degree of a vertex v_i is just the sum of entries in the row corresponding to it in X. As we know, a connected graph is a cycle if and only if the graph is 2-regular. That is,

$$\mathcal{C}(X) = \begin{cases} 1, \text{ if } \begin{cases} I + X + X^2 + \cdots + X^{n-1} > 0 \\ \sum_{j=1}^{n} x_{ij} = 2 \text{ for } i = 1,2,\cdots,n \end{cases} \\ 0, \text{ otherwise.} \end{cases}$$

Obviously, the function $\mathcal{C}(X)$ is a Boolean function. Then it follows from Theorem 2.5 that the theorem is proved. □

Remark 3.4 *If the uncertain random graph $\mathbb{G}$ becomes a random graph, then the cycle index of $\mathbb{G}$ is*

$$\mu(\mathbb{G}) = \sum_{X \in \mathbb{X}} \left(\prod_{1 \le i < j \le n} \nu_{ij}(X) \right) \mathcal{C}(X),$$

where

$$\mathbb{X} = \left\{ X \,\middle|\, \begin{array}{l} x_{ij} = 0 \text{ or } 1,\ i,j = 1,2,\cdots,n \\ x_{ij} = x_{ji},\ i,j = 1,2,\cdots,n \\ x_{ii} = 0,\ i = 1,2,\cdots,n \end{array} \right\}.$$

Remark 3.5 *If the uncertain random graph $\mathbb{G}$ becomes an uncertain graph, then the cycle index of $\mathbb{G}$ is*

$$\mu(\mathbb{G}) = \begin{cases} \sup\limits_{X \in \mathbb{X}, \mathcal{C}(X)=1} \min\limits_{1 \le i < j \le n} \nu_{ij}(X), & \text{if } H < 0.5 \\ 1 - \sup\limits_{X \in \mathbb{X}, \mathcal{C}(X)=0} \min\limits_{1 \le i < j \le n} \nu_{ij}(X), & \text{if } H \ge 0.5, \end{cases}$$

where

$$H = \sup_{X \in Y^*, \mathcal{C}(X)=1} \min_{(v_i,v_j) \in \mathcal{U}} \nu_{ij}(X),$$

$$\mathbb{X} = \left\{ X \,\middle|\, \begin{array}{l} x_{ij} = 0 \text{ or } 1,\ i,j = 1,2,\cdots,n \\ x_{ij} = x_{ji},\ i,j = 1,2,\cdots,n \\ x_{ii} = 0,\ i = 1,2,\cdots,n \end{array} \right\},$$

which is equivalent to the result that presented in Gao [21].

It is quite well known that if a graph is cycle then it must be connected. Thus we have Corollary 3.6.

Corollary 3.6 *Let $\mathbb{G}$ be an uncertain random graph, then the cycle index of $\mathbb{G}$ is less than or equal to the connectivity index of $\mathbb{G}$.*

A property that can be used to determine the cycle index of uncertain graph, which was given by Gao [21], can be extended into uncertain random graph. Hence, we have Lemma 3.7.

Lemma 3.7 *Let $\mathbb{G}$ be an uncertain random graph with the vertex set $V(\mathbb{G}) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E(\mathbb{G}) = \{\xi_1, \xi_2, \cdots, \xi_n\}$. Let $\mathbb{C}$ be a spanning cycle with $V(\mathbb{C}) = \{v_1, v_2, \cdots, v_n\}$ and $E(\mathbb{C}) = \{\xi_{c_1}, \xi_{c_2}, \cdots, \xi_{c_{n-1}}\}$. The cycle index of $\mathbb{G}$ is*

$$\mu(\mathbb{G}) = \sum_{Y \in \mathbb{X}} \left(\prod_{(v_i,v_j) \in \mathcal{R}} \nu_{ij}(Y) \right) \mathbb{C}^*(Y),$$

where

$$\mathbb{C}^*(Y) = \sup_{\mathbb{C}(X)=1} \min_{(v_i,v_j) \in \mathcal{U}} \left\{ \mu(\mathbb{C}), 1 - \max\nolimits_{(v_i,v_j) \in E(\mathbb{G}) \setminus E(\mathbb{C}))} \alpha_{ij} \right\}. \tag{5}$$

Proof. Based on Theorem 3.3, for any given Y, we only need to prove Equation (5) holds. According to the results presented in Gao [21], Equation (5) can be obtained directly. □

Given an uncertain random graph $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$, $\mathcal{U} = \{(v_i, v_j) | (v_i, v_j)$ are uncertain edge$\}$, $\mathcal{R} = \{(v_i, v_j) | (v_i, v_j)$ are random edge$\}$ and an uncertain random adjacency matrix $\mathcal{T} = [\alpha_{ij}]$. According to Lemma 3.7, we summarize a method to calculate the cycle index of an uncertain random graph as presented in Algorithm 1.

Algorithm 1. Algorithm for calculating the cycle index of uncertain random graph

Step 1. Set $Q = \{\emptyset\}, k = 0, \mu' = 0$.
If there exists an adjacency matrix $X_k = [x_{ij}]$ such that $\mathbb{G}$ is a cycle, then go to Step 2. Otherwise, stop, $\mu(\mathbb{G}) = 0$.

Step 2. For each X_k, determine matrix $Y_k = [y_{ij}]$. The element $y_{ij} = 1$ if (v_i, v_j) in $\mathcal{R}$, and otherwise, $y_{ij} = 0$.

Step 3. Determine the value $\nu_{ij}(Y_k)$, where $\nu_{ij}(Y_k) = \alpha_{ij}$ if $y_{ij} = 1$, and $\nu_{ij}(Y_k) = 1 - \alpha_{ij}$ if $y_{ij} = 0$ for all $(v_i, v_j) \in \mathcal{R}$.
Calculate $\delta = \prod_{(v_i,v_j)\in\mathcal{R}} \nu_{ij}(Y_k)$.

Step 4. Calculate $\min\limits_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X_k)$, where $\nu_{ij}(X_k) = \alpha_{ij}$ if $x_{ij} = 1$, and $\nu_{ij}(X_k) = 1 - \alpha_{ij}$ if $x_{ij} = 0$ for $(v_i, v_j) \in \mathcal{U}$.
Set $\mu' = \sup\left\{\mu', \min\limits_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X_k)\right\}$, $Q = Q \cup \{X_k\}$ and $k = k + 1$.

Step 5. If there exists other adjacency matrix $X_k \in \bar{Q}$ such that $\mathbb{G}$ is a cycle, then go to Step 3. Otherwise, stop, and calculate $\mu(\mathbb{G}) = \sum\limits_{X_k\in\mathbb{G}} \delta\mu'$.

Next, we give an example to illustrate the application of the theoretical result and the proposed algorithm.

Example 3.8 A salesman will travel from a place (vertex) 1, visit another places exactly one times, and back to the place 1. The possible route is presented in the uncertain random graph: $\mathbb{G} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ of order 4, where

$$\mathcal{U} = \{(v_2, v_3), (v_2, v_4), (v_3, v_4)\},$$

$$\mathcal{R} = \{(v_1, v_2), (v_1, v_3), (v_1, v_4)\}.$$

The uncertain random adjacency matrix of $\mathbb{G}$ is

$$\mathcal{T} = \begin{pmatrix} 0 & 0.7 & 0.6 & 0.1 \\ 0.7 & 0 & 0.3 & 0.7 \\ 0.6 & 0.3 & 0 & 0.2 \\ 0.1 & 0.7 & 0.2 & 0 \end{pmatrix}.$$

The route needed by the salesman is a cycle of the uncertain random graph $\mathbb{G}$. What is the belief degree that the salesman can find a cycle in the uncertain random graph $\mathbb{G}$? To answer this problem, we need to determine the cycle index of the uncertain random graph $\mathbb{G}$.

In **Step 1**, we determine an adjacency matrix such that $\mathbb{G}$ is a cycle. Since the uncertain random graph $\mathbb{G}$ contains $\varepsilon = \{(i,j) | 1 \le i < j \le n$ and $0 < \alpha_{ij} < 1\}$ and $|\varepsilon| = 6$. It follows from Gao et al. [26] that $\mathbb{G}$ has 2^6 realizations (models). Among 64 models, if a model has one of the following three adjacency matrices

$$X_0 = \begin{pmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0&1&1&0 \\ 1&0&0&1 \\ 1&0&0&1 \\ 0&1&1&0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0&0&1&1 \\ 0&0&1&1 \\ 1&1&0&0 \\ 1&1&0&0 \end{pmatrix},$$

then it is cycle. Otherwise, the model is not a cycle.

In **Step 2**, we determine a matrix Y_k for each X_k.

For $X_0 = \begin{pmatrix} 0&1&0&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&1&0 \end{pmatrix}$, the matrix Y_0 is $\begin{pmatrix} 0&1&0&1 \\ 1&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}$.

In **Step 3**, we determine $\nu_{ij}(Y_k)$ as follows: $\nu_{12}(Y_0) = 0.7, \nu_{13}(Y_0) = 0.4, \nu_{14}(Y_0) = 0.1$. After that, the value $\delta = \prod_{(v_i,v_j)\in\mathcal{R}} \nu_{ij}(Y_1) = 0.7 \times 0.4 \times 0.1 = 0.028$ is obtained.

In **Step 4**, we calculate $\mu' = \min_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X_1) = 0.2$. Since there exists other adjacency matrix such that $\mathbb{G}$ is a cycle, then go back to step 3.

For $X_1 = \begin{pmatrix} 0\,1\,1\,0 \\ 1\,0\,0\,1 \\ 1\,0\,0\,1 \\ 0\,1\,1\,0 \end{pmatrix}$, the matrix Y_1 is $\begin{pmatrix} 0\,1\,1\,0 \\ 1\,0\,0\,0 \\ 1\,0\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix}$, we get

$$\delta = \prod_{(v_i,v_j)\in\mathcal{R}} \nu_{ij}(Y_1) = 0.378, \quad \text{and}$$

$$\mu' = \min_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X_1) = 0.2.$$

For $X_2 = \begin{pmatrix} 0\,0\,1\,1 \\ 0\,0\,1\,1 \\ 1\,1\,0\,0 \\ 1\,1\,0\,0 \end{pmatrix}$, the matrix Y_2 is $\begin{pmatrix} 0\,0\,1\,1 \\ 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 1\,0\,0\,0 \end{pmatrix}$, we obtain

$$\delta = \prod_{(v_i,v_j)\in\mathcal{R}} \nu_{ij}(Y_2) = 0.018, \quad \text{and}$$

$$\mu' = \min_{(v_i,v_j)\in\mathcal{U}} \nu_{ij}(X_2) = 0.3.$$

In **Step 5**, we obtain the cycle index of the uncertain random graph $\mathbb{G}$ as follows:

$$\mu(\mathbb{G}) = 0.028\times0.2+0.378\times0.2+0.018\times0.3 = 0.0866.$$

The result can be interpreted as follows: the belief degree that the salesman can find a cycle in the uncertain random graph $\mathbb{G}$ is 0.0866.

4. Conclusions

This paper mainly studied the cycle problem of an uncertain random graph. Firstly, we proposed a cycle index to show the chance measure that an uncertain random graph being a cycle. Following that, a formula for calculating the cycle index was presented. In addition, some simplified forms of the formula were also discussed, and an algorithm for obtaining the cycle index was designed. Finally, a numerical example was given to illustrate the proposed method.

Further research will focus on the following aspects. First, we can investigate how likely an uncertain random graph is Hamiltonian and discuss the connectedness strength of two vertices for an uncertain random graph. Second, some efficient algorithms can be employed to improve the efficiency in the process of finding the cycle index of an uncertain random graph when the size of the graph becomes larger. Finally, the diameter of uncertain random graph is an uncertain random variable (not a constant number), so studying the distribution function of the diameter of uncertain random graph may be an interesting work in the future.

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