

## ON ACCURACY OF INVARIANT NUMERICAL SCHEMES

MARX CHHAY

LEPTIAB, Université de La Rochelle  
Avenue Michel Crépeau, 17000 La Rochelle, France

AZIZ HAMDOUNI

LEPTIAB, Université de La Rochelle  
Avenue Michel Crépeau, 17000 La Rochelle, France

(Communicated by the associate editor name)

**ABSTRACT.** In this paper we present a method of construction for invariant numerical schemes for partial differential equations. The resulting schemes preserve the Lie-symmetry group of the continuous equation and they are at least as accurate as the original scheme. The improvement of the numerical properties thanks to the Lie-symmetry preservation is illustrated on the Burgers' equation.

**1. Introduction.** Geometric integration is a relatively recent area that has occurred in the development of numerical methods. It consists in preserving some of the geometric properties of the equations (conservations laws, variational structures, symplecticity, symmetry, and so on). The geometric integrators for variational systems of ordinary differential equations (ODE) perform stupendous numerical properties [4] [25] [38]. Symplectic methods overcome classical methods in the description of Hamiltonian systems over a long time integration [26] [24] [6] [45] [44]. The exact conservation of the mechanical energy can be obtained by time adaptative methods (including asynchronous time steps methods) [21] [46] [29] [30] [8] [33] [34]. However the transposition to infinite dimensional systems [22] [23], namely to partial differential equations (PDE), encounters some limitations. For example, the canonical multisymplectic approach [2] [40] has a limited area of applications. Indeed there exists only few PDE that can be recast in the multi-Hamiltonian form. The variational integrators, preserving the energy and the momentum of Lagrangian PDE [36] [37] [39] thanks to time and space steps adaptation [10] [11] [9], present difficulties of implementation. For any PDE, admitting no variational structure, numerical methods preserving Lie-symmetry are among the most promising strategy. Pioneer study has been made by Yanenko and Shokin [48] [47] [27]. Then other geometric integrators have emerged. Budd *et al.* proposes a mesh adaptative method based on the preservation of the scale transformation [5] [3]. In the work of Dorodnitsyn and his coworkers, it is suggested to construct a scheme that preserves the Lie group of an equation from its set of discrete differential invariants [1] [15] [16] [17] [18]. More recently, Fels and Olver have developed a modern theory of moving frames

---

2000 *Mathematics Subject Classification.* Primary: 58J70, 35A30; Secondary: 65D18.

*Key words and phrases.* Invariant methods, Lie symmetry, order of accuracy, moving frames, parametrized scheme, geometric integration.

[19] [20]. Thanks to this concept, it is possible to confer the symmetry invariance to a classical scheme [41]. It can be done by the process of normalization due to É. Cartan [7]. The resulting invariant scheme may inherit robust behaviour, as it has been implemented by Kim [31] [32] and compared with other integrators by Dawes [14]. However one of the major limitation of the process of normalization in the construction of invariant schemes is that there is no constructive criterion to pick up relevant moving frames. Indeed the resulting invariant scheme can even be less accurate than the original scheme. It means that the preservation the Lie-symmetry group of the continuous equation by this way can imply the loss of fundamental properties.

The aim of this paper is to propose an algorithmic process for the construction of invariant numerical schemes that allows an explicit control on the order of accuracy of the resulting scheme. In Section 2, some mathematical preliminaries are introduced. The high numerical potential of invariant schemes is illustrated with the convection-diffusion equation. It is also noted that the construction of invariant scheme by normalization may yield to average, or even weak scheme. We propose a method of parametrization of the scheme to overcome this lack. This constitutes the main result of this Note. It is presented in Section 3. Illustration with the Burgers' equation is performed in Section 4. The theoretical order of accuracy is confirmed numerically. The gain of invariance property is pointed out with the capture of self-similar solutions, and the respect of the galilean transformation. We conclude by suggesting further developments for the method.

## 2. Mathematical preliminaries.

**2.1. Lie-symmetry basic definitions.** Consider a  $(p + q)$  dimensional manifold  $M = X \times \Psi$  whose elements are noted in local coordinates  $z = (x, u)$ , where the independent variables are  $x = (x^1, \dots, x^p) \in X$  and the dependent variables are  $u = (u^1, \dots, u^q) \in \Psi$ . The derivatives of  $u$  of order  $k$  with respect to  $x$  are noted by  $u^{(k)}$ . A partial differential equation (PDE) can be cast into the general formulation:

$$\mathcal{F}(x, u^{(k)}) = 0 \quad (1)$$

It can be seen as a function over the jet space of order  $k$ ,  $M^{(n)} = X \times \Psi^{(k)} \rightarrow \mathbb{R}$ . This space is the space whose coordinates are the independent variables, the dependent variables and the successive derivatives of the dependent variables up to the order  $k$ .

**Definition 2.1** (Symmetry group). A Lie-symmetry group for a PDE (1) is a group of transformation  $G$  depending continuously on the real parameter  $\varepsilon$  and acting on  $X \times \Psi$  such that, if  $u = f(x)$  is a solution of (1) and  $g_\varepsilon \in G$ , then  $g_\varepsilon \cdot f(x)$  is also a solution of (1).

**Example 1.** Let's consider a transport equation described by the convection-diffusion equation:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad (2)$$

The independent variables  $t$  and  $x$  respectively represent the variable of time and the variable of space. The dependent variable  $u$  represents the scalar field which is convected at the velocity  $c$ .

The symmetry group is composed by the one-parameter subgroups:

- Spatial translation:  
 $\mathbf{G}_1 : (x, t, u) \mapsto (x + \varepsilon_1, t, u)$
- Time translation:  
 $\mathbf{G}_2 : (x, t, u) \mapsto (x, t + \varepsilon_2, u)$
- Scale transformation:  
 $\mathbf{G}_3 : (x, t, u) \mapsto (x, t, ue^{\varepsilon_3})$
- Scale transformation:  
 $\mathbf{G}_4 : (x, t, u) \mapsto (xe^{2\nu\varepsilon_4}, te^{4\nu\varepsilon_4}, ue^{-\frac{c}{2\nu}(x(1-e^{2\nu\varepsilon_4})-\frac{ct}{2}(1-e^{4\nu\varepsilon_4}))})$
- Galilean boost:  
 $\mathbf{G}_5 : (x, t, u) \mapsto (x - 2t\nu\varepsilon_5, t, ue^{\varepsilon_5((x-ct)-t\nu\varepsilon_5)})$
- Projection:  
 $\mathbf{G}_6 : (x, t, u) \mapsto (\frac{x}{1-4t\nu\varepsilon_6}, \frac{t}{1-4t\nu\varepsilon_6}, u\sqrt{1-4t\nu\varepsilon_6}e^{-\frac{\varepsilon_6(x-ct)^2}{1-4t\nu\varepsilon_6}})$

The real parameters  $\varepsilon_i$ ,  $i = 1, \dots, 6$  are the symmetry parameters of each transformation.

It is usually simpler to determine whether or not a group is a symmetry group for a PDE by using the infinitesimal generators. Suppose  $\mathbf{V}$  is an infinitesimal generator of a connected local group of transformation  $G$  acting on an open subset of  $X \times \Psi$ , that is,  $\mathbf{V}$  is a vector field defined on an open subset of  $X \times \Psi$ :

$$\mathbf{V} = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{j=1}^q \eta^j(x, u) \frac{\partial}{\partial u^j} \quad (3)$$

Then  $G$  is a symmetry group for a nondegenerate PDE (1) if and only if for every infinitesimal generator  $\mathbf{V}$ :

$$\mathcal{F}(x, u^{(k)}) = 0 \Rightarrow \mathbf{V}^{(k)} \cdot \mathcal{F}(x, u^{(k)}) = 0 \quad (4)$$

where  $\mathbf{V}^{(k)}$  is the prolongation of order  $k$  of the vector field.

The reader may look at classical references to Lie-group theory for deeper explanations and applications. See, for example [43] [28] [35] [42].

**Example 2.** The symmetry group of the equation (2) has 6 finite dimensional subgroups given by the following infinitesimal generators:

$$\mathbf{V}_1 = \frac{\partial}{\partial x} \quad (5)$$

$$\mathbf{V}_2 = \frac{\partial}{\partial t} \quad (6)$$

$$\mathbf{V}_3 = u \frac{\partial}{\partial u} \quad (7)$$

$$\mathbf{V}_4 = 2x\nu \frac{\partial}{\partial x} + 4t\nu \frac{\partial}{\partial t} + uc(x-ct) \frac{\partial}{\partial u} \quad (8)$$

$$\mathbf{V}_5 = -2t\nu \frac{\partial}{\partial x} + u(x-ct) \frac{\partial}{\partial u} \quad (9)$$

$$\mathbf{V}_6 = -4xt\nu \frac{\partial}{\partial x} - 4t^2\nu \frac{\partial}{\partial t} + u((x-ct)^2 + 2t\nu) \frac{\partial}{\partial u} \quad (10)$$

The symmetry group is also composed of an infinite dimensional subgroup generated by:

$$\mathbf{V}_\infty = \alpha(x, t) \frac{\partial}{\partial u} \quad (11)$$

where  $\alpha$  is a solution of (2).

**2.2. Numerical schemes preserving Lie symmetry.** Taking  $M$  as the manifold whose elements are the dependent and the independant variables, its joint product  $M^{\diamond n}$  is defined to be the cartesian product  $M^{\times n}$  of  $M$  without the  $n$ -tuples of identical points:

$$M^{\diamond n} = \{\mathbf{z} = (z_1, \dots, z_n) \mid z_i \in M, z_i \neq z_j \forall i \neq j\} \quad (12)$$

A numerical scheme for a PDE  $\mathcal{F} = 0$  can be seen as a couple of functions  $(N, \phi)$  defined over the joint product such that  $(N, \phi)$  verify some condition of *coalescent limit* over a set of *exact* points: a set of  $n$  points  $\mathbf{z} = (z_1, \dots, z_n) \in M^{\diamond n}$  is exact for  $\mathcal{F} = 0$  if an exact solution of the equation whose graph contains  $z_1, \dots, z_n$  exists [32]. Then a numerical scheme for an equation  $\mathcal{F} = 0$  can formally be defined as a couple of numerical applications  $(N, \phi)$  which verifies:

$$\lim_{S \ni \mathbf{z} \rightarrow \mathbf{z}^*} N(\mathbf{z}) = 0 \quad (13)$$

where  $S$  is the set of all exact points for  $\mathcal{F} = 0$  in the stencil defined by  $\phi$ :

$$S \subset \{\mathbf{z} = (z_1, \dots, z_n) \in M^{\diamond n} \mid \phi(\mathbf{z}) = 0\} \quad (14)$$

and  $\mathbf{z}^* \in \{(z, \dots, z) \mid z \in M\} \subset M^{\times n}$ .

The application  $N$  represents the discrete equation (the finite differences equation), and  $\phi$  determines the equations of the stencil.

**Example 3.** Let's consider on a regular and orthogonal mesh the Forward in Time and Centered in Space scheme (FTCS) for the convection-diffusion equation (2):

$$N(\mathbf{z}) = \frac{u_j^{n+1} - u_j^n}{\Delta t} + c \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \quad (15)$$

where  $\Delta t$  and  $\Delta x$  are the time and the space step sizes. The mesh equations are given by  $\phi$ :

$$\begin{cases} \phi_1(\mathbf{z}) = x_j^{n+1} - x_j^n & (16a) \\ \phi_2(\mathbf{z}) = x_{j+1}^n - 2x_j^n + x_{j-1}^n & (16b) \\ \phi_3(\mathbf{z}) = t_{j+1}^n - t_j^n & (16c) \\ \phi_4(\mathbf{z}) = t_j^{n+1} - 2t_j^n + t_j^{n-1} & (16d) \end{cases}$$

The associated basic stencil is composed with  $\mathbf{z} = (z_j^{n+1}, z_j^n, z_{j\pm 1}^n)$ , where  $z_j^n = (x_j, t^n, u_j^n)$ .

Now let us consider a partial differential equation (PDE)  $\mathcal{F} = 0$ , its symmetry group  $G$  and a numerical scheme  $(N, \phi)$  for the PDE. The action of  $G$  can be naturally prolonged to the joint product as a product action, for  $g \in G$  and  $\mathbf{z} \in M^{\diamond n}$ .

$$g \cdot \mathbf{z} = (g \cdot z_1; \dots; g \cdot z_n) \quad (17)$$

**Definition 2.2** (Symmetric and invariant scheme [13]). With the notations introduced above, we say that  $(N, \phi)$  is a  $G$ -symmetric numerical scheme if,  $\forall g \in G$  :

$$N(g \cdot \mathbf{z}) = 0 \Leftrightarrow N(\mathbf{z}) = 0, \quad \phi(g \cdot \mathbf{z}) = 0 \Leftrightarrow \phi(\mathbf{z}) = 0, \quad (18)$$

A stronger property for numerical scheme is the  $G$ -invariance, which can be defined as  $\forall g \in G$  :

$$N(g \cdot \mathbf{z}) = N(\mathbf{z}), \quad \phi(g \cdot \mathbf{z}) = \phi(\mathbf{z}), \quad (19)$$

**Example 4.** The FTCS scheme (15), over a regular and orthogonal grid, is invariant under space and time translations  $\mathbf{G}_1$  and  $\mathbf{G}_2$ , it is also symmetric under scale transformation  $\mathbf{G}_3$ , but it breaks the projection symmetry  $\mathbf{G}_6$ .

In next sections a method of construction of invariant schemes is presented. It is based on the the concept of moving frames:

**Definition 2.3.** Let  $G$  be a Lie group acting on a manifold  $M$ . A (right) moving frame on  $M$  to  $G$  is a map  $\rho : M \mapsto G$  such that

$$\rho(g \cdot z) = \rho(z) g^{-1} \quad \forall g \in G \quad (20)$$

**Theorem 2.4.** A moving frame exists if and only if the action of the group  $G$  is regular and free.

The proof of this statement is in [41].

Moving frames allow to construct invariant counterpart of any numerical application as numerical schemes defined below. A straightforward computation using the equivariance property of  $\rho$  yields to the following proposition:

**Proposition 1.** Let  $(N, \phi)$  be a numerical scheme for  $\mathcal{F} = 0$ ,  $G$  a symmetry group, and  $\rho : M^{\infty} \rightarrow G$  a moving frame. Then the scheme  $(\bar{N}, \bar{\phi})$  defined as

$$N(\rho(\mathbf{z}) \cdot \mathbf{z}) = \bar{N}(\mathbf{z}) \quad (21)$$

$$\phi(\rho(\mathbf{z}) \cdot \mathbf{z}) = \bar{\phi}(\mathbf{z}) \quad (22)$$

is an invariant numerical scheme for  $\mathcal{F} = 0$ .

This fundamental result says that it is enough to construct equivariant moving frame to get invariance.

**2.3. Invariantization by normalization.** The construction of moving frames can be computed with the Cartan's method of normalization [7]: to each choice of a cross-section to the group orbit, moving frames are determined thanks to the implicit function theorem. This means that the choice of a moving frame can be reduced to the choice of a cross-section. We give an illustration on how the invariantization of numerical schemes by normalization can be realized, and we point out the limits of this process. We consider the convection diffusion equation (2), its symmetry group, and the FTCS scheme (15) over a regular and orthonogonal grid (16), as they are introduced above. To our purpose we focus our interest with the following transformation:

$$\bar{x} = \frac{(x + \varepsilon_1)}{1 - 4\nu\varepsilon_6(t + \varepsilon_2)} \quad (23)$$

$$\bar{t} = \frac{t + \varepsilon_2}{1 - 4\nu\varepsilon_6(t + \varepsilon_2)} \quad (24)$$

$$\bar{u} = u \sqrt{1 - 4\nu\varepsilon_6(t + \varepsilon_2)} e^{-\varepsilon_6 \frac{((x + \varepsilon_1) - c(t + \varepsilon_2))^2}{1 - 4\nu\varepsilon_6(t + \varepsilon_2)}} \quad (25)$$

It takes into account the translations, the galilean boost and the projection through the parameters  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_6$ . The transformation over the discrete points are:

$$\bar{x}_j^n = \frac{x_j + \varepsilon_1}{1 - 4(t^n + \varepsilon_2)\nu\varepsilon_6} \quad (26)$$

$$\bar{t}_j^n = \frac{t^n + \varepsilon_2}{1 - 4(t^n + \varepsilon_2)\nu\varepsilon_6} \quad (27)$$

$$\bar{u}_j^n = u_j^n \sqrt{1 - 4\nu(t^n + \varepsilon_2)\varepsilon_6} e^{-\frac{\varepsilon_6((x_j + \varepsilon_1) - c(t^n + \varepsilon_2))^2}{1 - 4(t^n + \varepsilon_2)\nu\varepsilon_6}} \quad (28)$$

According to the process of *normalization* of Cartan, the choice of three cross sections yields to the expression of  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\varepsilon_6$  thanks to the implicit function theorem. The translation parameters are choosen in order to conserve this basic stencil as simple as it can be. A simple choice for cross-sections is

$$\bar{x}_j^n = 0, \quad \bar{t}^n = 0, \quad (29)$$

This gives the moving frames associated to the spatial and time translations according to (26) and (27):

$$\varepsilon_1 = -x_j, \quad \varepsilon_2 = -t^n, \quad (30)$$

In order to avoid numerical wiggles degrading the solution, it may be relevant to minimize a discrete version of the curvature:

$$\frac{\bar{u}_{j+1}^n - 2\bar{u}_j^n + \bar{u}_{j-1}^n}{\Delta \bar{x}^2} = 0 \quad (31)$$

This cross section gives the expression of the moving frame, defined for  $u^n \neq 0$ :

$$\varepsilon_6 = \frac{1}{\Delta x^2} \ln \left( \frac{u_{j-1}^n + u_{j+1}^n}{2u_j^n} \right) \quad (32)$$

The invariant (for the considered transformations) scheme has the expression:

$$\begin{aligned} u_j^{n+1} = & \frac{1}{(1 - 4\nu\varepsilon_6\Delta t)^{\frac{3}{2}}} \left( (1 - 4\nu\varepsilon_6\Delta t)u_j^n - \frac{c\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)e^{-\varepsilon_6\Delta x^2} \right. \\ & \left. + \frac{\nu\Delta t}{\Delta x^2}((u_{j+1}^n + u_{j-1}^n)e^{-\varepsilon_6\Delta x^2} - 2u_j^n) \right) \end{aligned} \quad (33)$$

with  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\varepsilon_6$  as below.

**Remark 1** (Determination of the cross-section by normalization process). The choice of this cross section is made by an anticipation of the numerical solution for a given problem. In the present case, it is expected for the classical scheme to produce spurious oscillations. An *a priori* knowledge of the behaviour of the solution is requierd. This also constitutes a part of what the method lacks to be entirely systematic.

**2.3.1. Numerical applications.** Let's consider the convection-diffusion equation (2) on an unbounded domain  $\Omega$ . The initial condition is given by:

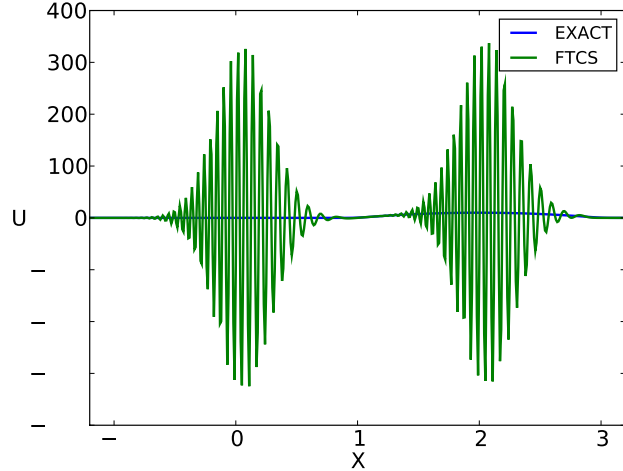
$$u_0(x) = \begin{cases} -10x^2 + 10 + \text{Cste} & \text{if } |x| \leq 1, \\ \text{Cste} & \text{otherwise} \end{cases} \quad (34)$$

The behaviour of the numerical solutions are illustrated in Fig.(1) out of the stability region of the classical FTCS scheme. As expected, the classical scheme (Fig.(1-a)) presents a completely degraded solution, while the invariant counterpart stays stable and accurate.

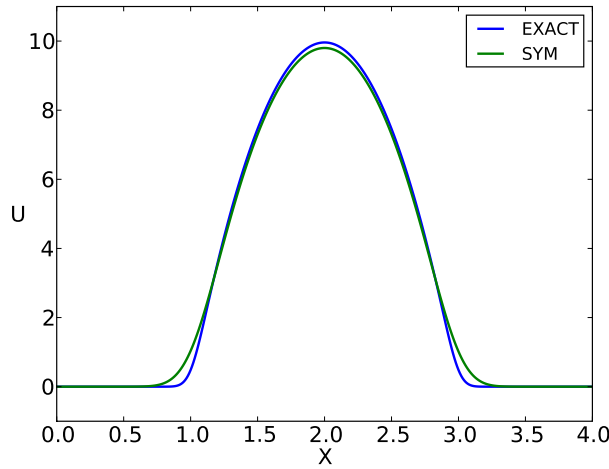
When the viscosity tends to zero, a standard process is to discretize the convective term with a backward Euler method to get the upwind scheme ( $c > 0$ ):

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) + \frac{\nu\Delta t}{\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \quad (35)$$

This scheme is well-known to be able to represent the solution beyond the stability region of the FTCS scheme. The counterpart is an additional numerical dissipation. Fig.(2) represents the behaviour of the classical upwind scheme and the invariant FTCS scheme. As the mesh Reynolds number  $Re_h = |u|\Delta x/\nu$  grows from 200 to



a) Classical FTCS scheme



b) Invariant FTCS scheme

FIGURE 1. *Convection-diffusion*.  $\nu = 10^{-3}$ ,  $c = 1$ ,  $\Delta x = 10^{-2}$  and CFL = 0.4 at time  $t = 2$ .

1000, the numerical dissipation increases for the upwind scheme. This effect does not emerge for the invariant scheme. It confirms the gain of sturdiness and accuracy that a scheme may get when it becomes invariant.

**2.4. Problem position.** Up to the present, the construction of moving frames is made according to the Cartan's method of normalization [7]. This algorithm reduces the determination of a moving frame to the choice of a cross-section to the group orbit. But as an infinity of submanifold transverse to the group orbit could exist, one could have such many choices for moving frames. Therefore the choice for an invariant scheme is very large and their numerical behaviour are not equal

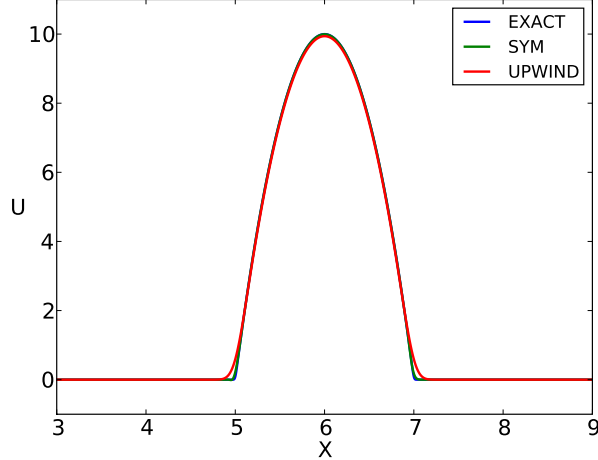
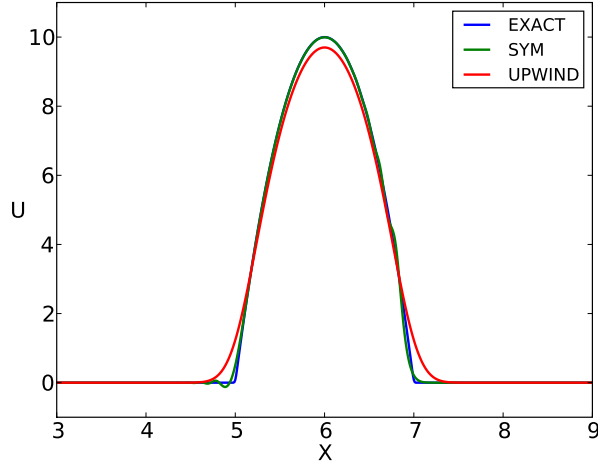
a)  $Re_h = 200$ b)  $Re_h = 1000$ 

FIGURE 2. *Convection-diffusion*. Upwind scheme and invariant FTCS scheme (SYM).  $\nu = 10^{-5}$ ,  $c = 1$  and  $CFL = 1/2$  at  $t = 7$ .

at all [32]. The authors have tested different choices of cross-section related to the symmetry parameter  $\varepsilon_6$  in the previous example. The results have shown a large variety of numerical behaviour. Some of the invariant schemes have performed very bad results. A simple explanation is that the process of invariantization does not take into account numerical properties. The moving frame constructed by geometrical arguments on the cross-section could give rise to an invariant scheme with low order of accuracy for example.



Our aim is to propose a method of construction of invariant numerical scheme using moving frames that assures the order of accuracy, so that the invariant scheme transports some of the fundamental geometric structure of the continuous equation and possesses good numerical properties.

**3. Scheme Parametrization Algorithm.** The algorithm we propose is not based on the normalization process. It consists in parametrizing the classical scheme following the symmetry transformation. Then we suppose an algebraic expression for the symmetry parameters. This expression depends on constant real coefficients. On one hand, we compute the equivariance equations with respect to the constant coefficients in order to get the invariance of the transformed scheme. On the other hand, we compute the consistency order of the transformed scheme with respect to the real constant coefficients to insure the accuracy order of the scheme. It can be formalized as follows:

1. Start with a numerical scheme:

$$\begin{aligned} N : U \subset M^{\diamond n} &\longrightarrow \mathbb{R} \\ \mathbf{z} &\mapsto N(\mathbf{z}) \end{aligned} \quad (36)$$

where  $U$  represents the stencil of the discrete equation

$$U = \{\mathbf{z} \in M^{\diamond n} \mid \phi(\mathbf{z}) = 0\} \quad (37)$$

2. From the prolonged action of the  $r$ -dimensional symmetry group  $G$  on  $M^{\diamond n}$ :

$$g \cdot \mathbf{z} = \bar{\mathbf{z}}(\varepsilon_1, \dots, \varepsilon_r, \mathbf{z}) \quad (38)$$

where  $\varepsilon_1, \dots, \varepsilon_r$  are the symmetry parameters, write down the expression of the transformed scheme:

$$N(g \cdot \mathbf{z}) = \tilde{N}(\varepsilon_1, \dots, \varepsilon_r, \mathbf{z}) = 0 \quad (39)$$

where the discrete points  $\mathbf{z}$  are in the transformed stencil  $\bar{U} = \{\mathbf{z} \in M^{\diamond n} \mid \phi(g \cdot \mathbf{z}) = \bar{\phi}(\varepsilon_1, \dots, \varepsilon_r, \mathbf{z}) = 0\}$

3. Then for each symmetry parameters, suppose an algebraic form that depends on constant real coefficients, that is for  $i = 1, \dots, r$ :

$$\varepsilon_i = f_i(a_i^{1_i}, \dots, a_i^{m_i}, \mathbf{z}) \quad (40)$$

The transformed discrete equation has an expression that depends on those real constant coefficients:

$$\bar{N}(f_1(a_1^{1_1}, \dots, a_1^{m_1}), \dots, f_r(a_r^{1_r}, \dots, a_r^{m_r}), \mathbf{z}) = 0 \quad (41)$$

for  $\mathbf{z} \in \bar{U}$ .

4. For each symmetry parameter, compute the equivariance relation:

$$\rho(\mathbf{z}) \cdot \mathbf{z} = \rho(\bar{\mathbf{z}}) \cdot \bar{\mathbf{z}} \quad (42)$$

5. Compute conditions over the constant coefficients from the order of accuracy of the scheme:

$$\bar{N}(f_1(a_1^{1_1}, \dots, a_1^{m_1}), \dots, f_r(a_r^{1_r}, \dots, a_r^{m_r}), \mathbf{z}) = O(\Delta x_1^{d_1}, \dots, \Delta x_p^{d_p}) \quad (43)$$

This process gives a computational way for determining relevant moving frames. The main difference with the invariantization using normalization is that our process does not involve a cross-section. It directly deals with moving frames. This can be summarized by the algorithm described in Fig.(3).

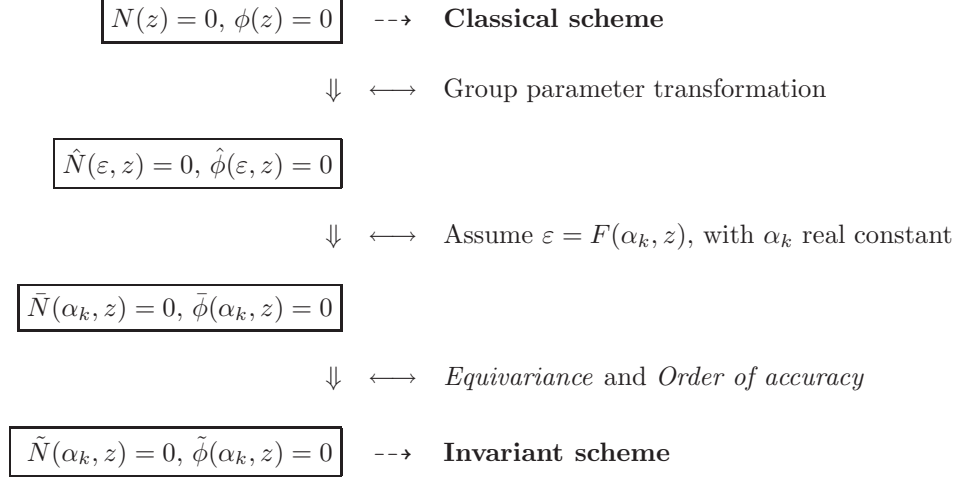


FIGURE 3. The scheme parametrization algorithm to control the order of accuracy of the invariant scheme.

**4. Numerical illustration: the Burgers equation.** Let's consider the Burgers equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad (44)$$

The symmetry group of the equation (44) has 5 finite dimensional subgroups given by the following infinitesimal generators:

$$\mathbf{v}_1 = \frac{\partial}{\partial x} \quad (45)$$

$$\mathbf{v}_2 = \frac{\partial}{\partial t} \quad (46)$$

$$\mathbf{v}_3 = xt \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} - (x + tu) \frac{\partial}{\partial u} \quad (47)$$

$$\mathbf{v}_4 = \frac{x}{2} \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - \frac{u}{2} \frac{\partial}{\partial u} \quad (48)$$

$$\mathbf{v}_5 = t \frac{\partial}{\partial x} - \frac{\partial}{\partial u} \quad (49)$$

There exists also a sixth subgroup of infinite dimension. Lie-symmetry transformations are given by the one-parameter maps:

- Spatial translation:  
 $\mathbf{G}_1 : (x, t, u) \mapsto (x + \varepsilon_1, t, u)$
- Time translation:  
 $\mathbf{G}_2 : (x, t, u) \mapsto (x, t + \varepsilon_2, u)$
- Projection:  
 $\mathbf{G}_3 : (x, t, u) \mapsto (\frac{x}{1 - \varepsilon_3 t}, \frac{t}{1 - \varepsilon_3 t}, (1 - \varepsilon_3 t)u + \varepsilon_3 x)$
- Scale transformation:  
 $\mathbf{G}_4 : (x, t, u) \mapsto (xe^{\varepsilon_4}, te^{2\varepsilon_4}, ue^{-\varepsilon_4})$
- Galilean boost:  
 $\mathbf{G}_5 : (x, t, u) \mapsto (x + \varepsilon_5 t, t, u + \varepsilon_5)$

Most classical numerical schemes are invariant under the spatial and the time translations, as the scale transformation. But projection and galilean boost are generally broken. We will focus on the numerical properties induced by conserving them.

**4.1. Construction of invariant numerical scheme.** We deal with the following transformation, depending on parameters  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$  and  $\varepsilon_5$ :

$$\bar{x} = e^{\varepsilon_4} \frac{(x + \varepsilon_1) + \varepsilon_5(t + \varepsilon_2)}{1 - \varepsilon_3(t + \varepsilon_2)} \quad (50)$$

$$\bar{t} = e^{2\varepsilon_4} \frac{t + \varepsilon_2}{1 - \varepsilon_3(t + \varepsilon_2)} \quad (51)$$

$$\bar{u} = e^{-\varepsilon_4} ((1 - \varepsilon_3(t + \varepsilon_2))u + \varepsilon_3(x + \varepsilon_1) + \varepsilon_5) \quad (52)$$

This transformation corresponds to all Burger's finite dimensional Lie-symmetry transformations.

**4.1.1. The FTCS scheme.** Let's construct a numerical scheme invariant from the explicit forward time and centered space (FTCS) numerical scheme, in its non-conservative form:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \left( \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right) = \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \quad (53)$$

It is well-known that this scheme gives average numerical results. It has a low order of accuracy  $(\Delta t, \Delta x^2)$ . Its explicit nature implies a strong limitation due to the stability condition. The numerical scheme (53) is computed with a cartesian regular and orthogonal grid. Following the notations of the last section, this classical method can be noted as:

$$N^{\text{cl}}(\mathbf{z}) = 0, \quad \phi^{\text{cl}}(\mathbf{z}) = 0, \quad (54)$$

where the discrete scheme is:

$$N^{\text{cl}}(\mathbf{z}) = \frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \quad (55)$$

and the grid equations are:

$$\phi_1^{\text{cl}}(\mathbf{z}) = x_j^{n+1} - x_j^n \quad (56)$$

$$\phi_2^{\text{cl}}(\mathbf{z}) = x_{j+1}^n - 2x_j^n + x_{j-1}^n \quad (57)$$

$$\phi_3^{\text{cl}}(\mathbf{z}) = t_{j+1}^n - t_j^n \quad (58)$$

$$\phi_4^{\text{cl}}(\mathbf{z}) = t_j^{n+1} - 2t_j^n + t_j^{n-1} \quad (59)$$

The equation associated to  $\phi_1^{\text{cl}}$  gives the orthogonality of the grid, whereas the equation associated to  $\phi_2^{\text{cl}}$  gives the spatial regularity of the grid. The meaning of  $\phi_3^{\text{cl}}$  and  $\phi_4^{\text{cl}}$  is then straightforward.

It can be easily seen that this method  $(N^{\text{cl}}, \phi^{\text{cl}})$  does not respect the symmetry of the continuous equation. For example, when applying the galilean transformation

$$\bar{x} = x + \lambda t, \quad \bar{t} = t, \quad \bar{u} = u + \lambda, \quad (60)$$

an additional advective term tied to the transformation parameter  $\lambda$  appears in the discrete equation:

$$N^{\text{cl}}(\bar{\mathbf{z}}) = N^{\text{cl}}(\mathbf{z}) + \lambda \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \quad (61)$$

In the same way, an orthogonal grid cannot preserve the galilean transformation:

$$\phi_1^{\text{cl}}(\bar{\mathbf{z}}) = \phi_1^{\text{cl}}(\mathbf{z}) + \lambda \Delta t \quad (62)$$

In the next sections, we focus our interest in the improvement of the numerical properties of this scheme when it becomes invariant.

*4.1.2. Invariant counterpart of the FTCS scheme.* Following the process described in the section above, the computation of the transformed scheme is:

$$\begin{aligned} 0 = e^{-3\varepsilon_4} & \left[ \frac{u_j^{n+1}(1 - \varepsilon_3 \Delta t) + \varepsilon_3(x_j^{n+1} - x_j^n) - u_j^n}{\Delta t} (1 - \varepsilon_3 \Delta t) \right. \\ & \left. + (u_j^n + \varepsilon_5) \left( \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \varepsilon_3 \right) - \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \right] \end{aligned} \quad (63)$$

with  $x_j^{n+1} - x_j^n = 0$  because the grid is time-space orthogonal. The nontrivial solutions of the discrete equation (63) do obviously not depend on the symmetry parameter  $\varepsilon_4$ . The resulting scheme will nevertheless preserve the scale transformation. Therefore for sake of simplicity, the scale factor will be omitted hereafter.

We now have to specify the expression of  $\varepsilon_3$  and  $\varepsilon_5$  to make invariant the transformed scheme. The fundamental result of the construction of invariant numerical scheme using moving frames in Prop.(1) guarantees that if  $\varepsilon_3$  and  $\varepsilon_5$  are moving frames for the actions (50), (51) and (52) then the transformed scheme (63) is an invariant scheme.

Construction of the moving frames. The condition of equivariance for  $\rho$  is:

$$\rho(z) \cdot z = \rho(\bar{z}) \cdot \bar{z} \quad (64)$$

For the left hand side of (64), and applied to points of the stencil used by the FTCS scheme:

$$\begin{aligned} \rho(z_j^n) \cdot z_j^n &= (0, 0, u_j^n + \varepsilon_5) \\ \rho(z_j^n) \cdot z_j^{n+1} &= \left( \frac{x_j^{n+1} - x_j^n + \varepsilon_5 \Delta t}{1 - \varepsilon_3 \Delta t}, \frac{\Delta t}{1 - \varepsilon_3 \Delta t}, u_j^{n+1}(1 - \varepsilon_3 \Delta t) + \varepsilon_3(x_j^{n+1} - x_j^n) + \varepsilon_5 \right) \\ \rho(z_j^n) \cdot z_{j\pm 1}^n &= (\Delta x, 0, u_{j\pm 1}^n + \varepsilon_3 \Delta x + \varepsilon_5) \end{aligned}$$

and for the right hand side:

$$\begin{aligned} \rho(\bar{z}_j^n) \cdot \bar{z}_j^n &= (0, 0, u_j^n + (\lambda + \bar{\varepsilon}_5)) \\ \rho(\bar{z}_j^n) \cdot \bar{z}_j^{n+1} &= \left( \frac{x_j^{n+1} - x_j^n + (\lambda + \bar{\varepsilon}_5) \Delta t}{1 - (\mu + \bar{\varepsilon}_3) \Delta t}, \frac{\Delta t}{1 - (\mu + \bar{\varepsilon}_3) \Delta t}, \right. \\ & \quad \left. u_j^{n+1}(1 - (\mu + \bar{\varepsilon}_3) \Delta t) + (\mu + \bar{\varepsilon}_3)(x_j^{n+1} - x_j^n) + (\lambda + \bar{\varepsilon}_5) \right) \\ \rho(\bar{z}_j^n) \cdot \bar{z}_{j\pm 1}^n &= (\Delta x, 0, u_{j\pm 1}^n \pm (\mu + \bar{\varepsilon}_3) \Delta x + (\lambda + \bar{\varepsilon}_5)) \end{aligned}$$

So that the equivariance condition (64) is equivalent to:

$$\bar{\varepsilon}_3 = \varepsilon_3 - \mu, \quad \bar{\varepsilon}_5 = \varepsilon_5 - \lambda, \quad (65)$$

In order to determine a family of moving frames allowing us to make the invariant numerical scheme, we suppose an algebraic expression of  $\varepsilon_3$ .

As  $[\varepsilon_3] = s^{-1}$ , one chooses a combinaison of discret variables which is dimensionally relevant. To keep the explicit form of the numerical scheme, there must not be any term in  $u^{n+1}$ . On the other hand, to keep the order of degree of the convective

term, the parameter of symmetrization must be at most of degree one. Finally, the construction of a moving frame from the application  $\varepsilon_3$  requires that there is no term in  $\Delta x$  and  $\Delta t$  alone. Suppose then the algebraic form:

$$\varepsilon_3 = \frac{au_{j+1}^n + bu_j^n + cu_{j-1}^n}{\Delta x} \quad a, b, c \in \mathbb{R} \quad (66)$$

Similar arguments for  $\varepsilon_5$  (of dimension  $m.s^{-1}$ ) allow us to suppose a general form like:

$$\varepsilon_5 = du_{j+1}^n + eu_j^n + fu_{j-1}^n \quad d, e, f \in \mathbb{R} \quad (67)$$

Conditions over  $a, b, c$  and  $d, e, f$  such that (66) and (67) become moving frames, and so verify (65), are:

$$c - a = 1, \quad a + b + c = 0, \quad d - f = 0, \quad d + e + f = -1, \quad (68)$$

The determination of the expressions of the family of moving frames associated to a given group action is, of course, independent of the considered numerical scheme. But when using moving frames for finite differences equations, numerical properties must be taken into account in the choice of the moving frames. We consider therefore the terms composing the discrete equation: the temporal term  $T_t$ , the convective term  $T_c$  and the diffusive term  $T_d$ :

$$T_t = \frac{u_j^{n+1} - u_j^n}{\Delta t}, \quad T_c = u_j^n \left( \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right), \quad T_d = \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \quad (69)$$

The condition of consistency of the invariant scheme is:

$$\bar{T}_t + \bar{T}_c - \bar{T}_d = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} + O(\Delta t^p, \Delta x^q) \quad p, q > 0 \quad (70)$$

where  $\bar{T}_t$ ,  $\bar{T}_c$  and  $\bar{T}_d$  are the invariant counterpart of the time, convection and diffusion terms:

$$\begin{aligned} \bar{T}_t &= \frac{1}{\Delta t} \left( 1 - \frac{au_{j+1}^n + bu_j^n + cu_{j-1}^n}{\Delta x} \Delta t \right) \\ &\quad \times \left( u_j^{n+1} \left( 1 - \frac{au_{j+1}^n + bu_j^n + cu_{j-1}^n}{\Delta x} \Delta t \right) + \frac{au_{j+1}^n + bu_j^n + cu_{j-1}^n}{\Delta x} (x_j^{n+1} - x_j^n) - u_j^n \right) \end{aligned} \quad (71)$$

$$\bar{T}_c = (u_j^n + du_{j+1}^n + eu_j^n + fu_{j-1}^n) \left( \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{au_{j+1}^n + bu_j^n + cu_{j-1}^n}{\Delta x} \right) \quad (72)$$

$$\bar{T}_d = \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \quad (73)$$

with coefficients  $a, b, c, d, e$  and  $f$  satisfying the equivariance condition (68). Taylor expansion gives:

$$\begin{aligned} \bar{T}_t + \bar{T}_c - \bar{T}_d &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} \\ &\quad - (a + c) \frac{\Delta x}{2} u \frac{\partial^2 u}{\partial x^2} + \frac{dx}{dt} \left( \frac{\partial u}{\partial x} + \dots \right) + O(\Delta t, \Delta x^2) \end{aligned} \quad (74)$$

In order to keep the same degree of order, it is necessary to impose:

$$a + c = 0 \quad (75)$$

There is no constrain over coefficients  $(d, e, f)$  of  $\varepsilon_5$ ; in fact  $\varepsilon_5$  only appears in the convective term. Or the taylor expansion of  $\bar{T}_c$  indicates that:

$$\bar{T}_c = O(\Delta x^2) \quad (76)$$

It means that the symmetrized convective term is identically null, therefore the convective phenomena are produced by the temporal term.

The moving frame associated to the projection is then determined in one way:

$$\varepsilon_3 = -\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \quad (77)$$

The expression  $\varepsilon_5$  is such that:

$$\varepsilon_5 = du_{j+1}^n + eu_j^n + fu_{j-1}^n, \quad d - f = 0, \quad d + e + f = -1, \quad (78)$$

**Remark 2** (Link with the normalization process). It would have been possible to obtain this expression for  $\varepsilon_3$  with the normalization process by taking the cross-section:

$$\bar{u}_{j+1}^n - \bar{u}_{j-1}^n = 0 \quad (79)$$

The parameter  $\varepsilon_5$  associated to the galilean boost does not occur when the numerical scheme is expressed in the regular and orthogonal original mesh. It is no more the case if the original grid is not orthogonal, nor if the numerical solution is expressed in the transformed frame of reference.

During the process of *invariantization* the orthogonality of the grid has been broken. Even if the invariant scheme is expressed finally in the original grid. Indeed, the process of construction of the invariant scheme has yield to a system of discrete equations  $(N^{\text{inv}}, \phi^{\text{inv}})$  that preserve the symmetry group of the Burgers equation.

$$N^{\text{inv}}(\mathbf{z}) = 0, \quad \phi^{\text{inv}}(\mathbf{z}) = 0, \quad (80)$$

where the discrete scheme is:

$$\begin{aligned} N^{\text{inv}}(\mathbf{z}) = & \frac{u_j^{n+1} - u_j^n}{\Delta t} - \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\ & \times \left( 2u_j^{n+1} - u_j^n - \frac{x_j^{n+1} - x_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \left( u_j^{n+1} - \frac{x_j^{n+1} - x_j^n}{\Delta t} \right) \right) \end{aligned} \quad (81)$$

and the grid equations are:

$$\begin{aligned} \phi_1^{\text{inv}}(\mathbf{z}) = & (x_j^{n+1} - x_j^n + (du_{j+1}^n + eu_j^n + fu_{j-1}^n)\Delta t) \\ & \times \left( 1 + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \Delta t \right)^{-1} \end{aligned} \quad (82)$$

$$\phi_2^{\text{inv}}(\mathbf{z}) = x_{j+1}^n - 2x_j^n + x_{j-1}^n \quad (83)$$

$$\phi_3^{\text{inv}}(\mathbf{z}) = t_{j+1}^n - t_j^n \quad (84)$$

$$\phi_4^{\text{inv}}(\mathbf{z}) = \Delta t \left( \left( 1 + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right) \Delta t \right)^{-1} - \Delta t \left( \left( 1 - \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right) \Delta t \right)^{-1} \quad (85)$$

In fact, when applying, for example, the galilean transformation to the invariant scheme, an easy computation leads to:

$$N^{\text{inv}}(\bar{\mathbf{z}}) = N^{\text{inv}}(\mathbf{z}) \quad (86)$$

And the same for the grid equations:

$$\phi_1^{\text{inv}}(\bar{\mathbf{z}}) = \phi_1^{\text{inv}}(\mathbf{z}) + \lambda(1 + (d + e + f))\Delta t \quad (87)$$

As the equivariant condition over the coefficient is exactly  $d + e + f = -1$ , this implies the preservation of the galilean boost by  $\phi_1^{\text{inv}}$ . The galilean invariance for  $\phi_2^{\text{inv}}$ ,  $\phi_3^{\text{inv}}$  and  $\phi_4^{\text{inv}}$  is obvious. The same verification can be done with the projective transformation. In that sense, the scheme constructed above is invariant.

At this stage, it is possible to directly compute the invariant scheme associated to the invariant grid. But from the implementation point of view, it is possible (and easier) to express the invariant scheme back to the original orthogonal and regular grid. It can be done without loosing the invariance property by injecting the grid transformation into the finite differences equation. That is, by simply replacing the expression of  $\Delta \bar{t}$  and  $\Delta \bar{x}$  wherever they occur in terms of the original time and space steps size  $\Delta t$  and  $\Delta x$ , and of the discrete velocity variables  $u_j^n, u_{j\pm 1}^n, \dots$ .

**4.2. Numerical applications.** The aim of the numerical tests performed in this section is to enforce the constatation that the property of invariance may improve intrinsically the scheme. The invariant counterpart of an average scheme could hardly be more accurate than a high order classical scheme. It is the reason why the numerical comparisons of the invariant FTCS scheme are made with the original FTCS scheme. In particular, we expect the invariant schemes to successfully reproduce the solutions tied to the symmetry group of the Burgers equation, whereas the classical FTCS scheme failed.

**4.2.1. Self-similar solutions.** Self-similar solutions have a particular importance in the fact that they allow to describe long-time behaviours or singularity. They can be computed by firstly reducing the PDE in some differential equation, and then solve the reduced equation. As standard numerical schemes -like FTCS- do not preserve every single symmetry, we expect they do not acquire the physical solutions as good as the invariant numerical schemes, unless having very thin grid.

Let's consider global invariants of the projection:

$$y = -\frac{x}{t} \quad (88)$$

$$w = ut - x \quad (89)$$

The partial derivatives of  $u$  in terms of new coordinates are:

$$\frac{\partial u}{\partial t} t^2 = -(w + y + y \frac{dw}{dy}) \quad (90)$$

$$\frac{\partial u}{\partial x} t = 1 + \frac{1}{t} \frac{dw}{dy} \quad (91)$$

$$\frac{\partial^2 u}{\partial x^2} t^3 = \frac{d^2 w}{dy^2} \quad (92)$$

Injecting those transformations into the burgers equation gives the reduced differential equation associated to the projection. It corresponds to the steady burgers equation:

$$w \frac{dw}{dy} = \nu \frac{d^2 w}{dy^2} \quad (93)$$

The general solution of this ODE is:

$$w(y) = D_0 \sqrt{2\nu} \tanh \left( \frac{D_0(y + D_1)}{\sqrt{2\nu}} \right) \quad (94)$$

where  $D_0$  and  $D_1$  are arbitrary real constants.

Substituting the reduction transformation, one obtains general self similar solution of Burgers' equation for the projection on the upper half plan  $\{t > 0\}$ :

$$u(x, t) = \frac{1}{t} \left( x - D_0 \sqrt{2\nu} \tanh \left( \frac{D_0 \left( \frac{x}{t} + K_1 \right)}{\sqrt{2\nu}} \right) \right) \quad (95)$$

In numerical example, boundary and initial conditions are given by a smooth solution ( $K_1 = 0$  and  $D_0 = (2\nu)^{-1/2}$ ) corresponding to a viscous shock:

$$u_{\text{exa}}(x, t) = \frac{1}{t} \left( x - \tanh \left( \frac{x}{2\nu t} \right) \right) \quad (96)$$

This viscous shock tends to a shock when the viscosity  $\nu$  becomes closer to 0.

Let's consider the problem over the domain  $\Omega = ]-1, 1[$  whose boundary and initial conditions are given by the shock solution:

$$u(x, t) = \frac{-\sinh \frac{x}{2\nu}}{(\cosh \frac{x}{2\nu} + \exp(-\frac{t}{4\nu}))} \quad (97)$$

The Fig.(4-a) shows the numerical behaviour of both classical and invariant scheme. The classical scheme presents some wriggles around the shock, although the invariant scheme reproduces the exact solution accurately. When the viscosity becomes smaller (but with constant CFL), the classical solution blows up, which is not the case for the invariant numerical scheme as shown in Fig.(4-b). The process of invariance has made the scheme more stable without any loss of accuracy.

**4.2.2. Galilean invariance.** A fundamental principle of physics is the galilean invariance. The systems are equivalent if they differ by a galilean translation. From the numerical point of view, it corresponds to apply a galilean boost of velocity  $\lambda$  (Fig. 5):

$$\bar{x} = x + \lambda t, \quad \bar{t} = t, \quad \bar{u} = u + \lambda,$$

The frame moves at constant velocity while the solution is translated up to  $\lambda$ . In order to test how a numerical scheme can reproduce this property, let's consider the problem over the domain  $\Omega = ]-2; 10[$ , whose boundary conditions are given by the exact solution:

$$u_{\text{exact}}(x, t) = \frac{\frac{x-2t}{t+0.1}}{1 + \nu \sqrt{t+0.1} \exp \frac{(x-2t)^2}{4\nu(t+0.1)}} + 2 \quad (98)$$

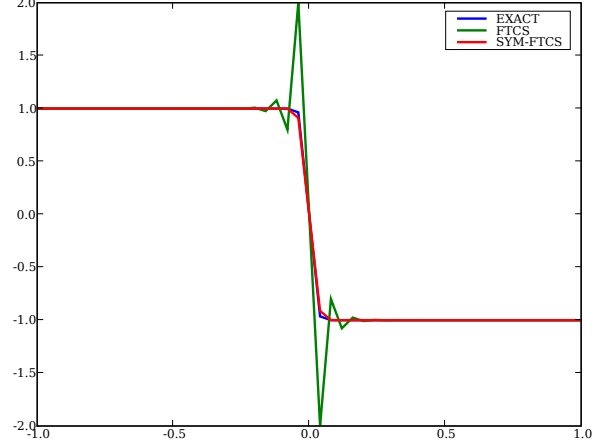
This solution represents a damped sinusoidal wave.

In Fig.(6-a), the numerical solution associated to the classical FTCS scheme shows an extra convection velocity growing up to the galilean boost. The galilean shift has produced error in evaluation of the velocity of the travelling wave. When comparing with the invariant scheme under galilean transformation in Fig.(6-b), we observe that the behaviour is insensitive up to the galilean shift as expected. The accuracy of the invariant scheme is evaluated in function of the value of the galilean boost  $\lambda$  in Tab.(1), with the discrete error:

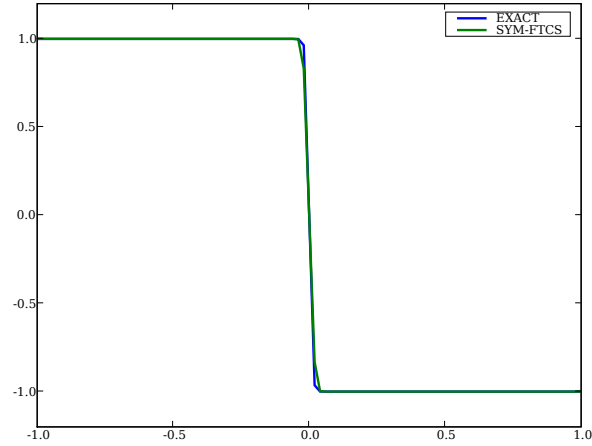
$$\text{Err } L_{\text{abs}} = \frac{1}{N} \sum_j^N |u_{\text{exact}}(x_j^n, t^n) - u_j^n| \quad (99)$$

The error of the invariant scheme is the same at  $10^{-6}$ , whatever the value of  $\lambda$  is. The exact solution equ.(98) is enough smooth for low viscosity. The loss of galilean invariance produces error in the convective velocity for classical scheme. But when





a) Invariant FTCS, and classical FTCS.  $\nu = 10^{-2}$ ,  $\Delta t = 10^{-2}$  and  $\Delta x = 4 \cdot 10^{-2}$ .



b) Invariant FTCS.  $\nu = 5 \cdot 10^{-3}$ ,  $\Delta t = 10^{-2}$  and  $\Delta x = 2 \cdot 10^{-2}$ .

FIGURE 4. *Pseudo-shock*. Numerical solutions for self-similar solution. CFL = 1/2, at time  $t = 1$ .

$\lambda$	0.	0.2	0.4	0.6	0.8	1.
Err $L_{\text{abs}} (\times 10^{-3})$	4.62526	4.6253	4.62535	4.6254	4.62544	4.6255

TABLE 1. *Damped sinusoidal wave*. Errors in function of galilean boost  $\lambda$  for the damped sinusoidal wave

the solution allows shock as the self-similar solution associated to the projection equ.(97), some numerical wiggles may appear even if the numerical conditions are

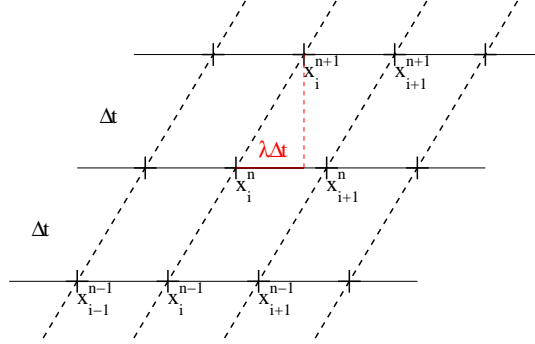


FIGURE 5. *Evolution of the frame.* Straight and uniform displacement of the frame for a fixed boost velocity  $\lambda$ .

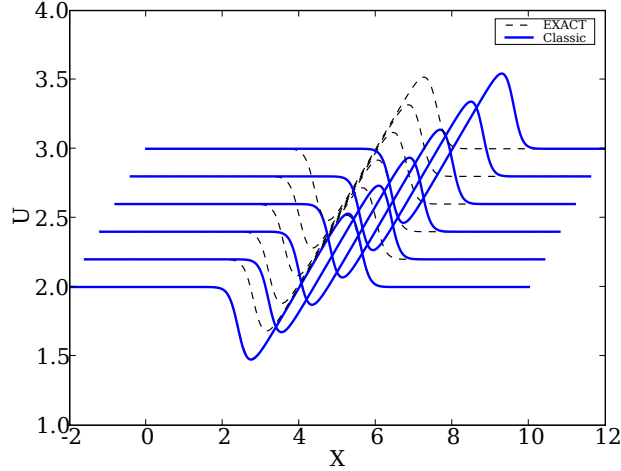
effective enough in the original referential. Fig.(7-a) represents the classical solution for different galilean boost  $\lambda$ . Unphysical wiggles appear around the shock as  $\lambda$  grows. The numerical solution stays advected by an additional velocity, so that the velocity of the shock depends on the galilean boost too. Fig.(7-b) shows the solution associated to the invariant scheme. The solution stays the same whatever the value of  $\lambda$  is. The evolution of behaviour of the invariant scheme is quite satisfying. The  $L^2$  error with the exact solution of the invariant scheme over the time interval is reported in Tab.(8):

$$\text{Err } L^2 = \left( \frac{1}{N} \sum_j^N |u_{\text{exact}}(x_j^n, t^n) - u_j^n|^2 \right)^{\frac{1}{2}} \quad (100)$$

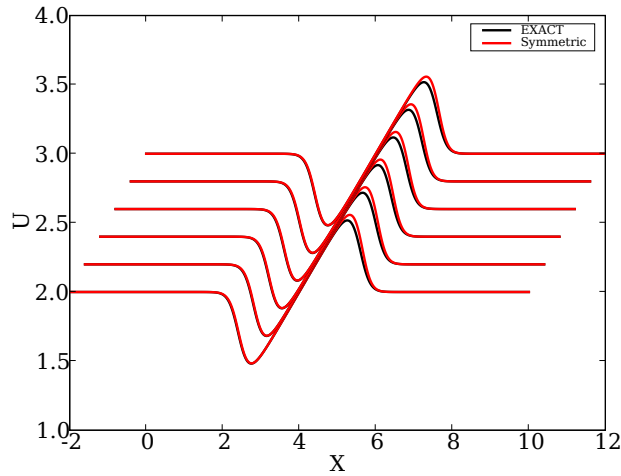
**4.3. Order of convergence.** We present the order of convergence of the invariant FTCS scheme. The stability condition is given by the ratio between the CFL number  $|u| \frac{\Delta t}{\Delta x}$  and the mesh Reynolds number  $Re_h = |u| \frac{\Delta x}{\nu}$ . For a viscosity  $\nu = 0.05$ , and an average velocity  $u = 1$ , the diffusion term is predominant. Therefore the grid is refined such that the ratio between the time and the space steps  $\Delta t$  and  $\Delta x^2$  is conserved. The relative error is computed with the problem of the damped sinusoidal wave, in the  $L^2$ -norm:

$$\text{Err } L_{\text{rel}}^2 = \sqrt{\frac{\sum_j^N |u_{\text{exact}}(x_j, t^n) - u_j^n|^2}{\sum_j^N |u_{\text{exact}}(x_j, t^n)|^2}} \quad (101)$$

In Fig. (9-a), the relative error is computed for a variable time step size. The affine fitting function has a slope almost equal to 1. In Fig. (9-b), the relative error is computed for a variable space step size. The slope of the fitting function is very close to 2. These numerical tests confirm the theoretical order of accuracy  $O(\Delta t, \Delta x^2)$  of the invariant scheme.



a) Standard FTCS scheme.

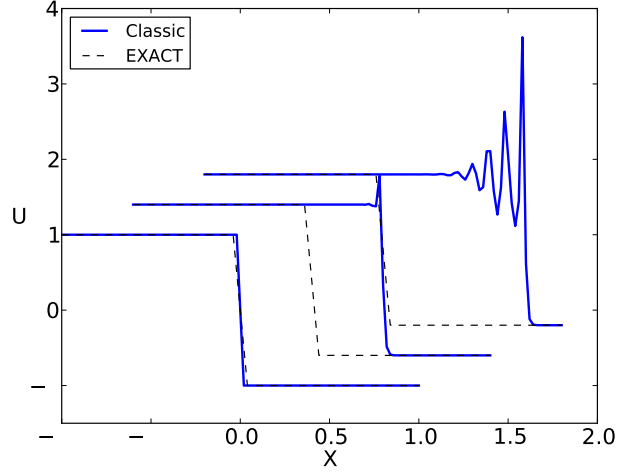


b) Invariant FTCS scheme.

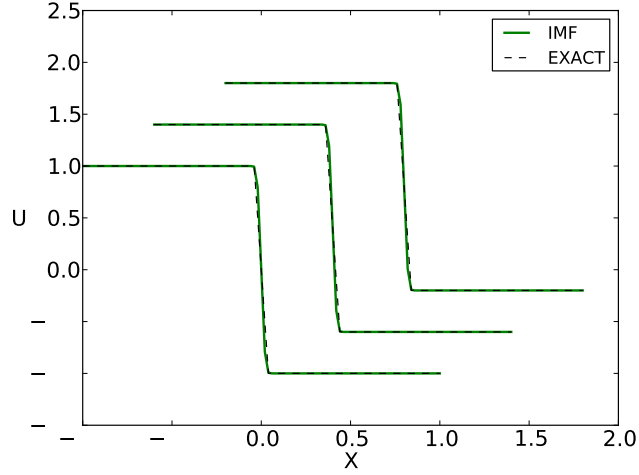
FIGURE 6. *Damped sinusoidal wave.* Galilean parameter  $\lambda$  has value from 0 to 1.  $\nu = 5.10^{-2}$ , CFL = 0.1 in the referential frame,  $\Delta x = 1.10^{-2}$  at time  $t = 1$ .

**5. Conclusion.** We have presented a new method of construction of invariant numerical schemes using moving frames. Our invariant scheme is by construction invariant under finite dimensional Lie-symmetry. The invariant scheme is constructed following the Prop.(1), telling that if a scheme is transformed under the action of a moving frame, then the scheme becomes invariant. Taking the basic definition of a moving frame (i.e. an equivariant transformation), we perform a way to choose some coefficients such that the transformation parameters:

1. Verify the equivariance property, and then the scheme becomes invariant,



a) Standard FTCS scheme



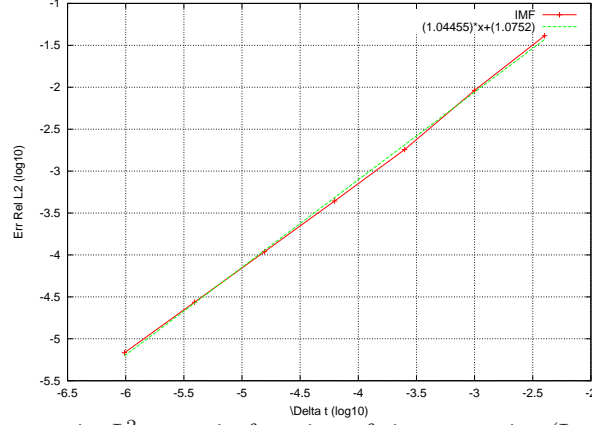
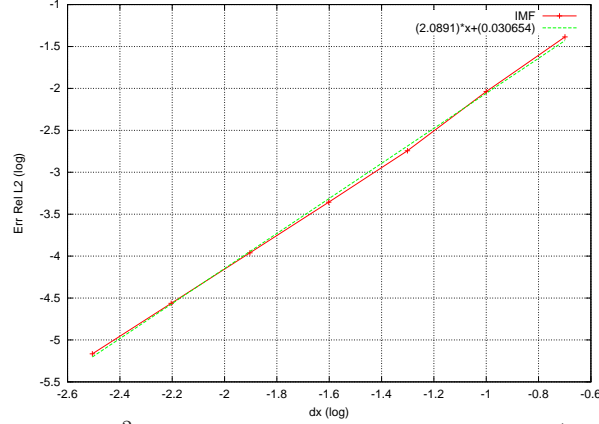
b) Invariant FTCS scheme

FIGURE 7. *Pseudo-shock*. Galilean parameter  $\lambda$  has value from 0 to 0.8.  $\nu = 10^{-2}$ ,  $CFL = 1/2$  in the referential frame,  $\Delta x = 2.10^{-2}$  at time  $t = 1$ .

Time	$t = 0.25$	$t = 0.5$	$t = 0.75$	$t = 1$	$t = 1.5$	$t = 2$
Err $L^2 (\times 10^{-3})$	6.34265	6.19552	6.19524	6.19524	6.19524	6.19524

FIGURE 8. *Pseudo-shock*. Error  $L^2$  for the invariant scheme in function of time  $t$  for the pseudo-shock test.

2. The order of accuracy of the scheme stays the same

a) Relative error in  $L^2$ -norm in function of time step size (Log-Log scale)b) Relative error in  $L^2$ -norm in function of space step size (Log-Log scale)FIGURE 9. *Damped sinusoidal wave.*

Therefore our process is based on geometric considerations and include as a parameter of construction the order of accuracy of the invariant scheme. The main motivation is to get rational arguments to choose moving frames leading to accurate schemes. This problem is one of the major limitation of the normalization process. It also points out that attributing geometrical features to schemes can break fundamental numerical properties. In this paper we propose to fix the order of accuracy of invariant scheme to be the same as the original classical scheme. Applied on the FTCS scheme for the Burgers equation, a family of invariant schemes has been pointed out by this process. It corresponds to the equations (63), (66), (67) and (68). To pick up from this family the invariant schemes that are as accurate as the original classical scheme, it is necessary to get  $a + c = 0$ . One could include other fundamental numerical conditions as stability conditions, dispersive relations, and so on. Through the numerical tests, we have illustrated how the conservation of symmetry by numerical scheme has its importance for the capture of self-similar solutions and the respect of galilean invariance. One of the main

trump of such schemes is in succeeding in *naturally* capturing global features of equations described in the symmetry group. Comparable results could be obtained with classical scheme by dealing with a quite thinner but more expansive grid.

## REFERENCES

- [1] M.I. Bakirova, V.A. Dorodnitsyn and R.V. Kozlov, *Symmetry-preserving differences schemes for some heat transfer equations*, J. Phys. A: Math. Gen., **30** (1997), 8139–8155.
- [2] T. J. Bridges, Multi-symplectic structures and wave propagation, Math. Proc. Camb. Phil. Soc., **121** 1997, 147–190.
- [3] C. J. Budd and V. A. Dorodnitsyn, *Symmetry adapted moving mesh schemes for the nonlinear Schrödinger equation*, J. Phys., **34** (48) 2001, 10387–10400.
- [4] C.J. Budd and M. Piggott., *Geometric integration and its applications.*, Handbook of numerical analysis, North-Holland, (2000) 35–139
- [5] C.J. Budd and G.J. Collins, *Symmetry based numerical methods for partial differential equations*, Numerical analysis, Notes Math., **380** 1998, 16–36.
- [6] M. P. Calvo and E. Hairer, *Accurate long-term integration of dynamical systems*, Journal of Applied Mathematics, **18** 1995, 95–105.
- [7] É. Cartan, *La Méthode du Repère Mobile, La Théorie des Groupes Continues, et Les Espaces Généralisés*, publisher = Hermann, **5** 1935, Exposés de Géométrie
- [8] J-B. Chen, H. Y. Guo and K. Wu, *Discrete total variation calculus and Lee’s discrete mechanics*, Applied Mathematics and Computation, **177** 2006, 226–234.
- [9] J-B. Chen and M. Z. Qin, *Multisymplectic composition integrators of high orders*, J. Comput. Math, **21** 2003, 647–656.
- [10] J-B. Chen, M. Z. Qin and Y-F Tang, *Symplectic and multisymplectic methods for the non-linear Schrödinger equation*, Comput. Math. Appl., **43** 2002, 1095–1106.
- [11] J-B. Chen, *Total Variation in Discrete Multisymplectic Field and Multisymplectic-Energy-Momentum Integrators*, Letters in Mathematical Physics, **61** 2002, 63–73.
- [12] M. Chhay and A. Hamdouni, *A new construction for invariant numerical schemes using moving frames*, C. R. Acad. Sci. Mecanique, **in press**.
- [13] G. Cicogna, *A discussion on the different Notions of symmetry of differential equations*, Proceedings of Institute of Mathematics of NAS of Ukraine, **50** 2004, 77–84.
- [14] A. S. Dawes, *Invariant numerical methods*, Int. Journ. for Numer. Meth. in Fluids, **56** (8) 2008, 1185–1191.
- [15] V. A. Dorodnitsyn, *Finite-difference models entirely inheriting continuous symmetry of original differential equations*, International Journal of Modern Physics C, (Physics and Computers), **5** (4) 1994, 723–734,
- [16] V. A. Dorodnitsyn, *Some new invariant difference equations on evolutionary grids*, IMACS World Congress of Computational and Applied Mathematics, **1** 1994, Proceedings of 14-th.
- [17] V. A. Dorodnitsyn, *Continuous symmetries of finite-difference evolution equations and grids*, Proceedings of Workshop on Symmetries and Integrability of Difference Equations, CRM, **9** 1996, 103–112.
- [18] V. A. Dorodnitsyn, *Group theoretical methods for finite difference modeling*, Proceedings of the First World Congress of Nonlinear Analysts, 1996, 979–990.
- [19] M. Fels and P.J. Olver, *Moving coframes I. a practical algorithm*, Acta Appl. Math., **51** 1998, 161–213.
- [20] M. Fels and P.J. Olver, *Moving coframes II. regularization and theoretical foundations*, Acta Appl. Math., **55** 1999, 127–208.
- [21] Z. Ge and J.E. Marsden, *Lie-Poisson Hamilton-Jacobi theory and Lie-Poisson integrators*, Physics Letters A, **133** 1988, 134–139.
- [22] M. J. Gotay, "A multisymplectic framework for classical field theory and the calculus of variations I: Covariant Hamiltonian formalism", Mechanics, Analysis and Geometry: 200 Years after Lagrange., 1991, 203–235.
- [23] M. J. Gotay, "A multisymplectic framework for classical field theory and the calculus of variations II: Space + Time decomposition", Diff. Geom. Appl., **1** 1991, 375–390.
- [24] E. Hairer, *Variable time step integration with symplectic methods*, Applied Numerical Mathematics: Transactions of IMACS, **25** 2-3 1997, 219–227.
- [25] E. Hairer and C. Lubich and G. Wanner, *Geometric Numerical Integration*, Springer-Verlag, 2002.

- [26] E. Hairer, S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations*, Springer-Verlag - 2nd Ed., 1993.
- [27] E. Hoarau, C. David, P. Sagaut and T.H. Le, *Lie group stability of finite difference schemes*, Discrete and continuous dynamical systems, 2007, 1–10.
- [28] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations, Vol. I: Symmetries, Exact Solutions, and Conservation Laws*, CRCC Press: Boca Raton, 1994.
- [29] C. Kane, J. E. Marsden and M. Ortiz, *Symplectic energy-momentum integrators*, J. Math. Phys., **40** 1999, 3353–3371.
- [30] C. Kane, J. E. Marsden, M. Ortiz and M. West, *Variational integrators and the Newmark algorithm for conservative and dissipative mechanical systems*, International Journal for Numerical Methods in Engineering, **49** (10) 2000, 1295–1325.
- [31] P. Kim, *Invariantization of Numerical Schemes using Moving Frames*, BIT Numerical Mathematics, **47** (3) 2007, 525–546.
- [32] P. Kim, *Invariantization of the Crank Nicolson method for Burgers equation*, Physica D: Nonlinear Phenomena, **237** (2) 2008, 243–254.
- [33] A. Lew, J. E. Marsden, M. Ortiz and M. West, *Asynchronous variational integrators*, Archive for Rational Mechanics and Analysis, **2** 2003, 85–146.
- [34] A. Lew, J. E. Marsden, M. Ortiz and M. West, *Variational time integrators.*, International Journal for Numerical Methods in Engineering, 2003,
- [35] J. E. Marsden and T. S. Ratiu *Introduction to Mechanics and Symmetry*, **17** 1999, Springer-Verlag, Texts in Applied Mathematics 2nd.
- [36] J. E. Marsden, G. W. Patrick, and S. Shkoller, *Multisymplectic geometry, variational integrators, and nonlinear PDEs*, Communication in Mathematical Physics, **199** (2) 1998, 351–395.
- [37] J. E. Marsden and S. Shkoller, *Multisymplectic geometry, covariant Hamiltonians and water waves*, Math. Proc. Camb. Phil. Soc., **125** 1999, 553–575.
- [38] J. E. Marsden and M. West, *Discrete mechanics and variational integrators*, Acta Numerica, **10** 2001, Cambridge University Press 357–515.
- [39] J. E. Marsden, S. Pekarsky, S. Shkoller and M. West, *Variational methods, multisymplectic geometry and continuum mechanics*, J. Geom. Phys., **38** 3-4 2001, 253–284.
- [40] B. E. Moore and S. Reich, *Multi-symplectic integration methods for Hamiltonian PDEs*, Future Gener. Comput. Syst., **19** (3) 2003, 395–402.
- [41] P.J. Olver, *Moving frames*, J. Symb. Comp., **3** 2003, 501–512.
- [42] P.J. Olver, *The Concise Handbook of Algebra Lie Groups and Differential Equations*, Kluwer Acad. Publ. 2002, Dordrecht, Netherlands, 92–97.
- [43] P.J. Olver., *Applications of Lie Groups to Differential Equations.*, Springer-Verlag., 1993, 2nd
- [44] G.R.W. Quispel and C. Dyt, *Solving ODE's Numerically while Preserving Symmetries, Hamiltonian Structure, Phase Space Volume or First Integrals*, Proceedings IMALS, **2** 1997, 601–607.
- [45] J. M. Sanz-Serna, *Symplectic Integrators for Hamiltonian Problems: An Overview*, Acta Numerica, **1** 1992, 243–286.
- [46] J. C. Simo and N. Tarnow and K. K. Wong, *Exact energy-momentum conserving algorithms and symplectic schemes for nonlinear dynamics*, Computer Methods in Applied Mechanics and Engineering, **10** 1992, 63–116.
- [47] Y.I. Shokin, *The Method of Differential Approximation*, Springer-Verlag, 1983
- [48] N.N. Yanenko and Yu. Shokin, *Group classification of difference schemes for a system of one-dimensional equations of gas dynamics*, Amer. Math. Soc. Transl., **104** (2) 1976, 259–265.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: nchhay01@univ-lr.fr

E-mail address: aziz.hamdouni@univ-lr.fr