

# “Perpetuum Mobile” as an Effective Example to Illustrate Instability of some Numerical Integration Methods

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## 1 Introduction

In this document, instability of the forward Euler integration rule for numeric solution of ordinary differential equations is illustrated by designing a “perpetuum mobile” which is going to be verified by simulation. The analysis is focused to the parallel resonant circuit shown in Fig. 1, which is a stable circuit. However, with some help of forward Euler, discrete model of the circuit is going to be brought to the verge of instability, and output of a constant amplitude will be generated, as verified by simulation.

## 2 Continuous Time Model

To construct an exact, continuous time model of the circuit of Fig. 1, sparse tableau model is going to be constructed first, being reduced to the state space model by algebraic manipulations next, and finally reduced to a second order ordinary differential equation over the node voltage.

To form the tableau model, there is one equation over the Kirchhoff's current law

$$i_1 + i_2 + i_3 = 0 \quad (1)$$

three equations over the Kirchhoff's voltage law

$$u_1 - v_1 = 0 \quad (2)$$

$$u_2 - v_2 = 0 \quad (3)$$

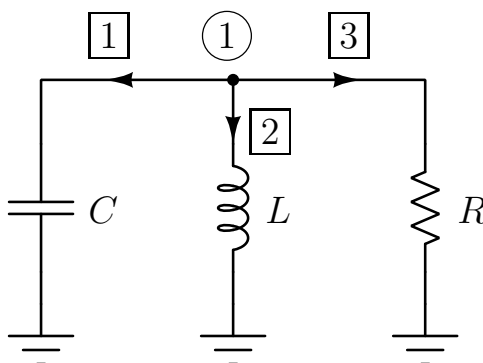


Figure 1: The circuit.

$$u_3 - v_3 = 0 \quad (4)$$

and three equations over constitutive relations of the network elements

$$i_1 - C \frac{d u_1}{dt} = 0 \quad (5)$$

$$L \frac{d i_2}{dt} - u_2 = 0 \quad (6)$$

$$R i_3 - u_3 = 0. \quad (7)$$

Reducing the system to state variables,  $u_1$  and  $i_2$ , i.e. algebraically eliminating all the variables except the state variables, it reduces to

$$\frac{d u_1}{dt} = -\frac{u_1}{RC} - \frac{i_2}{C} \quad (8)$$

and

$$\frac{d i_2}{dt} = \frac{u_1}{L}. \quad (9)$$

The system of state equations could be reduced to a second order equation over one state variable, which expressed in terms of the node voltage  $v_1 = u_1$  becomes

$$\frac{d^2 v_1}{dt^2} + \frac{1}{RC} \frac{d v_1}{dt} + \frac{1}{LC} v_1 = 0. \quad (10)$$

with the initial conditions

$$v_1(0) = u_1(0) \quad (11)$$

and

$$\dot{v}_1(0) = \left. \frac{d v_1}{dt} \right|_{t=0} = -\frac{u_1(0)}{RC} - \frac{i_2(0)}{C}. \quad (12)$$

Characteristic polynomial of the considered second order ordinary differential equation is

$$s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0 \quad (13)$$

which for

$$R < \frac{1}{2} \sqrt{\frac{L}{C}} \quad (14)$$

has real roots

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\frac{1}{(2RC)^2} - \frac{1}{LC}}. \quad (15)$$

In the opposite case, the roots are complex conjugate pair

$$s_{1,2} = -\frac{1}{2RC} \pm j \sqrt{\frac{1}{LC} - \frac{1}{(2RC)^2}} \quad (16)$$

Analysis of the root locations (15) and (16) results in conclusion that for  $R > 0$  the circuit of Fig. 1 is stable.

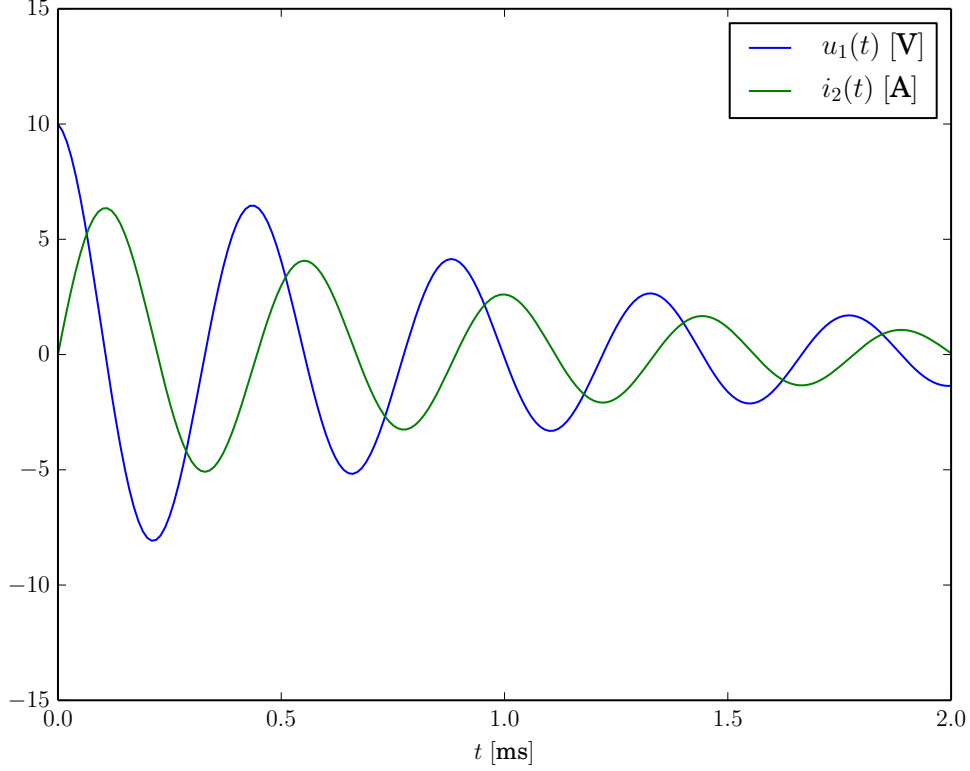


Figure 2: State variables, exact solution.

As an example, that our analysis would return to later, consider a circuit of Fig. 1 for  $R = 10 \Omega$ ,  $L = 100 \mu\text{H}$ , and  $C = 50 \mu\text{F}$ . For initial conditions  $u_1(0) = 10 \text{ V}$ ,  $i_2(0) = 0$ , the circuit response for the state variables is

$$u_1(t) = e^{-1000 \frac{1}{s} t} \left( 10 \cos \left( 1000 \sqrt{199} \frac{\text{rad}}{\text{s}} t \right) - \frac{10}{\sqrt{199}} \sin \left( 1000 \sqrt{199} \frac{\text{rad}}{\text{s}} t \right) \right) \text{ V} \quad (17)$$

and

$$i_2(t) = \frac{100}{\sqrt{199}} e^{-1000 \frac{1}{s} t} \sin \left( 1000 \sqrt{199} \frac{\text{rad}}{\text{s}} t \right) \text{ A} \quad (18)$$

which is plotted in Fig. 2. Diagrams of instantaneous power at the circuit elements are given in Fig. 3, and as expected they add up to zero.

### 3 Discrete Time Model

In order to perform numerical simulation of the circuit of Fig. 1, constitutive equation of the capacitor (5) is discretized according to the forward Euler discretization rule,

$$u_{1,n+1} = u_{1,n} + \frac{\Delta t}{C} i_{C,n}. \quad (19)$$

To simplify notation in the equations that follow, let us introduce the discretization resistance

$$R_C \triangleq \frac{\Delta t}{C}. \quad (20)$$

It should be noted that (19) is constitutive equation of an ideal voltage source, since the element voltage at the time instant indexed  $n + 1$  is completely specified from the circuit variables at the previous time instant,  $n$ .

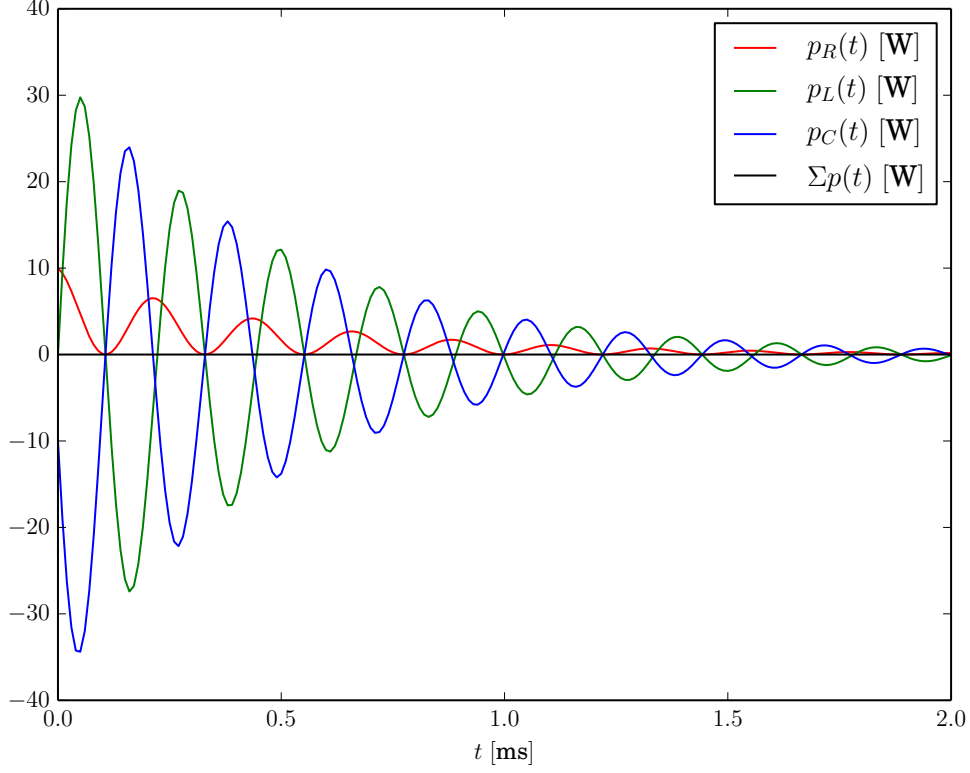


Figure 3: Power, exact solution.

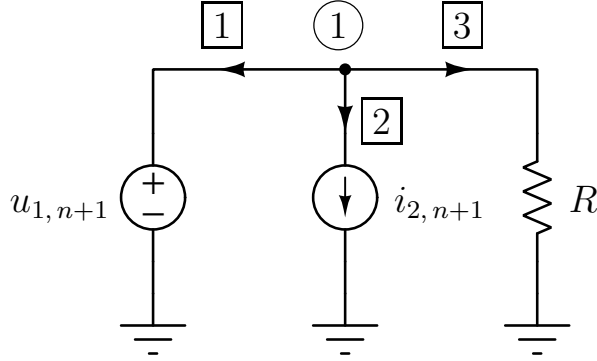


Figure 4: Discretized circuit.

In the same manner as for the capacitor, the inductor is discretized applying the forward Euler method as

$$i_{2,n+1} = i_{2,n} + \frac{\Delta t}{L} u_{L,n}. \quad (21)$$

To simplify notation, let us introduce the discretization conductance

$$G_L \triangleq \frac{\Delta t}{L}. \quad (22)$$

Equation (21) is constitutive equation of a current source, since the branch current at  $n + 1$  is specified by circuit variables at  $n$ .

Discretization of the reactive elements results in the discretized circuit model depicted in Fig. 4, in which the capacitor is represented by a voltage source specified by (19), and the inductor is represented by a current source specified by (21).

System of equations that describe the circuit of Fig. 4 is formed using the Modified Nodal

Analysis (MNA) technique, starting from

$$i_{1,n+1} + i_{2,n+1} + i_{3,n+1} = 0. \quad (23)$$

Except for  $i_1$ , the currents can be expressed in terms of the node voltage  $v_1$ , which provides the nodal equation

$$i_{1,n+1} + \frac{v_{1,n+1}}{R} = -i_{2,n} - G_L v_{1,n} \quad (24)$$

and an additional equation for the element whose current cannot be expressed in terms of the node voltage

$$v_{1,n+1} = v_{1,n} + R_C i_{1,n}. \quad (25)$$

In (24) and (25), the left hand sides contain nodal variables (nodal voltage(s) and added current(s)) at the time point indexed  $n + 1$ . To iterate the solution during the simulation of transients, it is convenient to have the same variables on the right hand side, but referred to the time point indexed  $n$ . In this goal,  $i_2$  is expressed as

$$i_{2,n} = -i_{1,n} - \frac{1}{R} v_{1,n}. \quad (26)$$

Finally, the system is reduced to MNA variables  $i_1$  and  $v_1$  as

$$i_{1,n+1} + \frac{1}{R} v_{1,n+1} = i_{1,n} + \left( \frac{1}{R} - G_L \right) v_{1,n} \quad (27)$$

and

$$v_{1,n+1} = R_C i_{1,n} + v_{1,n} \quad (28)$$

which is in matrix form given by

$$\begin{bmatrix} 1 & 1/R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ v_1 \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & 1/R - G_L \\ R_C & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ v_1 \end{bmatrix}_n. \quad (29)$$

The system (29) can be solved over the vector of MNA variables, resulting in

$$\begin{bmatrix} i_1 \\ v_1 \end{bmatrix}_{n+1} = \begin{bmatrix} 1 - R_C/R & -G_L \\ R_C & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ v_1 \end{bmatrix}_n \quad (30)$$

providing a closed-form solution

$$\begin{bmatrix} i_1 \\ v_1 \end{bmatrix}_k = \begin{bmatrix} 1 - R_C/R & -G_L \\ R_C & 1 \end{bmatrix}^k \begin{bmatrix} i_1 \\ v_1 \end{bmatrix}_0. \quad (31)$$

It should be noted that the solution of (31) is over MNA variables,  $v_1$  and  $i_1$ . In the case the state variables are needed,  $u_1$  and  $i_2$ , they can be computed as

$$u_{1,k} = v_{1,k} \quad (32)$$

and

$$i_{2,k} = -i_{1,k} - \frac{1}{R} v_{1,k}. \quad (33)$$

The MNA method is applied to analyze the discretized circuit since the same method is used in SPICE.

## 4 “Perpetuum Mobile”

Solution of the discrete system model (31) involves power of the matrix

$$M \triangleq \begin{bmatrix} 1 - R_C/R & -G_L \\ R_C & 1 \end{bmatrix} \quad (34)$$

which has eigenvalues

$$\lambda_{1,2} = 1 - \frac{R_C}{2R} \pm \sqrt{\left(\frac{R_C}{2R}\right)^2 - G_L R_C}. \quad (35)$$

Our goal is to provide a periodic response, thus the eigenvalues should be a complex conjugate pair. To meet this requirement

$$\left(\frac{R_C}{2R}\right)^2 < G_L R_C \quad (36)$$

should be satisfied, which reduces to

$$\frac{1}{2} \sqrt{\frac{R_C}{G_L}} < R \quad (37)$$

or by substituting (20) and (22) to

$$\frac{1}{2} \sqrt{\frac{L}{C}} < R. \quad (38)$$

In this case, the eigenvalues are

$$\lambda_{1,2} = 1 - \frac{R_C}{2R} \pm j \sqrt{G_L R_C - \left(\frac{R_C}{2R}\right)^2} \quad (39)$$

with the real part

$$r \triangleq \Re(\lambda_{1,2}) = 1 - \frac{R_C}{2R} \quad (40)$$

and the imaginary part

$$i \triangleq \Im(\lambda_1) = -\Im(\lambda_2) = \sqrt{G_L R_C - \left(\frac{R_C}{2R}\right)^2}. \quad (41)$$

To provide a periodic response, according to diagonalization of (34) the eigenvalues should satisfy

$$|\lambda_1| = |\lambda_2| = 1 \quad (42)$$

which reduces to

$$r^2 + i^2 = 1. \quad (43)$$

Substituting (40) and (41) results in

$$\left(1 - \frac{R_C}{2R}\right)^2 + G_L R_C - \left(\frac{R_C}{2R}\right)^2 = 1 \quad (44)$$

which can be simplified to the periodic response condition

$$R G_L = 1. \quad (45)$$

According to (22), the time step of forward Euler discretization that provides periodic response is specified by

$$\Delta t = \frac{L}{R}. \quad (46)$$

This choice for  $\Delta t$  results in

$$\lambda_{1,2} = e^{\pm j\varphi} \quad (47)$$

where

$$\tan \varphi = \frac{i}{r} = \frac{\sqrt{4 \frac{RC}{\Delta t} - 1}}{2 \frac{RC}{\Delta t} - 1}. \quad (48)$$

According to the definition of  $\varphi$  given by (47), the number of data points  $n$  per period  $T$  provided by the numerical simulation is

$$n \triangleq \frac{T}{\Delta t} = \frac{2\pi}{\varphi} \quad (49)$$

which is not necessarily a whole number. Dependence of  $n$  on  $\frac{RC}{\Delta t}$  is depicted in Fig. 5.

## 5 “Perpetuum Mobile” Design Guidelines

Let us assume that input data for our design are resistance of the load,  $R$ , the power we want to extract  $P_R$ , the time step  $\Delta t$  in which the data points are produced, and the number of data points per period  $n$ . The goal of this section is to provide closed form expressions for required inductance  $L$ , capacitance  $C$ , and the initial conditions  $u_1(0)$  and  $i_2(0)$ .

First, according to (46), the inductance is given by

$$L = R \Delta t. \quad (50)$$

To simplify the notation, let us introduce an auxiliary variable  $a$  as

$$a \triangleq \tan \varphi = \tan \frac{2\pi}{n}. \quad (51)$$

Solving (48) in terms of  $a$  over  $\frac{RC}{\Delta t}$  provides the expression for the capacitance

$$C = \frac{\Delta t}{R} \frac{\sqrt{a^2 + 1} + a^2 + 1}{2a^2}. \quad (52)$$

Since the power on  $R$  is

$$P_R = \frac{V_0^2}{2R} \quad (53)$$

required initial condition for the capacitor voltage to provide

$$u_{R,k} = V_0 \cos(k\varphi) \quad (54)$$

is

$$u_1(0) = V_0 = \sqrt{2RP_R} \quad (55)$$

while the initial current of the inductor is

$$i_2(0) = -\frac{V_0}{R} = -\sqrt{\frac{2P_R}{R}}. \quad (56)$$

The things left to do to design and verify the “perpetuum mobile” are to plug in figures and to run the simulation.

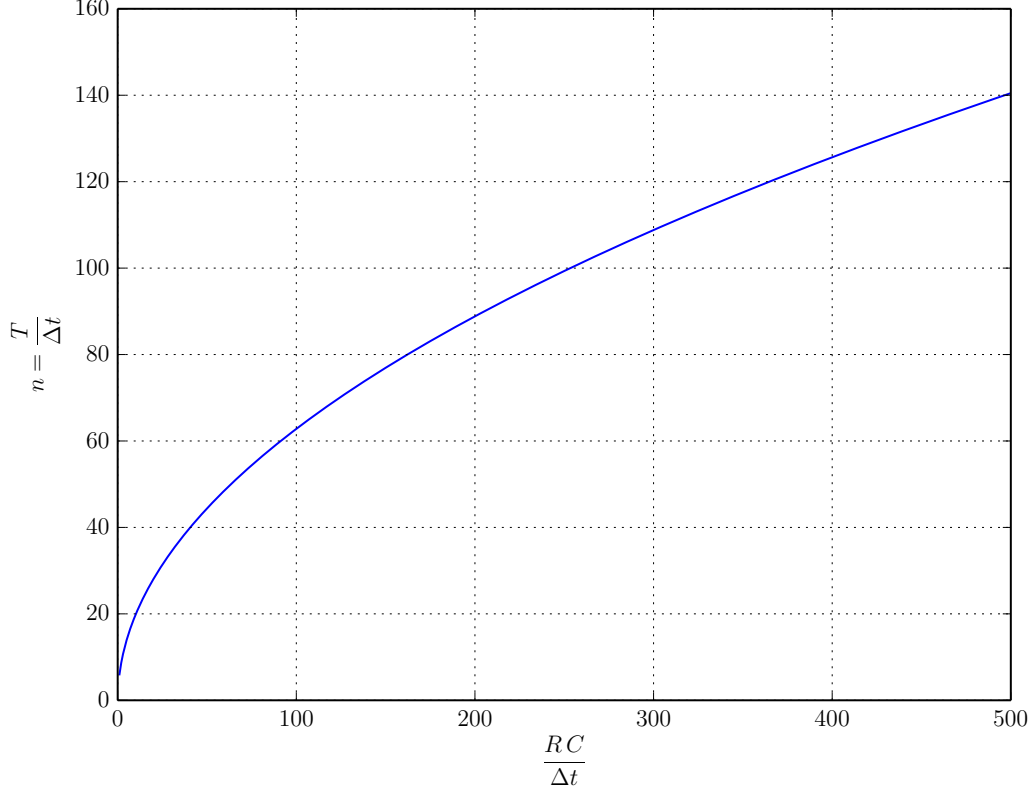


Figure 5: Period number.

## 6 Simulation Results

To illustrate proposed “perpetuum mobile” design technique, which unfortunately works only in simulation, the analytical example of Fig. 1 with the response given by (17) and (18) is simulated applying (31) with  $\Delta t = 10 \mu s$ , resulting in  $n = 44.39$ , using the same initial conditions,  $u_{1,0} = 10 \text{ V}$  and  $i_{2,0} = -1 \text{ A}$ . The simulation results for the state variables are presented in Fig. 6, where dotted lines represent the analytical solution of Fig. 2. In contrast to the exact solution, the simulation result does not vanish and preserves a constant amplitude. Corresponding diagrams of power are presented in Fig. 7. The simulation results provides average power flow to the load of

$$P_R = 5.025 \text{ W} \quad (57)$$

which is supported by the power provided by the capacitor model

$$P_C = -2.512 \text{ W} \quad (58)$$

and the inductor model

$$P_L = -2.512 \text{ W}. \quad (59)$$

This “justifies” the “perpetuum mobile” by simulation.

## 7 Conclusions

In this document, instability of the forward Euler integration rule is utilized to construct a perpetual motion machine, which is verified by simulation, and unfortunately, only by simulation. The example is used to illustrate that fundamentally wrong conclusions could be deduced if simulation methods are applied without thorough understanding.



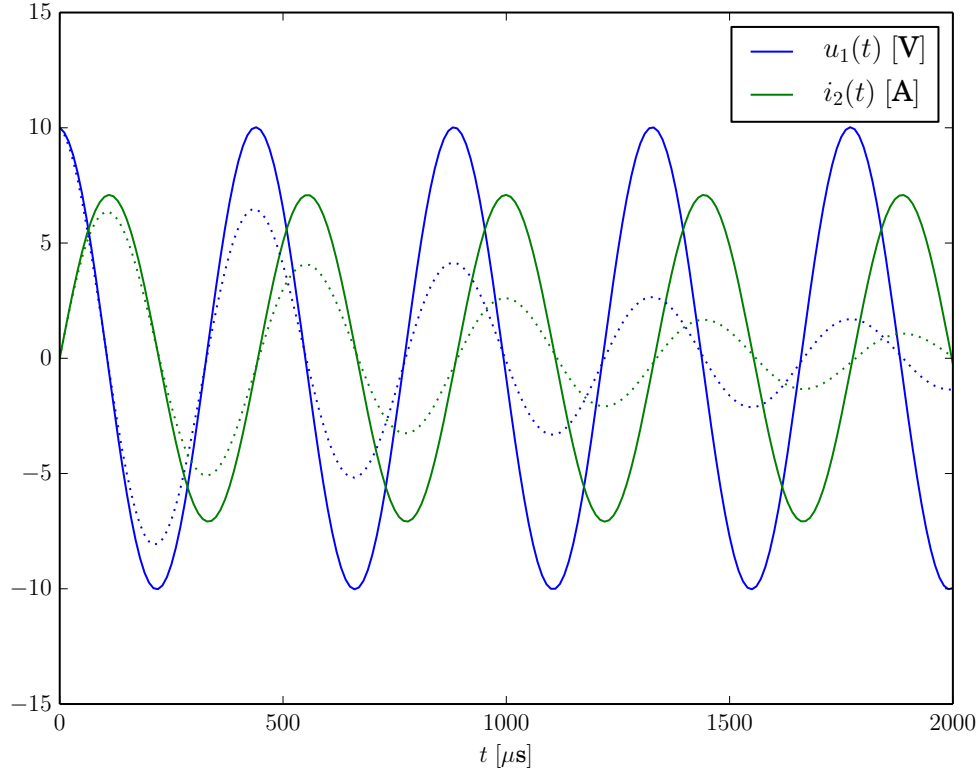


Figure 6: State variables: solid line — numerical solution; dotted line — exact solution.

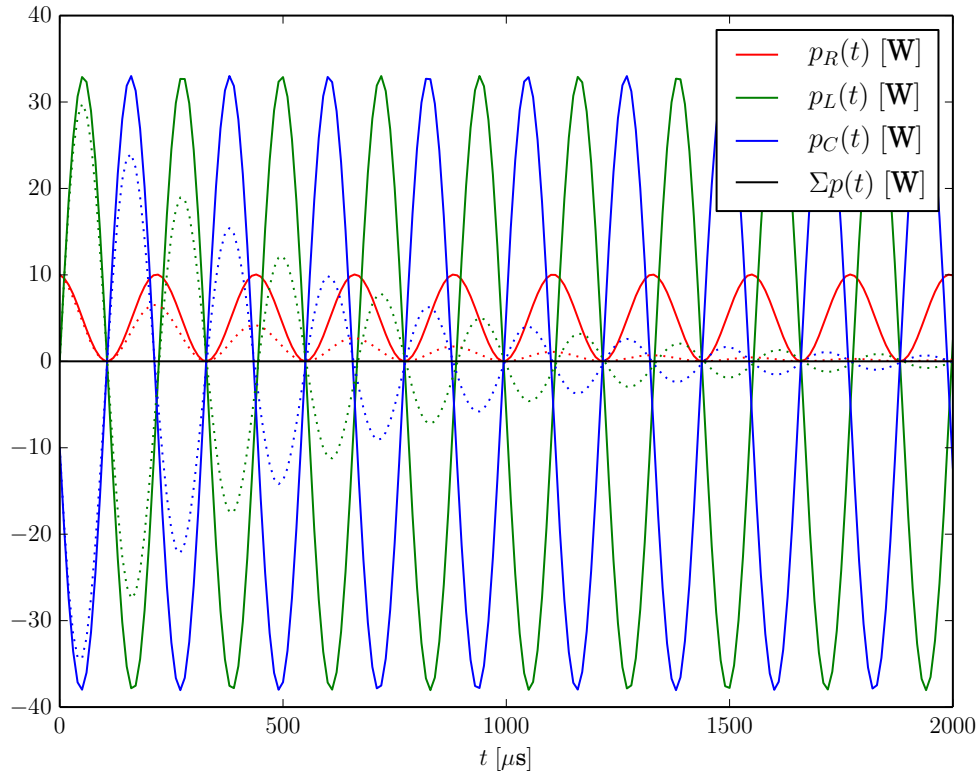


Figure 7: Power: solid line — numerical solution; dotted line — exact solution.