The Ambiguity Function for Broadband Signals with Application to Objects Undergoing Non-Uniform Motion

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ABSTRACT

This paper discusses an alternative ambiguity function for broadband signals. This new ambiguity function is derived in this paper. An example of the broadband waveform for a gaussian signal that is reflected from a reflecting surface undergoing a constant acceleration is presented. The problem of reconstruction of the law of motion from the ambiguity function is then considered.

1. INTRODUCTION

The ambiguity function (AF) [1-4] was first introduced in radar by Woodward [1] in an attempt to solve the problem of determining the ideal waveform to solve the measurement problem of radar, i.e. to determine the range, velocity, acceleration, phase, amplitude, polarization properties, etc [5] of the scattered waveform. The AF can be viewed as either a direct or indirect attempt to solve this problem. Given a specific signal, the AF can be computed and the measurement properties of the waveform can be determined. Or given a specific set of desired measurement properties, these sets of properties can be used to attempt the synthesis of an AF. The synthesis problem is not solvable, even theoretically, but approximate solutions are realizable in some cases.

There are several different interpretations of the AF [6], but the one that will be used is the maximum likelihood estimator of the Doppler parameter a and delay factor τ . This amounts to representing the return signal $s_2(t)$ in terms of the broadcast signal $s_1(t)$ by

$$s_2(t) = A s_1(a \cdot t - \tau), \quad (1)$$

where c is the speed of propagation $a = \frac{c - v}{c + v}$,

$\tau = \frac{2R_0}{c + v}$, $R = R_0 + vt$ and R_0 is the initial position.

Assuming conservation of energy, and normalizing the signal, then $A = \sqrt{a}$ and the received signal can be written as

$$s_2(t) = \sqrt{a} s_1(a(t - \tau')), \quad (2)$$

where $\tau' = \frac{2R_0}{c - v}$. This form is independent of any narrowband assumptions, so it is the broadband signal representation for the Doppler effect. Using the form of the received signal in (2), the wideband AF is [5]

$$A(a, \tau') = \sqrt{a} \int_{-\infty}^{\infty} s_1(t) s_1^*(a(t - \tau')) e^{-j\omega t} dt. \quad (3)$$

The AF in (3) can not be extended to more general forms of the law of motion. Kelly [6] has formulated the problem in a form that is independent of the law of motion, but this is not based on sound physical principles as shown in [7]. Subsequent formulations based on Kelly's work are limited [8,9,10,11].

2. THEORY

In a previous paper, [7], a general method was developed for calculating the received waveform, $g(t)$, for an arbitrary transmitted waveform, $f(t)$, scattered from a perfectly reflecting surface that is undergoing a law of motion $r(t)$. This method is based upon results derived by Censor [12], Cooper [13]. They independently arrived at a general method for treating the arbitrary motion of mirrors, which is easily examined in the Fourier domain. An exact expression for the reflected waveform, which is true for an arbitrary waveform $f(\cdot)$, is given by

$$g(\tau) = -\frac{d}{d\tau} F(2h(\tau) - \tau), \quad (4)$$

where,

$$F(y) = \int_{-\infty}^{\infty} f(y') dy'. \quad (5)$$

and with $h(\tau)$ satisfying the two relations:

For radar applications, the first order relativistic approximation, which is the same order correction

$$h(\tau) + \frac{r(h(\tau))}{c} = \tau, \quad (6)$$

$$h(\tau) = t. \quad (7)$$

as the velocity, is sufficient for terrestrial applications, so the expression for the received spectrum for an arbitrary waveform is

$$g_1(\tau) = -\frac{d}{d\tau} F\left(\tau - \frac{2r(\tau)}{c}\right). \quad (8)$$

It is now possible to derive the AF from the expression for the waveform in (8).

For simplicity of formulation, we will assume $r(t=0) = 0$, so the delayed waveform is

$$g_1(t) = -\frac{d}{dt} F\left(t - \frac{2r(t)}{c} - \tau\right) \quad (9)$$

where $\tau = \frac{2R_0}{c}$. Then applying the definition of the AF as a matched filter response to the return signal enables the broadband AF for an arbitrary law of motion $r(x)$ to be written as

$$\Psi(\omega, \tau) = -\int_{-\infty}^{\infty} f(x) \frac{d}{dx} F^*\left(x - \frac{2r(x)}{c} - \tau\right) e^{-j\omega x} dx. \quad (10)$$

Note that for $r'(x) = \text{const.}$, the definition in (10) is the same as that in (3). The primary difference between (10) and previously mentioned results is the derivative in the integral. It is impractical to transform the broadband AF in (10) to the usual symmetric form. A further modification is necessary, the received waveform is not normalized for non-uniform motion. If we let E denote the energy of g_1 , the AF can be written as

$$\Psi_s(\omega, \tau) = \frac{1}{\sqrt{E}} \Psi(\omega, \tau) \quad (11)$$

It is evident from (10) that $r'(x) = \text{const.}$ gives the same AF for either case, hence the distinction 'a' for other than these two cases.

Let s denote the original signal and s' be the modified signal. The AF for the modified signal expressed in terms of the original signal is $\Lambda_s(\omega, \tau)$, which obeys the following:

1. Modulation Property:

$$s'(t) = s(t - \Delta) \rightarrow \Lambda_{s'}(\omega, \tau) = \Lambda_s(\omega, \tau) e^{j\omega\Delta}$$

(Note that this property does not hold for the AF $\Psi_s(\omega, \tau)$ since $r(x + \Delta) \neq r(x) + r(\Delta)$.)

2. Symmetry Property: $\Lambda(\omega, \tau) = \Lambda(-\omega, -\tau)$ (Note this property holds for $\Psi_s(\omega, \tau)$)

3. Maximum Property:

$|\Lambda(\omega, \tau)| \leq |\Lambda(0, 0)|$ (Note that this property holds for $\Psi_s(\omega, \tau)$)

4. Scaling Property: $s'(t) = s(at) \rightarrow$

$\Lambda_{s'}(\omega, \tau) = \frac{1}{a} \Lambda_s(\omega, \tau)$ (Note that this property does not hold for the AF $\Psi_s(\omega, \tau)$ since $r(ax) \neq ar(x)$. The one exception is for homogenous functions.)

5. Quadratic Phase Property: For

$$s'(t) = s(t) e^{j\pi a t^2} \rightarrow \Lambda_{s'}(\omega, \tau) = \Lambda_s(\omega, \omega - \pi\tau)$$

(Note that this property does not hold for $\Psi_s(\omega, \tau)$.)

6. Volume Property:

$\iint |\Lambda(\omega, \tau)|^2 d\omega d\tau = |\Lambda(0, 0)|^2$ (Note that this property holds for $\Psi_s(\omega, \tau)$.)

3. EXAMPLES OF WAVEFORM

We now consider a specific example. Examples of the frequency spectrum for different laws of motion and using a CW waveforms are contained in [7]. For a constant acceleration law of motion, the new AF for an arbitrary waveform becomes

$$\Psi_s(\omega, \tau) = \frac{1}{\sqrt{E}} \int_{-\infty}^{\infty} (2\gamma t - \beta_-) s(t) s^*(\beta_- t - \gamma t^2 - \tau) e^{-j\omega t} dt, \quad (12)$$

where $\beta_- = 1 - 2\beta$, $\gamma = a_0/c$, $\beta = v_0/c$ and the energy is

$$E = \int_{-\infty}^{\infty} (\beta_- - 4\gamma y)^{1/2} |g_1(y)|^2 dy \quad (13)$$

The example now considered is that of a modulated gaussian waveform with a functional dependence

$$s(x) = \left(\frac{a}{\pi}\right)^{1/4} e^{-ax^2/2 + j\omega_0 x} \quad (14)$$

The AF can be written as

$$\Psi_s(\omega, \tau) = \frac{1}{\sqrt{E}} \int_{-\infty}^{\infty} g(t) e^{j\omega_0 x(t)} dt, \quad (15)$$

where

$$g(x) = \frac{1}{\sqrt{E}} (2\gamma x - \beta) e^{(-\pi/2)(x^2 + (\beta x - \gamma x^2 - \tau)^2)}, \quad (16)$$

$$f(x) = \gamma x^2 - 2\beta x - \tau - \omega/\omega_0 x \quad (17)$$

There are several different approaches to evaluating the integral in (15), but the best method for the high frequency limit, which is good for all radar applications, is the method of stationary phase [14]. The solution to (15) is therefore

$$\Psi_+(\omega, \tau) \sim \sqrt{\frac{2\pi}{2\gamma\omega_0}} g(t_1) e^{j(\omega_0 f(t_1) - j\pi/4)}, \quad (18)$$

where $t_1 = (2\beta\omega_0 - \omega) / (2\gamma\omega_0)$. This is graphed in Figure I and Figure II for different values of a .

4. THE RECONSTRUCTION PROBLEM

An interesting problem to consider is: given a law of motion $r(x)$, what is the "best" waveform $f(x)$ to enable one to determine $r(x)$. Part of the problem is to determine what "best" means. There are two functional forms for $r(x)$ where the answer is believed to be known; $r(x) = r_0$ and $r(x) = bx$. The best waveform to determine the constant position is the pulse function that is ideally a delta function $f(x) = A\delta(x)$. The reason this appears to be the best waveform is that the effect of the law of motion on the waveform is a translation, i.e. $g(x) = A' \delta(x - r_0)$. This translation is easily detectable as a sharp peak at r_0 . For the constant velocity model, the best waveform is $f(x) = e^{j\omega_0 x}$, which when Fourier transformed appears as a translated delta function again. The reason the delta function is the ideal function for detection is that it takes all of the energy distributed in one domain and maps it into a narrow region in the co-domain of the dynamic variable. A working hypothesis is therefore: the best waveform is the one for which $r(x)$ appears as a translation of the incident waveform.

The problem with this hypothesis is that the properties of the new AF indicate that it isn't true. The translation and scaling properties do not hold for the new AF; therefore it is difficult to determine a waveform where $f(x)$ doesn't obey these properties while $F(x)$ does. An alternative approach

to the problem is to recast it as an integral problem: Given that we want to transform $F(x)$ so that $r(x)$ appears as a translated delta function in some co-domain; then we are trying to determine the solution to the integral equation given by

$$A(\tau) \delta(\sigma - \sigma_0) = \int \frac{d}{dx} F(x - \tau - \frac{2r(x)}{c}) K(x, \sigma) dx. \quad (19)$$

This is almost a Fredholm integral of the first kind, if we know apriori what $K(x, \sigma)$ should be. Since c is a large parameter compared to $r(x)$ in some interval $[a, b]$, we consider functions which are almost separable: those that have the property

$$F(x - \frac{r(x)}{c}) = F(x) g(\frac{r(x)}{c}) \quad (20)$$

If $F(x)$ obeys this property, then it can be removed from the integral equation in (19), by modifying the definition, so (19) becomes

$$A(\tau) \delta(\sigma - \sigma_0) = \int g'(-\frac{2r(x)}{c}) k(x, \sigma) dx. \quad (21)$$

Note that the gaussian waveform is an example of a weakly separable function.

An approach to solving (21) is to approximate $r(x)$ by a truncated Taylor series and assume the delta function becomes a finite series of delta functions:

$$A(\tau) \sum_{j=0}^n \delta(\sigma - \sigma_j) = \int g'(-\sum b_j x^j) k(x, \sigma) dx. \quad (22)$$

where the coefficients b_j are to be associated with the spacing of the delta functions σ_j . This problem has now been transformed into a wavelet problem, which will be discussed further elsewhere.

5. CONCLUSION

In this paper, we derived a new AF based on the electromagnetic treatment of non-uniform motion. An example of a gaussian waveform was produced. The problem of reconstructing the law of motion was considered, and shown to be

equivalent to a wavelet problem.

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Figure 1. $|\psi_a(\omega, \tau)|$ with $a = 1 \text{ g}$, $v = 400 \text{ m/s}$, and $\alpha = 10^{-9}$

Figure 2. $|\psi_a(\omega, \tau)|$ with $a = 10 \text{ g}$, $v = 400 \text{ m/s}$, and $\alpha = 10^{-9}$