A Test for Equality of Exponential Distributions Based on Type-II Censored Samples

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Reader Aids—

Purpose: Widen state of the art

Special math needed for explanations: Complex analysis and statistics

Special math needed to use results: Statistics

Results useful to: Reliability theoreticians and statisticians

Abstract—Moments of the likelihood ratio test (LRT) criterion for testing the equality of several exponential distributions based on type-II censored samples of unequal size are obtained. They are used to obtain the distribution in a computational form for the first time. A table of selected significance points of LRT is presented.

1. INTRODUCTION

Most models for component & system life & reliability are based on the exponential distribution which plays much the same role in reliability theory that the s -normal distribution plays in sampling theory. Under some reasonably general conditions the distribution of the time between failures tends to an exponential as the complexity and the time of operation increase [2]. Epstein & Sobel [5-7] and Epstein [3, 4] gave numerous results and popularized the exponential as a lifetime distribution, especially in the area of industrial life testing. This paper addresses the problem of testing the equality of several exponential distributions based on type-II censored samples of unequal size. The general moment of the likelihood ratio test (LRT) criterion is obtained and then this is used to obtain its distribution in a computational form. A short table of selected significance points of LRT is presented.

2. PRELIMINARIES

Notation

p number of samples
 i sample serial number, $i = 1, 2, \dots, p$

n_i number of observations in sample i
 r_i number of failed items in sample i , predetermined prior to life testing, $1 \leq r_i \leq n_i$
 $t_{i(1)} \leq t_{i(2)} \leq \dots \leq t_{i(r_i)}$ first r_i observations in the sample i
 Σ_i sum over i from 1 to p
 Π_i product over i from 1 to p
 R $r_1 + r_2 + \dots + r_p$; total number of failed items
 T_i $\sum_{j=1}^{r_i} t_{i(j)} + (n_i - r_i)t_{i(r_i)}$, total observed lifetime, or total time on test for sample i
 λ likelihood ratio
LRT likelihood ratio test
 L $\lambda^{p/R}$
 H_0 null hypothesis

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

Assumptions

1. The p s -independent type II censored samples are available and sample i has been drawn from an exponential distribution with pdf:

$$\exp(t; \theta_i) = \theta_i^{-1} \exp(-t/\theta_i), \quad \text{for } t > 0, \quad \theta_i > 0;$$

$$i = 1, 2, \dots, p$$

$$= 0, \quad \text{otherwise} \quad (2.1)$$

2. Each sample has unequal number of observations. The LRT for testing the hypothesis

$$H_0: \theta_1 = \theta_2 = \dots = \theta_p \quad (2.2)$$

against the general alternatives, is:

$$\lambda = (\Pi_i \hat{\theta}_i^{r_i}) / \hat{\theta}^R \quad (2.3)$$

$$\hat{\theta}_i = T_i / r_i \quad (2.4)$$

$$\hat{\theta} = \Sigma_i T_i / R \quad (2.5)$$

The appendix shows that moment h of λ is:

$$E\{\lambda^h\} = K \cdot R^{Rh} \cdot \Pi_i [r_i^{-hr_i} \Gamma\{r_i(1+h)\}] / \Gamma\{R(1+h)\} \quad (2.6)$$

$$K \equiv \Gamma(R) / \Pi_i \Gamma(r_i) \quad (2.7)$$

It follows from (2.6) that moment h of L is:

$$E\{L^h\} = K \cdot R^{ph} \Pi_i[r_i^{-phr_i/R} \Gamma\{r_i + phr_i/R\}]/\Gamma(R + ph) \quad (2.8)$$

3. EXACT DISTRIBUTION OF L

Notation

δ convergence factor
 pm $R - \delta$
 k_i r_i/R , $k_1 + k_2 + \dots + k_p = 1$
 $B_r(\cdot)$ Bernoulli polynomial of degree r and order one
 $\text{betf}(\cdot; c, d)$ beta Cdf, $\text{betf}(x; p, q) \equiv \int_0^x y^{p-1}(1-y)^{q-1} dy/B(p, q)$, for $p > 0$, $q > 0$.

Nomenclature

Mellin transform The Mellin integral transform of a function $f(x)$, defined only for $x > 0$ is:

$M\{f(x)|s\} = E\{x^{s-1}\} = \int_0^\infty x^{s-1} f(x) dx$ where s is any complex variable.

$O(t)$ $f(t)$ is $O(t)$ if the function $f(t)$ is bounded by some constant multiple of t for large t .

Apply the inverse Mellin transform [9] to the moment function of L in (2.8); the pdf $\{L\}$ is:

$$f(\ell) = K \cdot (2\pi i)^{-1} \int_{-i\infty}^{i\infty} \ell^{-h-1} \Pi_i[(k_i)^{-phk_i} \Gamma\{(pm + ph + \delta)k_i\}] dh / \Gamma(pm + \delta + ph) \quad (3.1)$$

Define

$$t \equiv m + h \quad (3.2)$$

Equation (3.3) results from (3.1) and (3.2):

$$f(\ell) = K_1 \cdot (2\pi i)^{-1} \int_{m-i\infty}^{m+i\infty} \ell^{-t} \phi(t) dt \quad (3.3)$$

$$K_1 \equiv K \cdot \ell^{m-1} \Pi_i(k_i)^{pmk_i} \quad (3.4)$$

$$\phi(t) \equiv \Pi_i[(k_i)^{-ptk_i} \Gamma\{(pt + \delta)k_i\}] / \Gamma(pt + \delta) \quad (3.5)$$

Use the asymptotic expansion for log gamma function [1, pp 204]. Then

$$\phi(t) \equiv K_2 \cdot t^{-\nu} [1 + q_1/t + q_2/t^2 + \dots] \quad (3.6)$$

$$K_2 \equiv (2\pi/p)^\nu \Pi_i(k_i)^{\delta k_i - 1/2} \quad (3.7)$$

$$V \equiv (p-1)/2 \quad (3.8)$$

The coefficients q_r are recursively determined using (3.9) and (3.10):

$$q_r \equiv \sum_{k=1}^r k A_k q_{r-k}/r, \quad q_0 = 1 \quad (3.9)$$

$$A_r \equiv (-1)^r [B_{r+1}(\delta) - \sum_i \{B_{r+1}(\delta k_i)/(k_i)^r\}]/r(r+1)p^r \quad (3.10)$$

Equation (3.6) shows that:

$$\phi(t)/K_2 = O(t^{-\nu}) \quad (3.11)$$

with real part of t tending to infinity; therefore $\phi(t)$ has the following exact representation as a factorial series [8]:

$$\phi(t) = K_2 \cdot \sum_{k=0}^{\infty} R_i \{\Gamma(t+a)/\Gamma(t+a+\nu+k)\}, \quad R_0 = 1 \quad (3.12)$$

where a is a convergence factor chosen such that $R_1 = 0$ and the coefficients R_i are obtained using the following recurrence relations from [8]:

$$\sum_{j=0}^i R_{i-j} d_{i-j} = q_i \quad (i = 1, 2, \dots) \quad (3.13)$$

$$d_{ir} = \sum_{k=1}^r k c_{ik} d_{i-r-k}/r, \quad d_{i0} = 1 \quad (3.14)$$

$$c_{ir} = (-1)^{r-1} [B_{r+1}(a) - B_{r+1}(a+\nu+i)]/r(r+1) \quad (3.15)$$

Use (3.12) in (3.3); the pdf $\{L\}$ is [8]:

$$f(\ell) = K_3 \cdot \sum_{i=0}^{\infty} R_i \ell^{m+a-1} (1-\ell)^{\nu+i-1} / \Gamma(\nu+i) \quad (3.16)$$

$$K_3 \equiv (2\pi/p)^\nu \Gamma(R) \Pi_i[k_i^{Rk_i-1/2} / \Gamma(Rk_i)]. \quad (3.17)$$

In (3.10), choose δ such that $A_1 = 0$. Then —

$$\delta(\sum_i k_i^{-1} - 1)/6(p-1). \quad (3.18)$$

In (3.12), choose a such that $R_1 = 0$. Then —

$$a = (3-p)/4. \quad (3.19)$$

From (3.16), the Cdf $\{L\}$ is:

$$F(\ell) = K_3 \cdot \sum_{i=0}^{\infty} R'_i \text{betf}(\ell, m+a, \nu+i) \quad (3.20)$$

$$R'_i \equiv R_i \{\Gamma(m+a)/\Gamma(m+a+\nu+i)\} \quad (3.21)$$

4. NUMERICAL COMPUTATION

The Cdf $\{L\}$ in (3.20) is used to obtain s -significance points of L for $p = 2(1)4$, $\alpha = 0.005, 0.01, 0.025, 0.05, 0.1$, and with equal number of failures in each sample. The computations were carried out on Cyber 74 using double precision arithmetic and the results, to four significant

TABLE 1
Percentage Points of $L = \lambda^{p/R}$ when $p = 2$

r	0.1	0.05	0.025	0.01	0.005
3	0.613751	0.500527	0.405047	0.303874	0.243356
4	0.698207	0.600707	0.513780	0.415398	0.352607
5	0.752728	0.668244	0.590433	0.498895	0.438088
6	0.790694	0.716537	0.646769	0.562583	0.505173
7	0.818608	0.752684	0.689725	0.612381	0.558654
8	0.839978	0.780716	0.723487	0.652237	0.602059
9	0.856858	0.803074	0.750687	0.684790	0.637889
10	0.870525	0.821313	0.773052	0.711846	0.667915
11	0.881814	0.836472	0.791757	0.734671	0.693414
12	0.891295	0.849267	0.807629	0.754175	0.715323
13	0.899370	0.860209	0.821261	0.771027	0.734340
14	0.906329	0.869674	0.833096	0.785731	0.750997
15	0.912390	0.877941	0.843466	0.798669	0.765703
16	0.917714	0.885223	0.852626	0.810141	0.770780
17	0.922429	0.891606	0.860775	0.820302	0.790483
18	0.926633	0.897461	0.868072	0.829577	0.801015
19	0.930405	0.902652	0.874644	0.837880	0.810543
20	0.933808	0.907343	0.880593	0.845414	0.819204
25	0.946814	0.925338	0.903500	0.874576	0.852867
30	0.955550	0.937483	0.919039	0.894493	0.875981
35	0.961821	0.946230	0.930269	0.908956	0.892827
40	0.966542	0.952831	0.938765	0.919935	0.905647
45	0.970224	0.957988	0.945416	0.928552	0.915730
50	0.973176	0.962129	0.950764	0.935495	0.923868
55	0.975595	0.965527	0.955158	0.941208	0.930572
60	0.977614	0.968366	0.958832	0.945993	0.936192
65	0.979325	0.970772	0.961949	0.950057	0.940971
70	0.980793	0.972839	0.964638	0.953552	0.945084
75	0.982066	0.974632	0.966954	0.956591	0.948661
80	0.983181	0.976204	0.968894	0.959256	0.941800
85	0.984165	0.977592	0.970796	0.961613	0.954578
90	0.985040	0.978827	0.972400	0.963717	0.957054
95	0.985824	0.979933	0.973837	0.965594	0.959273
100	0.986530	0.980929	0.975132	0.967296	0.961274

p = number of samples

r = failures per sample

λ = likelihood ratio

R = total number of failed items (=pr)

TABLE 2
Percentage Points of $L = \lambda^{p/R}$ when $p = 3$

r	0.1	0.05	0.025	0.01	0.005
4	0.5448	0.4540	0.3704	0.2975	0.2401
5	0.6183	0.5351	0.4631	0.3827	0.3313
6	0.6717	0.5959	0.5288	0.4515	0.4007
7	0.7122	0.6431	0.5807	0.5075	0.4583
8	0.7439	0.6806	0.6223	0.5536	0.5065
9	0.7694	0.7111	0.6571	0.5921	0.5472
10	0.7902	0.7362	0.6859	0.6246	0.5819
11	0.8077	0.7574	0.7103	0.6525	0.6119
12	0.8224	0.7755	0.7312	0.6765	0.6379
13	0.8351	0.7911	0.7493	0.6975	0.6607
14	0.8461	0.8046	0.7652	0.7159	0.6809
15	0.8557	0.8165	0.7791	0.7323	0.6988
16	0.8642	0.8271	0.7916	0.7469	0.7148
17	0.8718	0.8365	0.8027	0.7600	0.7293
18	0.8785	0.8449	0.8126	0.7718	0.7423
19	0.8846	0.8525	0.8217	0.7825	0.7542
20	0.8901	0.8594	0.8299	0.7923	0.7650
25	0.9113	0.8861	0.8617	0.8304	0.8075
30	0.9256	0.9043	0.8835	0.8567	0.8370
35	0.9359	0.9175	0.8994	0.8759	0.8587
40	0.9437	0.9274	0.9114	0.8907	0.8753
45	0.9499	0.9353	0.9209	0.9022	0.8884
50	0.9548	0.9416	0.9286	0.9116	0.8990
55	0.9588	0.9468	0.9349	0.9194	0.9078
60	0.9622	0.9511	0.9401	0.9258	0.9152
65	0.9651	0.9548	0.9446	0.9314	0.9215
70	0.9675	0.9579	0.9485	0.9361	0.9269
75	0.9697	0.9607	0.9519	0.9403	0.9316
80	0.9715	0.9631	0.9548	0.9439	0.9357
85	0.9732	0.9653	0.9574	0.9471	0.9394
90	0.9747	0.9672	0.9597	0.9499	0.9427
95	0.9759	0.9689	0.9618	0.9526	0.9456
100	0.9772	0.9704	0.9637	0.9549	0.9483

figures, appear in tables 1-3. Percentage points of L for the case of unequal number of failures are given in table 4 for $p = 2$ and 3 and particular values of k_i .

TABLE 3
Percentage Points of $L = \lambda^{p/R}$ when $p = 4$

r	0.1	0.05	0.025	0.01	0.005
5	0.5215	0.4433	0.3780	0.3072	0.2631
6	0.5834	0.5099	0.4468	0.3763	0.3309
7	0.6314	0.5629	0.5029	0.4343	0.3892
8	0.6697	0.6059	0.5492	0.4833	0.4392
9	0.7009	0.6414	0.5879	0.5249	0.4822
10	0.7268	0.6711	0.6206	0.5605	0.5194
11	0.7486	0.6963	0.6486	0.5914	0.5519
12	0.7672	0.7180	0.6728	0.6183	0.5804
13	0.7832	0.7368	0.6940	0.6419	0.6056
14	0.7972	0.7533	0.7126	0.6629	0.6279
15	0.8095	0.7679	0.7291	0.6815	0.6480
16	0.8204	0.7808	0.7438	0.6983	0.6661
17	0.8302	0.7924	0.7571	0.7134	0.6824
18	0.8389	0.8028	0.7689	0.7271	0.6972
19	0.8468	0.8123	0.7798	0.7395	0.7107
20	0.8539	0.8209	0.7897	0.7508	0.7231
25	0.8815	0.8542	0.8282	0.7955	0.7719
30	0.9004	0.8771	0.8548	0.8266	0.8062
35	0.9141	0.8938	0.8743	0.8495	0.8315
40	0.9244	0.9065	0.8892	0.8671	0.8510
45	0.9326	0.9165	0.9009	0.8811	0.8665
50	0.9391	0.9245	0.9104	0.8923	0.8791
55	0.9445	0.9312	0.9182	0.9016	0.8894
60	0.9491	0.9367	0.9248	0.9095	0.8982
65	0.9529	0.9415	0.9304	0.9162	0.9057
70	0.9562	0.9455	0.9352	0.9219	0.9121
75	0.9591	0.9491	0.9394	0.9269	0.9177
80	0.9616	0.9522	0.9431	0.9314	0.9227
85	0.9638	0.9549	0.9463	0.9353	0.9271
90	0.9658	0.9574	0.9493	0.9388	0.9310
95	0.9676	0.9596	0.9519	0.9419	0.9345
100	0.9691	0.9616	0.9542	0.9447	0.9377

TABLE 4
Percentage Points of $L = \lambda^{p/R}$, $\alpha = .05$

$p = 2$					
R					
k_1	k_2	10	20	60	100
0.50	0.50	0.66824365	0.82131333	0.93748256	0.96212923
0.40	0.60	0.66755956	0.82109744	0.93745503	0.96212238
0.30	0.70	0.66514658	0.82032925	0.93735617	0.96208578
0.20	0.80	0.65934591	0.81843068	0.93710946	0.96199434
0.10	0.90	0.64306243	0.81251018	0.93630917	0.96169657

$p = 3$						
R						
k_1	k_2	k_3	10	20	60	100
0.33	0.33	0.33	0.38374797	0.62865293	0.85946462	0.91350174
0.30	0.30	0.40	0.38329741	0.62846305	0.85943505	0.91349039
0.20	0.20	0.60	0.37670923	0.62564201	0.85899219	0.91332019
0.20	0.30	0.50	0.38021160	0.62714428	0.85922828	0.91341034
0.10	0.30	0.60	0.36943764	0.62211411	0.85840698	0.91309410

APPENDIX

To obtain the moment h of λ , we essentially use the method in [10] and the fact that under H_0 : $\theta_1 = \theta_2 = \dots = \theta_p = \theta$, T_i are s -independently distributed as gammas with parameters r_i and θ . For this consider the function $g(\mu)$ where:

$$g(\mu) = E[\Pi_i(T_i)^{hr_i} \exp\{\mu \sum_i T_i\}] \quad (A.1)$$

$$= \theta^{Rh} \cdot \prod_i \frac{\Gamma\{r_i(1+h)\}}{\Gamma(r_i)} (1-\mu\theta)^{-(1+h)R} \quad (A.2)$$

The $E\{\lambda^h\}$ is then obtained by taking derivative r of $g(\mu)$ and then setting $r = -Rh$ and $\mu = 0$. This gives (2.6) of section 2.

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