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The symmetric solutions for boundary value problems of second-order singular differential equation

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Abstract—In this paper, by constructing a special operator and using fixed point index theorem of cone, we get the sufficient conditions for symmetric positive solution of a class of nonlinear singular boundary value problems with p-Laplace operator, which improved and generalized the result of related paper.

Keywords—Banach space, cone, fixed point index, singular differential equation, p-Laplace operator, symmetric solutions.

I. Introduction

THE boundary value problems with p-Laplace operator arises in a variety of applied mathematics and physics, and they are widely applied in studying for non-newtonian fluid mechanics, cosmological physics, plasma physics, and theory of elasticity, etc. In recent years, some important results have been obtained by a variety of method(see[1-4]). On the other hand, the study for the symmetric and multiple solutions to this problem is more and more active (see[5-6]). In paper [5], Sun study for the problem

$$\begin{cases} (u)'' + a(t)(t)f(t,u(t)) = 0, t \in (0,1) \\ u(0) = \alpha u(\eta) = u(1), \end{cases}$$

where $\alpha \in (0,1), \eta \in (0, \frac{1}{2}]$, by using spectrum theory, Sun get the existence of symmetric and multiple solution. But when $p \neq 2$, $\phi_p(u)$ is nonlinear, so the method of the paper [5] is not suitable to p-laplace operator. In paper [6], Tian and Liu study for the problem

$$\begin{cases} (\phi_p(u'))' + a(t)(t)f(t,u(t)) = 0, t \in (0,1) \\ u(0) = \alpha u(\eta) = u(1), \end{cases}$$

where $\phi(s)$ is p-Laplace operator. Motivated by paper [5,6], we consider the existence of solution for the following problems:

$$\begin{cases} (\phi_p(u'))' + h_1(t)f(u,v) = 0, \\ (\phi_p(v'))' + h_2(t)g(u) = 0, \\ u(0) = \gamma u(\eta) = u(1), \\ v(0) = \gamma v(\eta) = v(1), \end{cases} \tag{1}$$

where $t \in (0,1), \gamma \in (0,1), \eta \in (0, \frac{1}{2}], \phi(s)$ is a p-Laplace operator, i.e. $\phi_p(s) = |s|^{p-2}s, p > 1$. Obviously, if $\frac{1}{p} + \frac{1}{q} = 1$, then $(\phi_p)^{-1} = \phi_q$.

Compare with above paper, our method is different. By constructing a new operator, and using fixed point index theorem, we get the sufficient condition of the existence of symmetric solution, which improved and generalized the result of paper [5,6,7].

In this paper, we always suppose that the following conditions hold:

(H_1) $f \in C([0,+\infty) \times [0,+\infty), [0,+\infty)), g \in C([0,+\infty), [0,+\infty))$.

(H_2) $h_i \in C((0,1), [0,+\infty)), h_i(t) = h_i(1-t), t \in (0,1)$, for any subinterval of $(0,1)$, $h_i(t) \not\equiv 0$, and $\int_0^1 h_i(t)dt < +\infty (i = 1,2)$.

(H_3) There exists $\alpha \in (0,1]$, such that $\liminf_{u \to +\infty} \frac{g(u)}{u^{\frac{p-1}{\alpha}}} = +\infty$ and $\liminf_{v \to +\infty} \frac{f(u,v)}{v^{(p-1)\alpha}} > 0$ hold uniformly to $u \in R^+$.

(H_4) There exists $\beta \in (0,+\infty)$, such that $\limsup_{u \to 0^+} \frac{g(u)}{u^{\frac{p-1}{\beta}}} = 0$ and $\limsup_{v \to 0^+} \frac{f(u,v)}{v^{(p-1)\beta}} < +\infty$ hold uniformly to $u \in R^+$.

(H_5) There exists $n \in (0,1]$, such that $\liminf_{u \to 0^+} \frac{g(u)}{u^{\frac{p-1}{n}}} = +\infty$ and $\liminf_{v \to 0^+} \frac{f(u,v)}{v^{(p-1)n}} > 0$ hold uniformly to $u \in R^+$.

(H_6) $f(u,v)$ and $g(u)$ are nondecreasing with respect to u and v, and there exists $R > 0$, such that $\frac{\gamma}{1-\gamma} \int_0^{\frac{1}{2}} \phi_q(k_1(s))ds f(R, \frac{\gamma}{1-\gamma} \int_0^{\frac{1}{2}} \phi_q(k_1(s))ds \times g(R)) < R$, where $k_i(s) = \int_s^{\frac{1}{2}} h_i(\tau)d\tau, i = 1,2$.

For convenience, we list the following definitions and lemmas:

Definition 1.1 If $u(t) = u(1-t), t \in [0,1]$, we call $u(t)$ is symmetric in $[0,1]$.

Definition 1.2 If (u,v) is a positive solution of problem (1), and u,v is symmetric in $[0,1]$, we call (u,v) is symmetric positive solution of problem (1).

Definition 1.3 If $u(\lambda t_1 + (1-\lambda)t_2) \geq \lambda u(t_1) + (1-\lambda)u(t_2)$, we call $u(t)$ is concave in $[0,1]$.

Let $E = C[0,1]$, define the norm $||u|| = \max_{t \in [0,1]} |u(t)|$, obviously $(E, ||.||)$ is a Banach space.

Let $K = \{u \in E | u(t) > 0, u(t)$ is a symmetric concave function, $t \in [0,1]\}$, then K is a cone in E. By $(H_1), (H_2)$, the solution of problem (1) is equivalent to the solution of system of equation (2).

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$$\begin{cases} u(t)=\begin{cases} \int_0^t \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds+ \\ \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds, \\ 0\le t\le \frac{1}{2}, \\ \int_t^1 \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds+ \\ \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds, \\ \frac{1}{2}\le t\le 1, \end{cases} \\ v(t)=\begin{cases} \int_0^t \phi_q(\int_{\frac{1}{2}}^s h_2(\tau)g(u(\tau))d\tau)ds+ \\ \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds, \\ 0\le t\le \frac{1}{2}, \\ \int_t^1 \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds+ \\ \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds, \\ \frac{1}{2}\le t\le 1. \end{cases} \end{cases} \tag{2}$$

We define $T:K\to E:$

$$(Tu)(t)=\begin{cases} \int_0^t \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds+ \\ \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds, \\ 0\le t\le \frac{1}{2}, \\ \int_t^1 \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds+ \\ \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds, \\ \frac{1}{2}\le t\le 1, \end{cases} \tag{3}$$

where

$$v(t)=\begin{cases} \int_0^t \phi_q(\int_{\frac{1}{2}}^s h_2(\tau)g(u(\tau))d\tau)ds+ \\ \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds, 0\le t\le \frac{1}{2}, \\ \int_t^1 \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds+ \\ \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds, \frac{1}{2}\le t\le 1. \end{cases} \tag{4}$$

Obviously $Tu\in E$, it is easy to show if T has fixed point u, then by (4), problem (1) has a solution (u,v).

Lemma 1.1 Let $(H_1),(H_2)$, then $T:K\to K$ is completely continuous.

Proof $\forall u\in K$, by $(H_1),(H_2)$, we can get $(Tu)(t)\ge 0, t\in[0,1]$.

$$v^{'}(t)=\begin{cases} \phi_q(\int_t^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau), 0\le t\le \frac{1}{2}, \\ -\phi_q(\int_t^1 h_2(\tau)g(u(\tau))d\tau), \frac{1}{2}\le t\le 1, \end{cases}$$

correspondingly $(\phi_p(v^{'}))^{'}=-h_2(t)g(u)\le 0, 0<t<1$, so v is concave in $[0,1]$.

Next we show v is symmetric in $[0,1]$.

When $t\in[0,\frac{1}{2}], 1-t\in[\frac{1}{2},1]$, so

$$\begin{aligned} v(1-t) &= \int_{1-t}^1 \phi_q(\int_{\frac{1}{2}}^s h_2(\tau)g(u(\tau))d\tau)ds+ \\ &\quad \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds \\ &= \int_0^t \phi_q(\int_{\frac{1}{2}}^s h_2(\tau)g(u(\tau))d\tau)ds+ \\ &\quad \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds \\ &= v(t). \end{aligned}$$

Similarly, we have $v(1-t)=v(t), t\in[\frac{1}{2},1]$. So v is a symmetric concave function in $[0,1]$.

$$(Tu)^{'}(t)=\begin{cases} \phi_q(\int_t^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau), 0\le t\le \frac{1}{2}, \\ -\phi_q(\int_t^1 h_1(\tau)f(u(\tau),v(\tau))d\tau), \frac{1}{2}\le t\le 1, \end{cases}$$

so $(\phi_p((Tu)^{'}))^{'}=-h_1(t)f(u,v)\le 0, 0<t<1$, i.e.$Tu$ is concave in $[0,1]$.

Next we show Tu is symmetric in $[0,1]$. when $t\in[0,\frac{1}{2}], 1-t\in[\frac{1}{2},1]$, so

$$\begin{aligned} (Tu)(1-t) &= \int_{1-t}^1 \phi_q(\int_{\frac{1}{2}}^s h_1(\tau)f(u(\tau),v(\tau))d\tau)ds+ \\ &\quad \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds \\ &= \int_0^t \phi_q(\int_{\frac{1}{2}}^s h_1(\tau)f(u(\tau),v(\tau))d\tau)ds+ \\ &\quad \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)g(u(\tau),v(\tau))d\tau)ds \\ &= (Tu)(t). \end{aligned}$$

Similarly, we have $(Tu)(1-t)=(Tu)(t), t\in[\frac{1}{2},1]$. so Tu is concave in $[0,1]$, so $TK\subset K$. On the other hand, let D is a arbitrary bounded set of K, then there exist constant $c>0$, such that $D\subset\{u\in K|||u||\le c\}$. Let $b=\max\limits_{u\in[o,c]} g(u)$, so $\forall u\in D$, we have

$$\begin{aligned} ||v|| &= |\int_0^{\frac{1}{2}} \phi_q(\int_{\frac{1}{2}}^s h_2(\tau)g(u(\tau))d\tau)ds+ \\ &\quad \frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds| \\ &\le \frac{b^{q-1}}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)d\tau)ds=a. \end{aligned}$$

Let $L=\max\limits_{u\in[o,c],v\in[0,a]} f(u,v)$, so $\forall u\in D$, we have

$$\begin{aligned} ||Tu|| &= |\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds \\ &\quad +\frac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds,| \\ &\le \frac{L^{q-1}}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds. \end{aligned}$$

$$\begin{aligned}\|(Tu)'\| &= max\{|\phi_q(\int_0^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)|,\\ &|\phi_q(\int_{\frac{1}{2}}^1 h_1(\tau)f(u(\tau),v(\tau))d\tau)|\}\\ &\leq L^{q-1}\phi_q(\int_0^{\frac{1}{2}} h_1(\tau)d\tau).\end{aligned}$$

By Arzela-Ascoli theorem, we know TD is compact set. By Lebesgue dominated convergence theorem, it is easy to show T is continuous in K, so $T : K \to K$ is completely continuous.

Lemma 1.2 For any $0 < \varepsilon < \frac{1}{2}, u \in K$, we have
(1) $u(t) \geq \|u\|t(1-t)$, $\forall t \in [0,1]$;
(2) $u(t) \geq \epsilon^2\|u\|$, $t \in [\epsilon, 1-\epsilon]$. (the proof is elementary, we omit it.)

Lemma 1.3(see [8]) Let K is a cone of E in Banach space, Ω_1 and Ω_2 are open subsets in E, $\theta \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and $T : K\bigcap(\overline{\Omega_2}\backslash\Omega_1) \to K$ is a completely continuous operator, and satisfy one of the following conditions:

(1)$\|Tx\| \leq \|x\|, \forall x \in K\bigcap\partial\Omega_1, \|Tx\| \geq x, \forall x \in K\bigcap\partial\Omega_2$,

(2)$\|Tx\| \geq \|x\|, \forall x \in K\bigcap\partial\Omega_1, \|Tx\| \leq x, \forall x \in K\bigcap\partial\Omega_2$,

then A has at least one fixed point in $K\bigcap(\Omega_2\backslash\Omega_1)$.

Lemma 1.4(see [9]) Let K is a cone of E in Banach space, $K_r = \{x \in K| \parallel x \parallel\leq r\}$, suppose $A : K_r \to K$ is a completely continuous, and satisfy $Tx \neq x, \forall x \in \partial K_r$,

(1) If $\|Tx\| \leq x, \forall x \in \partial K_r$, then $i(T, K_r, K) = 1$,

(2) If $\|Tx\| \geq x, \forall x \in \partial K_r$, then $i(T, K_r, K) = 0$.

II. Conclusion

Theorem 2.1 Suppose $(H_1) - (H_4)$ hold, then problem (1) has at least one positive solution.

Proof By (H_3), there exist ν and a sufficient large number $M > 0$, such that

$$f(u,v) \geq \nu^{p-1}v^{(p-1)\alpha}, \forall u \in R^+, v > M, \tag{5}$$

$$g(u) \geq C_0^{p-1}u^{\frac{p-1}{\alpha}}, \forall u > M, \tag{6}$$

where $C_0 = max\{(\frac{\gamma}{1-\gamma}\int_\epsilon^\eta \phi_q(k_2(s))ds)^{-1}$, $(\frac{2}{\nu\gamma^\alpha\epsilon^2(\frac{1}{1-\gamma}\int_0^\eta \phi_q(k_1(s))^{\alpha+1}})^{\frac{1}{\alpha}}\}$. Let $N = (M+1)\epsilon^{-2}$, if $u \in K\bigcap\partial K_N$, by Lemma 2, $\min\limits_{\epsilon\leq t\leq 1-\epsilon} u(t) \geq \epsilon^2||u|| = \epsilon^2 N = M+1$, by (3)-(6) and the symmetric property, for any $t \in [\epsilon, 1-\epsilon]$

$$\begin{aligned}v(t) &= \int_0^t \phi_q(\int_{\frac{1}{2}}^s h_2(\tau)g(u(\tau))d\tau)ds+\\ &\tfrac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds\\ &\geq \tfrac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds\\ &\geq \tfrac{\gamma}{1-\gamma}\int_\epsilon^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds\\ &\geq \tfrac{C_0\gamma}{1-\gamma}\int_\epsilon^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)(u(\tau)^{\frac{p-1}{\alpha}})d\tau)ds\\ &\geq \tfrac{C_0\gamma}{1-\gamma}\int_\epsilon^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau))d\tau)ds(\epsilon^2||u||)^{\frac{1}{\alpha}}\\ &\geq \tfrac{C_0\gamma}{1-\gamma}\int_\epsilon^\eta \phi_q(\int_s^{\frac{1}{2}} h_2(\tau))d\tau)ds(M+1)^{\frac{1}{\alpha}}\\ &\geq M+1.\end{aligned}$$

$$\begin{aligned}||Tu|| &= |\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds+\\ &\tfrac{\gamma}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds,|\\ &\geq \tfrac{1}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau),v(\tau))d\tau)ds,\\ &\geq \tfrac{\nu}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)v(\tau)^{(p-1)\alpha}d\tau)ds,\\ &\geq \tfrac{\nu}{1-\gamma}\int_\epsilon^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds\times\\ &\quad(\tfrac{C_0\gamma}{1-\gamma}\int_\epsilon^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds)^\alpha\epsilon^2||u||\\ &= \nu C_0^\alpha\gamma^\alpha\epsilon^2(\tfrac{1}{1-\gamma}\int_0^\eta \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds)^{\alpha+1}||u||\\ &\geq 2||u||,\end{aligned}$$

so $||Tu|| > ||u||, \forall \in K\bigcap K_N$, by lemma 1.4, we can get

$$i(T, K\bigcap K_N, K) = 0. \tag{7}$$

On the other hand, by the second limit of H_4, there exists a sufficient small number $r_1 \in (0,1)$ such that

$$C_1^{p-1} = \sup\{\frac{f(u,v)}{v^{(p-1)\beta}}|u \in R^+, v \in (0, r_1]\} < +\infty. \tag{8}$$

Let $\varepsilon = min\{\frac{r_1(1-\gamma)}{\int_0^{\frac{1}{2}}\phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds}$, $(\frac{C_1}{1-\gamma}\int_0^{\frac{1}{2}}\phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds)^{\frac{-\beta-1}{\beta}}\}$, by the first limit of H_4, there exist a sufficient small number $r_2 \in (0,1)$ such that

$$g(u) \leq \varepsilon^{p-1}u^{\frac{p-1}{\beta}}, \forall u \in [0, r_2]. \tag{9}$$

Take $r = min\{r_1, r_2\}$, by (9), we can get

$$\begin{aligned} v(t) &= \int_0^{\frac{1}{2}} \phi_q(\int_{\frac{1}{2}}^{s} h_2(\tau)g(u(\tau))d\tau)ds + \\ &\tfrac{\gamma}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds \\ &\leq \tfrac{1}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds \\ &\leq \tfrac{\varepsilon}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)d\tau)ds||u||^{\frac{1}{\beta}} \\ &\leq r_1^{1+\frac{1}{\beta}} < r_1, \forall u \in K \bigcap \partial K_r, s \in [0,1]. \end{aligned}$$

By (8), we can get

$$\begin{aligned} ||Tu|| &\leq | \int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau), v(\tau))d\tau)ds + \\ &\tfrac{\gamma}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau), v(\tau))d\tau)ds, | \\ &\leq \tfrac{C_1}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds \times \\ &\quad (\tfrac{\varepsilon}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds)^{\beta}||u|| \\ &= C_1\varepsilon^{\beta}(\tfrac{1}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds)^{\beta+1}||u|| \\ &\leq ||u||, \forall u \in K \bigcap \partial K_r, t \in [0,1]. \end{aligned}$$

So $||Tu|| \leq ||u||, \forall u \in K \bigcap \partial K_r$, by lemma 1.4, we get

$$i(T, K \bigcap K_r, K) = 1. \tag{10}$$

By lemma 1.5, T has at least one fixed point in $K \bigcap (\overline{K_N} \backslash K_r)$, so problem (1) has at least a system positive solution.

Theorem 2.2 Suppose $(H_1), (H_2), (H_3), (H_5), (H_6)$ hold, then problem (1) has at least two systems positive solutions.

Proof By (H_5), there exists $\mu > 0$ and a sufficient small number $\xi \in (0, 1)$, such that

$$f(u, v) \geq \mu^{p-1}v^{n(p-1)}, \forall u \in R^+, 0 \leq v \leq \xi, \tag{11}$$

$$g(u) \geq (C_2 u)^{\frac{p-1}{n}}, \forall 0 \leq u \leq \xi, \tag{12}$$

where

$$C_2 = 2(\tfrac{\mu\epsilon^2}{1-\gamma}(\tfrac{\gamma}{1-\gamma})^n \int_{\epsilon}^{\eta} \phi_q(k_1(s))ds \int_{\epsilon}^{\eta} (\phi_q(k_2(s)))^n ds)^{-1}$$

since $g \in C(R^+, R^+)$, $g(0) \equiv 0$, so there exists $\sigma \in (0, \xi)$ such that $\forall u \in [0, \sigma]$, we have

$$g(u) \leq (\frac{1}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds)^{-1},$$

this imply

$$\begin{aligned} v(t) &\leq \tfrac{1}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds \\ &\leq \xi, \forall u \in K \bigcap \partial K_{\sigma}. \end{aligned} \tag{13}$$

By using Jensen inequality, $0 < q \leq 1$, and (11)-(13), we can get

$$\begin{aligned} (Tu)(\tfrac{1}{2}) &\geq \tfrac{\mu}{1-\gamma}\int_{\epsilon}^{\eta} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds \times \\ &(\tfrac{\gamma}{1-\gamma}\int_{\epsilon}^{\eta} \phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)ds)^n \\ &\geq \tfrac{\mu}{1-\gamma}\int_{\epsilon}^{\eta} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds \times \\ &(\tfrac{\gamma}{1-\gamma})^n \int_{\epsilon}^{\eta} (\phi_q(\int_s^{\frac{1}{2}} h_2(\tau)g(u(\tau))d\tau)^n ds \\ &\geq \tfrac{\mu C_2 \epsilon^2}{1-\gamma}(\tfrac{\gamma}{1-\gamma})^n \int_{\epsilon}^{\eta} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds \times \\ &\int_{\epsilon}^{\eta} (\phi_q(\int_s^{\frac{1}{2}} h_2(\tau)d\tau))^n ds||u|| \\ &= 2||u||, \forall u \in K \bigcap \partial K_{\sigma}. \end{aligned}$$

So $||Tu|| > ||u||, \forall u \in K \bigcap \partial K_{\sigma}$, by lemma 1.4, we can get

$$i(T, K \bigcap K_{\sigma}, K) = 0. \tag{14}$$

We can choose $N > R > \sigma$, such that (7),(14) hold together. On the other hand by (3),(4) and H_6 we can get

$$\begin{aligned} (Tu)(t) &< \tfrac{1}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)f(u(\tau), v(\tau))d\tau)ds \\ &\leq \tfrac{\gamma}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds \times \\ &f(R, \tfrac{\gamma}{1-\gamma}\int_0^{\frac{1}{2}} \phi_q(\int_s^{\frac{1}{2}} h_1(\tau)d\tau)ds g(R)) \\ &< R, \forall u \in K \bigcap K_R, \forall t \in [0,1]. \end{aligned}$$

So for any $u \in K \bigcap K_R$, by lemma 1.4, we can get

$$i(T, K \bigcap K_R, K) = 1. \tag{15}$$

By (7),(14),(15), we have

$$\begin{aligned} &i(T, K \bigcap (K_N \setminus \overline{K_R}), K) \\ &= i(T, K \bigcap K_N, K) - i(T, K \bigcap K_R, K) \\ &= -1. \end{aligned}$$

$$\begin{aligned} &i(T, K \bigcap (K_R \setminus \overline{K_{\sigma}}), K) \\ &= i(T, K \bigcap K_R, K) - i(T, K \bigcap K_{\sigma}, K) \\ &= 1 \end{aligned}$$

So T have at least two fixed points in $K \bigcap (K_N \setminus \overline{K_R}$ and $K \bigcap (K_R \setminus \overline{K_{\sigma}}$, by (4), problem (1) has at least two system solutions.

References

[1] Su H, Wei Z L, Wang B H. The existence of positive solutions four a nonlinear four-point singular boundary value problem with a p-Laplace operator. Nonlinear anal,2007,66:2204-2217.

[2] Ma D x, Han J X,Chen X G. Positive solution of boundary value problem for one-dimensional p-Laplacian with singularities.J Math Anal Appl,2006,324:118-133.

[3] Liu Y J, Ge W G.Multiple positive solutions to a three-point boundary value problems with p-Laplacian. J Math Anal Appl,2003,277:293-302

[4] Jin J X, Yin C H.Positive solutions for the boundary value problems of one-dimensional p-Laplacian with delay. J Math Anal Appl, 2007, 330:1238-1248.

[5] Sun Y P. Optimal existence criteria for symmetric positive solutions a three-point boundary differential equations. Nonlinear anal,2007,66:1051-1063
[6] Tian Yuansheng Liu Chungen. The existence of symmetric positive solutions for a three-point singular boundary value problem with a p-Laplace operator.Acta Mathematica scientia 2010,30A(3):784-792.
[7] Xie Shengli. Positive solutions of multiple-point boundary value problems for systems of nonlinear second order differential equations .Acta Mathematica scientia 2010,30(A):258-266.
[8] Guo Dajun Nonlinear functional analysis. Jinan. Shandong science and technology publishing house,2001 .
[9] Guo D,Lakshmikantham V. Nonlinear Problems in Abstract Cones.New YorkAcademic Press,1988