Short Papers

An Error Bound for Near-Field Array Synthesis

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Abstract—An expression for the upper bound of any component of the electric or magnetic field at any point in a region is derived in terms of a product of two surface field integrals. The result is most useful for bounding errors in near-field array synthesis, but might have other applications where upper bounds on field magnitudes are desired.

Key Words—Array synthesis, electromagnetic fields, error bound, hybrid vector, near field, surface fields.

Index Code—I7d, I11e.

I. INTRODUCTION

A method for synthesizing a desired field distribution in the near field of a phased array has been described [1], [2] and applied to electromagnetic susceptibility testing [3]. The method makes use of the fact that the electromagnetic fields inside a closed surface are uniquely specified by the values of a particular hybrid vector F over the closed surface. Thus it is sufficient to synthesize the desired values of F over the closed surface rather than dealing with the desired electric or magnetic field throughout the enclosed volume. (The definition of F in terms of the tangential electric and magnetic fields is given in Section II.) The advantage of working with surface fields rather than volume fields is computational efficiency. If L is some characteristic length of the enclosed region, then surface sampling of F requires on the order of L^2 points, whereas volume sampling requires on the order of L^3 points.

In practical synthesis problems F cannot be synthesized exactly because the array has a finite number of elements. The synthesis procedure described in [1] and [2] chooses the element excitations to minimize the integral of the squared error in F over the closed surface. However, it would also be useful to know how the errors in the electric and magnetic fields within the volume are related to the surface errors in F . The purpose of this short paper is to derive a simple upper bound for the field errors within the volume in terms of the surface errors in F , and to show that the errors are of the same order. This result is important because it shows that the small surface field errors which result from realistic near-field array synthesis [1]–[3] do not introduce large errors in the test volume.

Although the application here is to near-field array synthesis, the results have possible application to other EMC problems where an upper bound of the fields in an interior region is desired [4]. The results in this short paper show that an upper bound for fields in a volume can be obtained from surface values of F . Also, van den Berg and de Hoop [5] have developed a similar upper bound for errors in scattering problems where the boundary conditions are only approximately satisfied.

II. DERIVATION OF ERROR BOUND

We consider a homogeneous lossless region enclosed by a surface S as shown in Fig. 1. Let δE and δH be electric and magnetic error fields which are produced by sources outside S . All sources and fields

Fig. 1. A homogeneous region V bounded by a closed surface S . An electric dipole test source p has arbitrary orientation and location.

are assumed to have $\exp(j\omega t)$ time dependence. Thus δE and δH satisfy the source-free Maxwell's equations for time-harmonic fields. The tangential hybrid vector δF is defined on S as

$$\delta F = \delta E_{\tan} - \eta \hat{n} \times \delta H \quad (1)$$

where $\eta = (\mu/\epsilon)^{1/2}$. The subscript $\tan$ denotes the tangential component, and $\hat{n}$ is the inward unit normal. μ and ϵ are the permeability and permittivity of the medium, respectively, and are real for the lossless case. Thus the intrinsic impedance η of the medium is also real.

In the previous near-field array synthesis procedure [1], [2], a discretized version of the integrated squared error in δF was minimized. Thus we define an error integral I_F :

$$I_F = \oint_S |\delta F|^2 dS = \oint_S \delta F \cdot \delta F^* dS \quad (2)$$

where $*$ denotes complex conjugate. By substituting (1) into (2) and applying some vector identities, we can express I_F as

$$I_F = \oint_S (|\delta E_{\tan}|^2 + \eta^2 |\delta H_{\tan}|^2) dS + \oint_S 2\eta \operatorname{Re} [\hat{n} \cdot (\delta E \times \delta H^*)] dS \quad (3)$$

where Re denotes the real part. The second integral in (3) must be zero because it is proportional to the real part of the Poynting vector integrated over a surface which encloses a lossless source-free region. Thus I_F in (3) can be written

$$I_F = \oint_S (|\delta E_{\tan}|^2 + \eta^2 |\delta H_{\tan}|^2) dS. \quad (4)$$

We now turn to the problem of relating the values of δE and δH inside S to the error integral I_F . In order to accomplish this, we introduce electric and magnetic test currents, J' and M' , inside S . These sources radiate electric and magnetic test fields, E' and H' , which can be used in the following reciprocity relation [6]:

$$I_S = I_V \quad (5)$$

where

$$I_V = \iiint_V (\delta E \cdot J' - \delta H \cdot M') dV$$

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and

$$I_S = \oint_S (\delta \mathbf{E} \times \mathbf{H}' - \mathbf{E}' \times \delta \mathbf{H}) \cdot \hat{n} \, dS.$$

The volume integral I_V is over the volume V enclosed by S . The particular choice of $\mathbf{J}'$ and $\mathbf{M}'$ in I_V depends on how we wish to test $\delta \mathbf{E}$ or $\delta \mathbf{H}$. In general, the evaluation of the surface integral I_S is rather tedious, but a useful upper bound can be obtained in the following manner.

With use of a vector identity, I_S can be written

$$I_S = \oint_S [\mathbf{H}' \cdot (\hat{n} \times \delta \mathbf{E}) + \mathbf{E}' \cdot (\hat{n} \times \delta \mathbf{H})] \, dS. \quad (6)$$

The absolute value of I_S then has the upper bound

$$|I_S| \leq \oint_S |\mathbf{H}'_{\tan}| |\delta \mathbf{E}_{\tan}| \, dS + \oint_S |\mathbf{E}'_{\tan}| |\delta \mathbf{H}_{\tan}| \, dS. \quad (7)$$

An upper bound can be obtained for each integral in (7) from Schwarz's inequality, so that $|I_S|$ can be written

$$|I_S| \leq \left[\oint_S |\mathbf{H}'_{\tan}|^2 \, dS \oint_S |\delta \mathbf{E}_{\tan}|^2 \, dS \right]^{1/2} + \left[\oint_S |\mathbf{E}'_{\tan}|^2 \, dS \oint_S |\delta \mathbf{H}_{\tan}|^2 \, dS \right]^{1/2}. \quad (8)$$

By squaring both sides of (8) and performing some further algebra, we obtain

$$|I_S|^2 \leq \oint_S (|\delta \mathbf{E}_{\tan}|^2 + \eta^2 |\delta \mathbf{H}_{\tan}|^2) \, dS \cdot \oint_S (|\mathbf{E}'_{\tan}|^2 / \eta^2 + |\mathbf{H}'_{\tan}|^2) \, dS. \quad (9)$$

The upper bound in (9) is particularly useful because it separates the integral of the error fields from the integral of the test fields. In addition, the error integral is exactly the quantity defined in (2) and (4) which is minimized in the synthesis procedure [1], [2]. Thus we choose to rewrite (9) as follows:

$$|I_S|^2 \leq I_F I_t \quad (10)$$

where

$$I_t = \oint_S (|\mathbf{E}'_{\tan}|^2 / \eta^2 + |\mathbf{H}'_{\tan}|^2) \, dS.$$

We turn now to the evaluation of I_V in (5). To test the electric field at a point $\mathbf{r} = \mathbf{r}'$, we choose the following test sources as shown in Fig. 1:

$$\mathbf{M} = 0$$

and

$$\mathbf{J}_t = \mathbf{p} \delta(\mathbf{r} - \mathbf{r}') \quad (11)$$

where $\mathbf{p}$ is an electric dipole moment and δ is a delta function. ($\mathbf{p}$ would be more properly termed a current moment because it has units of ampere·meters.) Substituting (11) into (5), we obtain the following for I_V :

$$I_V = \mathbf{p} \cdot \delta \mathbf{E}|_{\mathbf{r}=\mathbf{r}'}. \quad (12)$$

If we let the magnitude of $\mathbf{p}$ be unity, then I_V is simply the component of $\delta \mathbf{E}$ in the $\mathbf{p}$ direction evaluated at $\mathbf{r} = \mathbf{r}'$:

$$I_V = \delta E_p(\mathbf{r}'). \quad (13)$$

From (5), (10), and (13), we obtain the desired upper bound:

$$|\delta E_p(\mathbf{r}')|^2 \leq I_F I_t \quad (14)$$

where I_F is given by (2) or (4) and I_t is given by (10).

A number of observations can be made regarding the upper bound in (14). It is possible to test all three components of $\delta \mathbf{E}_p$ by using three orthogonal orientations of $\mathbf{p}$, and it is possible to test at any location by varying $\mathbf{r}'$ throughout V . These variations do not affect the error integral I_F , which depends only on $\delta \mathbf{F}$. Therefore, from (13), if $I_F = 0$, then $\delta \mathbf{E} = 0$ everywhere inside S . This is an alternate proof of the uniqueness property of $\mathbf{F}$ as shown in [1] and [2]. Also, (14) shows that the squared error $|\delta E_p|^2$ inside S is of the same order as the integral of the squared error $|\delta \mathbf{F}|^2$ on S . This property was observed numerically in [1] and [2] and experimentally in [3], but was not shown analytically. In order to obtain the constant of proportionality in (14), it would be necessary to evaluate I_t as given by (10). This integral involves the tangential components of the test fields $\mathbf{E}'$ and $\mathbf{H}'$ which are the well-known fields of a short electric dipole [5].

In a completely analogous manner, we can test the magnetic field by choosing the following test sources:

$$\mathbf{J}_t = 0$$

and

$$\mathbf{M}_t = \mathbf{m} \delta(\mathbf{r} - \mathbf{r}') \quad (15)$$

where $\mathbf{m}$ is a magnetic dipole moment with units of volt·meters. If $\mathbf{m}$ is chosen to have unity magnitude, then we obtain

$$|\delta H_m(\mathbf{r}')|^2 \leq I_F I_t \quad (16)$$

where δH_m is the component of $\delta \mathbf{H}$ in the $\mathbf{m}$ direction. In this case, I_t is evaluated by using the fields of a short magnetic dipole.

III. A TWO-DIMENSIONAL EXAMPLE

In this section, we consider a simple two-dimensional example to illustrate how the upper bound of the error field compares with the actual value. The surface S is a circular cylinder of radius a as shown in Fig. 2. The electric field is assumed to have only a z component which is independent of z . Thus the error field δE_z must satisfy the scalar Helmholtz equation. In order to simplify the required integrations, we choose δE_z to be just a single term with no azimuthal dependence:

$$\delta E_z(\rho) = c_0 J_0(k\rho) \quad (17)$$

where $k = \omega(\mu\epsilon)^{1/2}$, J_0 is a zero-order Bessel function, and c_0 is a constant. The corresponding magnetic field is obtained from Maxwell's curl equation:

$$\delta \mathbf{H} = \frac{j}{\omega\mu} \nabla \times \hat{z} \delta E_z = \hat{\phi} \delta H_\phi \quad (18)$$

where

$$\delta H_\phi = \frac{1}{j\omega\mu} \frac{\partial}{\partial \rho} (\delta E_z).$$

Substituting (17) into (18), we obtain

$$\delta H_\phi = \frac{c_0}{j\eta} J'_0(k\rho) = \frac{j c_0}{\eta} J_1(k\rho) \quad (19)$$

where we use the identity $J'_0(x) = J_1(x)$ [5].

Fig. 2. A circular cylindrical region of radius a . The error field δF has only a z component.

By substituting (17) and (19) into (1), we can obtain δF :

$$\begin{aligned}\delta F &= [\hat{z}\delta E_z - \eta(-\hat{\rho}) \times \hat{\phi}\delta H_\phi]_{\rho=a} \\ &= \hat{z}c_0[J_0(ka) - jJ_1(ka)].\end{aligned}\quad (20)$$

By substituting (20) into (2), we can obtain the error integral I_F

$$I_F = \int_0^{2\pi} |\delta F|^2 a d\phi = 2\pi a |c_0|^2 [J_0^2(ka) + J_1^2(ka)]. \quad (21)$$

In obtaining (21), we used the fact that both $J_0(ka)$ and $J_1(ka)$ are real for ka real.

For the test source, we choose an electric-line source of strength I_0 located at the origin ($\rho = 0$). Thus the volume integral I_V in (5) is simply

$$I_V = I_0 \delta E_z(0) = I_0 c_0. \quad (22)$$

The test fields of the line source are [6]

$$E^t = \hat{z}E_z^t$$

and

$$H^t = \hat{\phi}H_\phi^t \quad (23)$$

where

$$E_z^t = \frac{-I_0 k \eta}{4} H_0^{(2)}(k\rho)$$

and

$$H_\phi^t = \frac{I_0 k}{4j} H_1^{(2)}(k\rho).$$

$H_0^{(2)}$ and $H_1^{(2)}$ are the zero- and first-order Hankel functions of the second kind [7]. By substituting (23) into (9), we can obtain the integral I_t :

$$\begin{aligned}I_t &= \int_0^{2\pi} (|E_z^t|^2/\eta^2 + |H_\phi^t|^2)_{\rho=a} a d\phi \\ &= 2\pi a (k|I_0|/4)^2 [|H_0^{(2)}(ka)|^2 + |H_1^{(2)}(ka)|^2].\end{aligned}\quad (24)$$

Now that I_V , I_F , and I_t are evaluated in (22), (21), and (24), we can substitute the expressions in the inequality

$$|I_V|^2 \leq I_F I_t. \quad (25)$$

This substitution yields

$$|I_0|^2 |c_0|^2 \leq (2\pi a)^2 |I_0|^2 |c_0|^2 (k/4)^2 [J_0^2(ka) + J_1^2(ka)] \cdot [|H_0^{(2)}(ka)|^2 + |H_1^{(2)}(ka)|^2]. \quad (26)$$

We can rewrite (26) as

$$|\delta E_z(0)|^2 \leq \left(\frac{\pi ka}{2}\right)^2 [J_0^2(ka) + J_1^2(ka)] [|H_0^{(2)}(ka)|^2 + |H_1^{(2)}(ka)|^2] |\delta E_z(0)|^2. \quad (27)$$

The inequality in (27) must hold for all ka , but this is most easily seen for small or large values of ka . For the small ka limit, we have [7]

$$[J_0^2(ka) + J_1^2(ka)] \approx J_0^2(ka) \approx 1$$

and

$$[|H_0^{(2)}(ka)|^2 + |H_1^{(2)}(ka)|^2] \approx |H_1^{(2)}(ka)|^2 \approx \left(\frac{2}{\pi ka}\right)^2. \quad (28)$$

When we substitute (28) into (27), the $(ka)^{-2}$ factor in (28) cancels the $(ka)^2$ factor in (27), and we have

$$|\delta E_z(0)|^2 \leq |\delta E_z(0)|^2. \quad (29)$$

Thus the upper bound for $|\delta E_z|$ at $\rho = 0$ becomes an equality as ka approaches zero. For large ka , we have [7]

$$[J_0^2(ka) + J_1^2(ka)] \approx \left(\frac{2}{\pi ka}\right) [\sin^2(ka) + \cos^2(ka)] = \frac{2}{\pi ka}$$

and

$$[|H_0^{(2)}(ka)|^2 + |H_1^{(2)}(ka)|^2] \approx \left(\frac{2}{\pi ka}\right) (1 + 1) = \frac{4}{\pi ka}. \quad (30)$$

When we substitute (30) into (27), the pair of $(ka)^{-1}$ factors in (30) cancel the $(ka)^2$ factor in (27), and we have

$$|\delta E_z(0)|^2 \leq 2|\delta E_z(0)|^2. \quad (31)$$

Thus the upper bound for $|\delta E_z|^2$ at $\rho = 0$ is twice the actual value when ka is large. For most applications this is still a very useful upper bound.

IV. CONCLUSIONS

An expression for the upper bound of any component of the electric or magnetic field at any point in a specified region has been derived in terms of a product of surface field integrals $I_F I_t$, as shown in (10). The convenient feature of the product form is that I_F depends only on the hybrid field vector on the closed surface as shown in (2) and is calculated only once. The test integral I_t depends on what component of the electric or magnetic field is to be evaluated, but it normally involves only the fields of a simple test source, such as an electric or magnetic dipole. The results are particularly useful in evaluating errors in near-field array synthesis [1], [2] because I_F is the error field integral which is minimized and is already known. Numerical results for two- and three-dimensional geometries are shown in [1]–[3]. The results obtained here also validate the previous synthesis procedure because it is shown that a small value of I_F produces small error fields in the enclosed region of interest.

Although our application involves evaluation of errors within an enclosed region in near-field array synthesis, the results might have other applications where the upper bound of fields in a region is desired. Such an upper bound might be useful in aperture penetration

or other interference problems [4]. In such cases, two forms of the surface field integral are available. One involves the hybrid vector F as shown in (2), but the other form involving tangential electric and magnetic fields, as shown in (4), might be more useful.

The results were derived for homogeneous media, but the same technique can be applied to inhomogeneous media. The complication which arises there is that the fields of the test source required to evaluate I_t in (10) will be more difficult to obtain.

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A Simple Recursive Definition for Walsh Functions

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Abstract—This paper presents a simple recursive definition for Walsh functions, which overcomes the shortcomings of other recursive definitions. A rule of thumb for writing down the explicit representation for Walsh functions of any order is also devised. In addition, this recursive formula is used to generate discrete Walsh functions in matrix form.

Key Words—Walsh functions, recursive formula, matrix form.

Index Code—P1d.

I. INTRODUCTION

Walsh functions form a complete orthonormal set of rectangular waveforms taking only two values $+1$ and -1 . Since there are two-valued functions, they can be easily generated using digital circuits, and can be used in the analysis of waveforms and in the transmission of information digitally in the same way as the sinusoidal functions are used in analog signal processing. In signal processing, the Walsh transform proves to be much easier and faster than the Fourier transform, since its computation involves only additions and subtractions [1].

In the literature, different definitions of Walsh functions are presented. The first definition is in terms of a product of Rademacher functions [2]. The second definition is based upon recursive

relationships in the form of difference equations [3], [4]. The third definition is based upon the binary expansion of the arguments [5]. Each of the above definitions has its own advantages and disadvantages.

In this paper, we present a simple recursive definition for Walsh functions, which overcomes the shortcomings of other recursive relationships. A rule of thumb for writing down the explicit representation for a Walsh function of any order is devised. Generation of discrete Walsh functions using the above recursive definition is also discussed.

II. SIMPLE RECURSIVE DEFINITION

In [3], Harmuth defined Walsh functions in terms of a recursive relation as follows:

$$\text{Wal}(2m+p, \theta) = (-1)^{([m/2]+p)} \left\{ \text{Wal}\left(m, 2\left(\theta + \frac{1}{4}\right)\right) + (-1)^{m+p} \text{Wal}\left(m, 2\left(\theta - \frac{1}{4}\right)\right) \right\} \quad (1)$$

where $[X]$ is the largest integer smaller than or equal to X

$$p=0 \text{ or } 1$$

$$m=0, 1, 2, \dots,$$

and

$$\text{Wal}(0, \theta) = \begin{cases} 1, & \frac{1}{2} \leq \theta < \frac{3}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

In [4], a simpler recursive relation is used to replace (1). The periodic Walsh functions are generated by the following formula:

$$\text{Wal}(4m+p, \theta) = (-1)^{([p+1]/2)[2\theta]} \text{Wal}(2m+[p/2], 2\theta) \quad (2)$$

where

$$p=0, 1, 2, \text{ or } 3$$

$$m=0, 1, 2, \dots,$$

and

$$\text{Wal}(0, \theta) = 1, \quad -\infty < \theta < \infty.$$

The formulas given in (1) and (2) have two shortcomings. First, the order of the Walsh function is not given as an explicit number but as an expression of two integer numbers. Second, the argument of the Walsh functions on the right-hand side of (1) is shifted by $\pm 1/4$ and then scaled by a factor of 2, while in (2), it is scaled by a factor of 2. In this paper, we arrive at a simpler formula, which will overcome the above shortcomings. The proposed recursive relation is given as follows:

$$\text{Wal}(n, \theta) = (-1)^{[2^m \theta]} \text{Wal}(2^m - 1 - n, \theta) \quad (3)$$

where

$$m = 1 + [\log_2 n]$$

$$n = 0, 1, 2, \dots, \quad (4)$$

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