Transient Response of an Infinite Cylindrical Antenna in a Dissipative Medium

KENNETH C. CHEN

Abstract—The transient solution of an infinite cylindrical antenna in a dissipative medium caused by an impulse excitation at a delta gap is obtained via an exact solution in the form of definite integrals and a simple asymptotic formula. This formula is used to obtain an integral for calculating the antenna current caused by a double exponential input voltage at the gap; furthermore, the resulting integral reduces to Sunde's classic result under the diffusion limit. These results are applied to the transient response of a buried wire subject to an electromagnetic pulse (EMP) or nearby lightning incident wave.

I. INTRODUCTION

THE TRANSIENT RESPONSE of an infinite cylindrical antenna in free space has been a subject of considerable interest [1]–[4]. Recently, it was shown that a simple formula provides, with remarkable accuracy, the description of the overall current waveform for a unit-step-voltage input at a delta gap [4].

The subject of this paper is the transient response of an infinite cylindrical antenna in a dissipative medium. Since the dissipative medium can be the earth, ocean, or lake water, and the infinite cylindrical antenna can be cable or wire, the transient response solution is significant for many applications. Historically, the transient current on a buried wire has been related to the lightning problem and has been and still is of great concern to telephone companies. Sunde's book summarizes the underlying theory on electromagnetic coupling to cables [5]. More recent applications are the calculation of the current on a buried wire caused by an incident time-domain wave or a localized voltage source. The incident electromagnetic field can be due to an indirect lightning source, a high-altitude burst (HAB), or a surface burst (SB) electromagnetic pulse (EMP). The buried wire can be a telephone line, a power line, or a very low frequency (VLF) antenna. The problem can also arise from the need to determine the transient current on a long dipole buried in the earth or submerged in water.

Sunde's investigation was concerned mostly with the attachment of a lightning stroke on a buried wire. Sunde used a diffusion approximation to obtain the current propagation function to calculate the wire current as a function of distance from the transmission point. This result has been widely used

in treating the problem of electromagnetic coupling to buried wires [6], [7].

The first objective of this paper is to obtain an exact solution to the transient problem of an infinite antenna in a dissipative medium from Maxwell's equations, to examine the steps of approximations required to reduce the exact solution to Sunde's result, and to assess the errors introduced in each step. The second objective is to provide an alternative formula that is accurate and easy to use. The third objective is to show the use of the theories in applications.

An integral equation for antenna current is derived in Section II. In Section III, three different formulas for an exterior antenna current $I(z, t)$ are given: the exact integral formula, an alternative numerically convenient integral formula, and a numerically accurate asymptotic formula. The method used for obtaining the asymptotic formula is general and is a potentially powerful technique. In Section IV, an integral formula for the antenna current due to a double exponential input voltage is derived for constructing the solution. The diffusion limit is used to show that this solution can reduce to Sunde's classic result. Numerical investigations have been made elsewhere; the discrepancies between the two solutions are briefly summarized. Section V deals with the effect of an air/ground interface under the diffusion limit, and the result is pertinent to applications. Section VI gives a numerical example applicable to a typical wire problem under the excitation of an SBEMP or nearby lightning, as described earlier. Section VII briefly summarizes the induced buried wire cable current due to an HABEMP. Section VIII concludes this paper. Interested readers should consult [8] and [9], where detailed derivations and extensive numerical results are given. Crucial derivations are included in the appendices. Appendix I derives the numerically efficient integral for exterior antenna current. Appendix II evaluates relevant definite integrals and then uses these integrals to derive the asymptotic formula. Appendix III clarifies an important discrepancy in the early-time solution for the wire current in free space. Appendix IV shows a rigorous transmission line solution. Appendix V gives a time-harmonic solution for currents on an infinite wire in a dissipative medium.

II. THE GOVERNING INTEGRAL EQUATION

It is well known that Maxwell's equations can be solved by scalar and vector potentials. Let

$$\mathcal{E} = -\frac{\partial \mathcal{Q}}{\partial t} - \nabla \phi \quad (1)$$

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The author is with the Electromagnetic Analysis Division, Sandia National Laboratories, P. O. Box 5800, Albuquerque, NM 87185.

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Fig. 1. Infinite tubular antenna with a gap generator embedded in a dissipative medium.

and the Lorentz gauge $\mathcal{Q}$ and ϕ are shown to be

$$\nabla^2 \mathcal{Q} - \frac{1}{c^2} \frac{\partial^2 \mathcal{Q}}{\partial t^2} - \sigma \mu \frac{\partial \mathcal{Q}}{\partial t} = -\mu \mathcal{J}. \quad (2)$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \sigma \mu \frac{\partial \phi}{\partial t} = -\frac{\rho}{\epsilon}. \quad (3)$$

and

$$\nabla \cdot \mathcal{Q} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} + \mu \sigma \phi = 0. \quad (4)$$

Equations (1)–(4) can be used to derive the governing equation for the current on an infinite antenna in a dissipative medium with parameters σ , μ , and ϵ .

Consider an infinite tubular antenna (with radius a) driven by a delta function voltage in time and a delta gap at the origin (Fig. 1) where

$$\mathcal{E}_z(a, \theta, z, t) = -V \delta(z) \delta(t). \quad (5)$$

An equation for $\mathcal{Q}_z$ on the antenna surface can be derived from (1) and (4) to give

$$\frac{\partial^2 \mathcal{Q}_z}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \mathcal{Q}_z}{\partial t^2} - \mu \sigma \frac{\partial \mathcal{Q}_z}{\partial t} = \left(\frac{1}{c^2} \frac{\partial}{\partial t} + \mu \sigma \right) \mathcal{E}_z. \quad (6)$$

Equation (2) has only the z -component and can be solved as

$$\mathcal{Q}_z(r, t) = \mu \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} \mathcal{G}(z', t') dt' \mathcal{K}(z - z', t - t') \quad (7)$$

where $\mathcal{K}(z - z', t)$ is defined in terms of Green's function $\mathcal{G}(\vec{r} - \vec{r}', t)$ for (2) as

$$\mathcal{K}(z - z', t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{G}(\vec{r} - \vec{r}', t) d\theta \quad (8)$$

and whose properties are listed below:

$$K(k, \omega) = \frac{i}{4} J_0(a\sqrt{k_1^2 - k^2}) H_0^{(1)}(a\sqrt{k_1^2 - k^2}), \quad (9)$$

for $\omega > 0$, any real k ,

where

$$k_1 = \left(\frac{\omega^2}{c^2} + i\omega\mu\sigma \right)^{1/2}, \quad (10)$$

and

$$K(\pm k, -\omega) = [K(\mp k, \omega)]^* \quad (11)$$

$$K(\pm r, -\omega) = [K(\mp r, \omega)]^* \quad (12)$$

where the asterisk is the complex conjugate.

Notice that (11) and (12) extend (9) to the domain where $\omega < 0$.

A complete integral equation for $\mathcal{G}(z, t)$ is obtained by combining (6), (7), and (8) as

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \mu \sigma \frac{\partial}{\partial t} \right) \int_{-\infty}^{\infty} dz' \cdot \int_{-\infty}^{\infty} \mathcal{G}(z', t') \mathcal{K}(z - z', t - t') dt' = - \left(\epsilon \frac{\partial}{\partial t} + \sigma \right) V \delta(z) \delta(t). \quad (13)$$

III. FORMULAS FOR $I(z, t)$

Define $I(z, t)$ as a solution to

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \mu \sigma \frac{\partial}{\partial t} \right) \int_{-\infty}^{\infty} dz' \cdot \int_{-\infty}^{\infty} dt' I(z', t') \mathcal{K}(z - z', t - t') = -\epsilon \delta(t) \delta(z). \quad (14)$$

Then, the solution to (13) is given by

$$\mathcal{G}(z, t) = V \left(\frac{\partial}{\partial t} + \frac{\sigma}{\epsilon} \right) I(z, t) \quad (15)$$

and

$$I(z, t) = -(2\pi)^{-2} \epsilon \int_{C_\omega} d\omega \int_{-\infty}^{\infty} dk [c^{-2}\omega^2 + i\sigma\mu\omega - k^2]^{-1} \cdot [K(k, \omega)]^{-1} \times \exp[i(kz - \omega t)] \quad (16)$$

where C_ω is the Bromwich contour, which is just above the real ω -axis, and $K(k, \omega)$ is given in (9). The simplest way to integrate (16) is to introduce the transformation $\xi = \sqrt{(\omega + i\sigma/\epsilon)^2 - k^2/c^2}$, change the variable of integration from k to ξ , interchange the order of integration, and then carry out the ω -integration. The result is

$$I(z, t) = \frac{4}{\pi \xi_0} e^{-(\sigma t/2\epsilon) R_e} \left[\left\{ - \int_0^{i\sigma/2\epsilon} + \int_0^\infty \right\} \left[\xi^2 + \left(\frac{\sigma}{2\epsilon} \right)^2 \right]^{-1} \cdot \left\{ J_0 \left[a \sqrt{\xi^2 + \left(\frac{\sigma}{2\epsilon} \right)^2 / c} \right] \cdot H_0^{(1)} \left[a \sqrt{\xi^2 + \left(\frac{\sigma}{2\epsilon} \right)^2 / c} \right] \right\}^{-1} J_0(\xi \tau_1) \xi d\xi \right] \quad (17)$$

for

$$\tau_1 = \sqrt{t^2 - \xi^2/c^2} > 0$$

where

$$\xi_0 = \sqrt{\mu/\epsilon}$$

and zero otherwise.

Equation (17) can be separated into three parts. The first part I_1 is the branch cut integration and is the dominant contribution to the exterior current for late time. The second part I_2 , is the integration along the real axis and contributes to the exterior current. The third part I_3 is the contribution due to poles located at zeroes of J_0 . Defining $\tau = \tau_1 c/a$ and $\alpha = \sigma a/2\epsilon c$, it is possible to express I_1 , I_2 , and I_3 as follows:

$$I_1 = \frac{4}{\pi \xi_0} e^{-(\sigma t/2\epsilon)} R_e \left[\int_0^{\sigma/2\epsilon} \left[\xi^2 + \left(\frac{\sigma}{2\epsilon} \right)^2 \right]^{-1} \cdot \left\{ J_0 \left[a \sqrt{\xi^2 + \left(\frac{\sigma}{2\epsilon} \right)^2 / c} \right] \cdot H_0^{(1)} \left[a \sqrt{\xi^2 + \left(\frac{\sigma}{2\epsilon} \right)^2 / c} \right] \right\}^{-1} J_0(\xi \tau_1) \xi d\xi \right] \\ = \frac{4}{\pi \xi_0} e^{-(\sigma t/2\epsilon)} \int_0^\alpha \frac{d\eta}{\eta} I_0(\tau \sqrt{\alpha^2 - \eta^2}) \{ J_0(\eta)^2 + Y_0(\eta)^2 \}^{-1} \quad (18)$$

$$I_2 = \frac{4}{\pi \xi_0} e^{-(\sigma t/2\epsilon)} R_e \left[\int_0^\infty \left[\xi^2 + \left(\frac{\sigma}{2\epsilon} \right)^2 \right]^{-1} \cdot \left\{ J_0 \left[a \sqrt{\xi^2 + \left(\frac{\sigma}{2\epsilon} \right)^2 / c} \right] \cdot H_0^{(1)} \left[a \sqrt{\xi^2 + \left(\frac{\sigma}{2\epsilon} \right)^2 / c} \right] \right\}^{-1} I_0(\xi \tau_1) \xi d\xi \right] \\ = \frac{4}{\pi \xi_0} e^{-(\sigma t/2\epsilon)} \int_\alpha^\infty \frac{d\eta}{\eta} J_0(\sqrt{\eta^2 - \alpha^2} \tau) \{ J_0(\eta)^2 + Y_0(\eta)^2 \}^{-1} \quad (19)$$

and

$$I_3 = -\frac{4}{\xi_0} e^{-(\sigma t/2\epsilon)} \sum_{j=1}^{\infty} \frac{J_0[\sqrt{\eta_j^2 - \alpha^2} \tau]}{\eta_j J_0'(\eta_j) Y_0(\eta_j)} \quad (20)$$

where η_j satisfies $J_0(\eta_j) = 0$.

Equation (20) denotes interior wave resonant modes and will be omitted in the ensuing discussion. Therefore,

$$I(z, t) = I_1 + I_2 \quad (21)$$

with I_1 and I_2 given by (18) and (19).

To improve the numerical convergence of (19), an approxi-

mate formula for I_2 is derived in Appendix I and is given by

$$I_2 \approx \frac{2}{\pi \xi_0} e^{-(\sigma t/2\epsilon)} \left\{ \ln \frac{\sqrt{\pi^2 + \left[\ln \left(\frac{\alpha_0}{2} \right) + \gamma \right]^2}}{- \left[\ln \left(\frac{\alpha_0}{2} \right) + \gamma \right]} \cdot K_0(\tau \sqrt{\alpha^2 + \alpha_0^2}) + \pi^2 \int_{\alpha_0}^\infty \frac{d\eta}{\eta} \frac{I_0(\eta)}{K_0(\eta)} \frac{K_0(\tau \sqrt{\alpha^2 + \eta^2})}{\{ [K_0(\eta)]^2 + \pi^2 [I_0(\eta)]^2 \}} \right\} \quad (22)$$

where $\alpha_0 \ll \alpha$.

Appendix II shows that an asymptotic formula for $I(z, t)$ as given in (21) is

$$I(z, t) \sim \frac{2}{\xi_0} e^{-(\sigma t/2\epsilon)} I_0(\alpha \tau) \cdot \arctan \left[\frac{-\pi}{\ln \left(\frac{\alpha}{\tau} \right) + \frac{K_0(\alpha \tau)}{I_0(\alpha \tau)} - \ln 2 + \gamma} \right] \quad (23)$$

When the bracket inside arctan is negative, the functional value of arctan is chosen such that (23) is a continuous function of τ .

Table I compares $I(z, t)$ as calculated from (18) and (22) and that calculated from (23). Notice that τ is in the units of antenna radius. For an antenna radius of 1 cm, this unit is approximately 1/30 ns in time. At $\tau \sim 1$ ns ($\tau = 30$ in Table I), the discrepancy of the two formulas is less than one-tenth of a percent. Therefore, unless the risetime of input voltage pulse is much less than 1 ns, (23) is all that is required for numerical computations. Finally, when $\sigma = 0$, or $\alpha = 0$, (19), (22), and (23) reduce to those for the lossless case [3], [4].

IV. AN INTEGRAL FORMULA FOR OBTAINING THE ANTENNA CURRENT CAUSED BY A DOUBLE EXPONENTIAL INPUT VOLTAGE

When the input voltage is

$$V_0(t) = e^{-t/t_0} u(t) \quad (24)$$

where $u(t)$ is the unit step function, the method of superposition can be applied to calculate the corresponding current:

$$g_0(z, t) = \int_{-\infty}^\infty \left[\left(\frac{\partial}{\partial t'} + \frac{\sigma}{\epsilon} \right) I(z, t') \right] \cdot e^{-(t-t')/t_0} u(t-t') dt' \quad (25)$$

Normalizing the variables in (25) to the unit of wire radius $t_n = ct/a$, $t_{0n} = ct_0/a$, and $z_n = z/a$, and evaluating the resulting integral by parts, yields

$$g_0 = I(z, t) + (2\alpha - t_{0n}^{-1}) \int_0^{t_n} I(z', t) e^{-(t_n - t')/t_{0n}} dt' \quad (26)$$

TABLE I
COMPARISON IN MILLIAMPERES OF A AND B FOR VARYING VALUES OF
 α , WHERE A IS OBTAINED FROM (18) AND (22) AND B IS OBTAINED FROM
(23); ENTRIES SHOWN ARE $I_N(\tau) \equiv I(z, t) \exp \alpha t / 2\epsilon - \alpha \tau$

τ	10^{-5}		10^{-4}		10^{-3}		10^{-2}	
	A	B	A	B	A	B	A	B
1.00	8.99621	8.33901	8.99540	8.33826	8.98732	8.33076	8.70306	8.25628
1.25	7.85090	7.58985	7.85002	7.58900	7.84120	7.58047	7.55037	7.49602
1.50	7.06864	6.99791	7.06769	6.99697	7.05816	6.98753	6.76104	6.89430
1.75	6.49615	6.52207	6.49513	6.52104	6.48492	6.51079	6.18174	6.40964
2.00	6.05635	6.13277	6.05527	6.13167	6.04439	6.12065	5.73536	6.01220
2.25	5.70616	5.80895	5.70500	5.80778	5.69348	5.79604	5.37879	5.68072
2.50	5.41950	5.53552	5.41829	5.53427	5.40613	5.52184	5.08595	5.40001
2.75	5.17969	5.30150	5.17841	5.30019	5.16563	5.28710	4.84012	5.15904
3.00	4.97549	5.09884	4.97415	5.09747	4.96075	5.08374	4.63005	4.94966
3.50	4.64472	4.76483	4.64326	4.76333	4.62868	4.74837	4.28794	4.60283
4.00	4.38667	4.50017	4.38509	4.49855	4.36936	4.48241	4.01903	4.32599
5.00	4.00588	4.10484	4.00408	4.10299	3.98614	4.08460	3.61773	3.90777
7.50	3.44329	3.51509	3.44097	3.51272	3.41788	3.48914	3.00912	3.26677
10.00	3.12213	3.17793	3.11933	3.17507	3.09148	3.14671	2.64742	2.88421
12.00	2.94492	2.99230	2.94175	2.98907	2.91028	2.95709	2.44073	2.68536
15.00	2.75124	2.79000	2.74753	2.78624	2.71088	2.74906	2.20675	2.41733
20.00	2.53331	2.56330	2.52876	2.65869	2.48394	2.51333	1.92991	2.12311
30.00	2.27510	2.29619	2.26898	2.29001	2.20912	2.22957	1.57653	1.74508
50.00	2.01221	2.02603	2.00319	2.01695	1.91641	1.92953	1.18189	1.31668
75.00	1.84097	1.85105	1.82862	1.83862	1.71197	1.72126	0.91106	1.01649
100.00	1.73524	1.74337	1.71974	1.72780	1.57612	1.58339	0.74685	0.83095
150.00	1.60426	1.61035	1.58284	1.58885	1.39157	1.39667	0.56092	0.61681
250.00	1.46343	1.46776	1.43112	1.43535	1.16238	1.16548	0.39925	0.42741
500.00	1.30496	1.30778	1.24839	1.25107	0.85064	0.85174	0.26484	0.27258
1000.00	1.17335	1.17524	1.07524	1.07694	0.56179	0.56138	0.17847	0.17906

Notice that $I(z, t)$ is given in the previous sections. One special case is of great interest. If $t_{0n} = (2\alpha)^{-1}$, then $g_0 \equiv I(z, t)$. This is the wire current due to an input voltage of

$$\mathcal{V}_0(t) = e^{-2\alpha t} u(t). \quad (27)$$

In free space, the input voltage reduces to a unit step function. This is the case discussed extensively in the literature [1], [3], [14].

The input voltage ((27)) contains a discontinuity at $t = 0$. This gives rise to the singularity in the current $I(z, t)$ at $ct - z = 0$ (Appendix III). In applications, the input voltage does not contain a discontinuity. In the following, a construction of the solution is shown for this case. Begin with (26), which is the current due to an input voltage of (24). The first term on the right side has been shown to be the response current to the voltage given by (27). Therefore, the second term on the right side of (26) is the current due to the input voltage of

$$\mathcal{V}(t) = (e^{-t/t_0} - e^{-\sigma t/\epsilon}) u(t) = (e^{-t_n t_{0n}} - e^{-2\alpha t_n}) u(t_n). \quad (28)$$

The corresponding current response is

$$\begin{aligned} g(z, t) &= \left(\frac{\sigma}{\epsilon} - t_0^{-1} \right) \int_0^t I(z, t') e^{-(t-t')/t_0} dt' \\ &= (2\alpha - t_{0n}^{-1}) \int_0^t \exp [-(t_n - \sqrt{\tau'^2 + z_n^2})/t_{0n}] \\ &\quad \cdot I(z, t') \frac{\tau' d\tau'}{\sqrt{\tau'^2 + z_n^2}} \end{aligned} \quad (29)$$

$$t'_n = \sqrt{\tau'^2 + z_n^2}.$$

The wavefront singularity of $I(z, t')$ is shown to be of $1/\tau'$ in Appendix III. Since it is multiplied by τ' in the integrand, it contributes negligibly to the integral. Therefore, (23) can be used for determining the numerical values of the wire current.

To obtain the current response for a double exponential input voltage, e.g.,

$$\mathcal{V}(t) = (e^{-t/t_f} - e^{-t/t_r}) u(t), \quad (30)$$

write (30) as

$$\mathcal{V}(t) = (e^{-t/t_f} - e^{-\sigma t/\epsilon}) u(t) - (e^{-t/t_r} - e^{-\sigma t/\epsilon}) u(t). \quad (31)$$

As a result, the corresponding current response can be constructed easily from (23). A simple computer program using (23) can be written to generate the wire current for any input voltage at the gap. However, the diffusion-limiting case of (23) plays an important role in applications. Universal curves for obtaining the current waveforms for this case are given below.

The diffusion limit of (23) for $I(z, t)$ yields accurate numerical results when two conditions are met. The first is that $I_0(\alpha\tau)$ be replaced by its leading asymptotic term. When the argument $\alpha\tau = 5$, the asymptote deviates less than 1 percent from $I_0(\alpha\tau)$. The second condition is that $t \gg z/c$; when $t = 3z/c$, $(t - z^2/2c^2t)$ deviates less than 0.5 percent from $\sqrt{t^2 - z^2/c^2}$. The arctangent function can also be replaced by its argument with minimum errors introduced.

Next, (23) is evaluated with $I(z, t)$ approximated by (12):

$$g(z, t) \sim (\sigma/\epsilon - t_0^{-1}) 2\epsilon(\pi/\mu\sigma)^{1/2} \int_0^t \frac{e^{-(z^2\sigma\mu/4(t-t')) - t'/t_0} dt'}{(t-t')^{1/2} \left\{ \ln \left[\frac{2(t-t')}{\sigma\mu a^2} \right] + \ln 2 - \gamma \right\}} \quad (32)$$

The second factor in the denominator of the above integrand is the impedance factor that is slowly varying and can be taken out of the integral. Thus,

$$g(z, t) \sim \frac{(\sigma/\epsilon - t_0^{-1}) 4\epsilon \left(\frac{\pi t_0}{\mu\sigma} \right)^{1/2} g}{\ln(2t/\sigma\mu a^2) + \ln 2 - \gamma} = \frac{4\sqrt{\pi(t_0 - t_{rel})} g}{\zeta_0(t_0 t_{rel})^{1/2} [\ln(2t/\sigma\mu a^2) + \ln 2 - \gamma]} \quad (33)$$

where

$$g = g(z, t) = e^{-t/t_0} \int_0^{\sqrt{t/t_0}} e^{-A/u^2 + u^2} du, \quad (34)$$

$$A = \frac{z^2\sigma\mu}{4t_0} = \frac{z^2}{4(ct_0)(ct_{rel})} = \frac{z_n^2}{2t_{0n}\alpha^{-1}},$$

and

$\epsilon/\sigma = t_{rel}$ = the relaxation time of the medium.

Notice that the last expression for A is in a normalized notation.

In Fig. 2, $g(z, t)$ is shown as a function of t/t_0 for $A = 10^{-4}, 10^{-2}, 10^{-1}, 1, 10$, and 100 . Similar curves were given by Sunde when he considered the propagating function for the current in a buried wire [6]. An extensive numerical comparison of the exact solution and Fig. 2 has been made [9]. The basic conclusion is that the early-time waveform cannot be obtained by a diffusion approximation such as that given by Fig. 2. However, peak values obtained from Fig. 2 are found to be not more than 15 percent less than that obtained from the present theory ((23)). When (32) is used to calculate the peak values, the deviations are found to be less than a few percent. Therefore, an accurate diffusion approximation is to include also the impedance variation as a function of time. Current waveforms obtained by the present theory are given in [9]. Since the arrival time for the wire current is z/c , approximate early-time waveforms can be obtained by interpolating from the arrival times and the peak values.

V. THE EFFECT OF AN AIR/GROUND INTERFACE

Since the diffusion limit of the wire current is shown to give a good approximate solution, the diffusion approximation can be pursued further. The effect of the air/ground interface is taken into account under that limit (Figs. 3(a) and 3(b)) by using an image consideration.

Appendix IV shows that a rigorous transmission line theory

Fig. 2. Universal curves for wire currents due to a localized voltage source. These curves are obtained by a diffusion limit.

Fig. 3. (a) Geometry of a buried wire with an air/ground interface. (b) Side view of a buried wire with its image.

can be obtained by considering a radial electric field of

$$E_\rho(\rho, z, t) = \frac{\partial I / \partial z}{2\pi\epsilon\rho} e^{-\rho^2/2\delta^2} \quad (35)$$

where I is given by (72). The electric field in the dissipative half-space can now be obtained by adding that due to the wire ((35)) and that due to the image located in the air as shown in Fig. 3(b). As in the case without interface, the transmission line voltage can be defined by integrating the electric field from the wire surface to infinity. The integration can be carried out most easily along the x -axis or the y -axis;

therefore, if $\mathcal{V}_0(z, t)$ is denoted as the voltage for the case when the interface is not present, the image term disappears.

$$\begin{aligned}\mathcal{V}_0(z, t) &= - \int_a^\infty E_\rho(\rho, z, t) d\rho \\ &= \frac{-1}{2\pi\epsilon} \frac{\partial I}{\partial z} E_1\left(\frac{a^2}{2\delta^2}\right) \\ &\sim \frac{1}{2\pi\epsilon} \frac{\partial I}{\partial z} \left[2 \ln \frac{\delta}{a} + \ln 2 - \gamma \right] \quad (36)\end{aligned}$$

where E_1 is the exponential integral. When the interface is present and the image is included, the transmission line voltage denoted by $\mathcal{V}(z, t)$ is

$$\begin{aligned}\mathcal{V}(z, t) &= - \int_a^\infty E_y(y, z, t) dy - \int_{2d+a}^\infty E_y(y, z, t) dy \\ &= - \frac{1}{2\pi\epsilon} \frac{\partial I}{\partial z} \left\{ E_1\left(\frac{a^2}{2\delta^2}\right) + E_1\left[\frac{(2d+a)^2}{2\delta^2}\right] \right\} \\ &\sim - \frac{1}{2\pi\epsilon} \frac{\partial I}{\partial z} \left[2 \ln \frac{\delta}{a} + \ln 2 - \gamma \right. \\ &\quad \left. + 2 \ln \frac{\delta}{2d} + \ln 2 - \gamma \right] \quad (37)\end{aligned}$$

The last step follows, because in considering a signal with a characteristic early time of 10^{-6} s and a characteristic late time of 10^{-3} s, the diffusion depth δ is found to be about 30 and 1000 m, respectively, for $\sigma = 10^{-3}$ S/m. Thus, a small argument approximation of E_1 is allowed. The ratio of $\mathcal{V}$ to $\mathcal{V}_0$, as given in (36) and (37), is the ratio of characteristic impedance for the two cases. The wire current is seen to be inversely proportional to this ratio, which is

$$R = \frac{2 \left[\ln \frac{\delta}{a} + \ln \frac{\delta}{2d} + \ln 2 - \gamma \right]}{2 \ln \frac{\delta}{a} + \ln 2 - \gamma} \quad (38)$$

Notice a small argument approximation is used in deriving (37) and (38). Therefore, (38) can only be used for $d/\delta \ll 1$, which is consistent with the late-time limit. When $\ln \delta/a \gg \ln 2d/a$, $R \sim 2$, the equivalent transmission line for the case with interface has half the shunt conductance of the case without the interface. When $\ln \delta/a \gg \ln \delta/2d$, $R \sim 1$, the interface has no effect on the wire current. This is the case described in most of the literature (see, for example, [6]).

Therefore, when the interface effect is included, the current on a wire buried d m from the interface and under an input voltage of (28) at a delta gap at origin is given by

$$g(z, t) \sim \frac{4\sqrt{\pi}(t_0 - t_{rel})g(z, t)}{\zeta_0(t_0 t_{rel})^{1/2} \left[\ln \frac{2t}{\sigma\mu a^2} + \ln \frac{2t}{\sigma\mu d^2} + 2 \ln 2 - 2\gamma \right]} \quad (39)$$

Fig. 4. The relative dielectric constant ϵ_r versus frequency for 10 percent by volume of water content (from [10]).

where $g(z, t)$ is given by (34). Equation (39) should be used only after $t > 2d/c$, since the reflected wave from the interface does not affect the wire current before that time.

VI. THE CURRENT FOR A WIRE BURIED IN A SOIL WITH A REALISTIC DIELECTRIC CONSTANT

Consider the problem of determining the current for a wire with a radius $a = 10^{-2}$ m buried 1 m below the air/ground interface when an input voltage is impressed at the gap (Figs. 3(a) and 3(b)). Assume the input voltage to be

$$\mathcal{V}(t) = (e^{-t/t_f} - e^{-t/t_r})u(t) \quad (40)$$

where

$$t_f = 10^{-3} \text{ s}$$

$$t_r = 10^{-6} \text{ s}.$$

The relative dielectric constant (Fig. 4) is more realistic than the dielectric constant used before [10]. The theoretical reasoning for its variation is given in [11]. The ground conductivity is assumed to be

$$\sigma = 2 \times 10^{-3} \text{ S/m}.$$

Due to the wide variation of the relative dielectric constant over the whole frequency range, it is necessary to decompose the input voltage into a superposition of many double exponential waveforms, so that each double exponential waveform is applied to the wire buried in an assumed medium with a uniform relative dielectric constant. The larger the time constant, the larger the response current. Therefore, begin with the decay portion of the double exponential. The relative

dielectric constant corresponding to $t_f = 10^{-3}$ s (or $f_f = 1/2\pi t_f \sim 160$ Hz) is $\epsilon_{rf} = 10^4$. To have an input voltage without discontinuity, let the dominating voltage be

$$\mathcal{V}_1(t) = (e^{-t/t_f} - e^{-t/t_1})u(t) \quad (41)$$

where

$$t_1 = \epsilon_f/\sigma = 4.428 \times 10^{-5} \text{ s.}$$

Next, begin with the second term in (41). The relative dielectric constant corresponding to $t_1 = 4.428 \times 10^{-5}$ ($f_1 \sim 3.6 \times 10^3$) is $\epsilon_{r1} = 10^3$. Thus, the next voltage to consider is

$$\mathcal{V}_2(t) = (e^{-t/t_1} - e^{-t/t_2})u(t) \quad (42)$$

where

$$t_2 = \epsilon_1/\sigma = 4.428 \times 10^{-6} \text{ s.}$$

Proceeding as before, the relative dielectric constant corresponding to t_2 is $\epsilon_{r2} = 200$, and the voltage to use is

$$\mathcal{V}_3(t) = (e^{-t/t_2} - e^{-t/t_3})u(t) \quad (43)$$

where

$$t_3 = \epsilon_2/\sigma = 8.855 \times 10^{-7} \text{ s.}$$

Notice that the input voltage $\mathcal{V}(t)$ can be approximated by

$$\mathcal{V}(t) \sim \mathcal{V}_1(t) + \mathcal{V}_2(t) + \mathcal{V}_3(t).$$

The current is the sum of the individual current due to $\mathcal{V}_i(t)$ in its corresponding medium ϵ_i and σ . The total wire current in amperes, which includes the interface effect for an input voltage given by (40) at the gap, is shown in Fig. 5. The figure shows the wire currents at locations $z = 100, 316, 1000$, and 3160 m from the source. The waveforms obtained are found to give very accurate results, especially for $z \geq 1000$ m. The errors introduced by the use of a superposition procedure and a late-time interface correction factor are less than a few percent.

VII. THE CURRENT FOR A WIRE BURIED BELOW AN AIR/GROUND INTERFACE AND UNDER A NORMAL INCIDENT HABEMP

Consider the problem of calculating the current on a wire with a wire radius $a = 10^{-2}$ buried 1 m below the air/ground interface, when an electric field of

$$E(t) = (e^{-t/t_f} - e^{-t/t_r})u(t) \quad (44)$$

where

$$t_f = 2.5 \times 10^{-7} \text{ s}$$

$$t_r = 10^{-8} \text{ s}$$

is normally incident on the interface.

This problem can be solved via the frequency domain by using the time-harmonic formula derived in Appendix V. To begin, all the time-harmonic transfer functions are listed. First, the transform of (43) is

$$\tilde{E}(\omega) = \frac{1}{-i\omega + t_f^{-1}} - \frac{1}{-i\omega + t_r^{-1}}.$$

Fig. 5. Wire currents for an input voltage at the gap given by (40). The location for the given currents is shown in the figure; from the left $z = 100, 316, 1000$, and 3160 m. The wire configuration is shown in Figs. 3(a) and 3(b).

The transmission coefficient of a normal incident wave from the air to the earth with ϵ and σ as its dielectric permittivity and conductivity, respectively, is

$$\tilde{T}(\omega) = \frac{2(\epsilon_r - \sigma/i\omega\epsilon_0)^{1/2}}{\epsilon_r - \sigma/i\omega\epsilon_0 + (\epsilon_r - \sigma/i\omega\epsilon_0)^{1/2}} \quad (45)$$

$$\sim 2\epsilon_r^{-1/2}(-i\omega)^{1/2}(-i\omega + \sigma/\epsilon)^{-1/2}. \quad (46)$$

The frequency-domain transfer function from just below the interface to a depth of d from the interface is

$$\tilde{T}_d(\omega) = e^{-ikd},$$

$$-ikc = [(-i\omega)(-i\omega + \sigma/\epsilon)]^{1/2}. \quad (47)$$

The wire current transfer function due to a localized voltage $E dz$ at z distance away is (Appendix V)

$$\tilde{J}(z, \omega) = (-i\omega + \sigma/\epsilon) \cdot (\text{equation (90)}). \quad (48)$$

The total wire current in the frequency domain due to a normal incident electric field is

$$\begin{aligned} \tilde{J}_T(\omega) &= \frac{4c}{\sqrt{\epsilon_r}} \left(\int_{-\infty}^0 dz - \int_0^{\infty} dz \right) e^{-ik(d+z)} \\ &\quad \cdot \arctan \left[\frac{-\pi}{\ln \frac{\alpha}{\langle \tau \rangle} + \frac{K_0(\alpha\langle \tau \rangle)}{I_0(\alpha\langle \tau \rangle)} - \ln 2 + \gamma} \right] \tilde{E}(\omega) dz \\ &= \frac{4c}{\sqrt{\epsilon_r}} \tilde{E}(\omega) \left\{ \frac{2e^{-ikd}}{\sqrt{\epsilon_r} ik} \right. \\ &\quad \cdot \arctan \left[\frac{-\pi}{\ln \frac{\alpha}{\langle \tau \rangle} + \frac{K_0(\alpha\langle \tau \rangle)}{I_0(\alpha\langle \tau \rangle)} - \ln 2 + \gamma} \right] \left. \right\}. \quad (49) \end{aligned}$$

The time-domain function for the frequency-domain function inside the parentheses is $I(d, t)$ in (23) with z replaced by d .

The time-domain total current, which is the inverse transform of (49), can be easily written as

$$\mathcal{G}_T(d, t) = \frac{4c}{\sqrt{\epsilon_r}} \int_0^t dt' I(d, t') [e^{-(t-t')/t_f} - e^{-(t-t')/t_r}]. \quad (50)$$

Finally, in a diffusion limit, $\mathcal{G}_T(d, t)$ can be written as

$$\mathcal{G}_T(d, t) = Z_T(t_f, t) g_{t_f}(d, t) - Z_T(t_r, t) g_{t_r}(d, t) \quad (51)$$

where

$$Z_T(t_0, t) = \frac{16(\pi t_0 t_e)^{1/2}}{\mu \left[2 \ln \left(\frac{\delta}{a} \right) + \ln 2 - \gamma \right]} \quad (52)$$

and $g_{t_0}(d, t)$ is given by (34) and is a function of t_0 because of $A = z^2 \sigma \mu / 4 t_0$.

Notice that t_e is defined as $t_e = \epsilon_0 / \sigma$. Equation (51) differs by a factor of two from the wire current given in [6]. It appears that a factor of two has been dropped on [6, p. 385]. The effect of the air/ground interface can be easily accounted for by using (38). An important comment is in order: the wire current solved in Section IV and that solved in this section have identical normalized admittance functions. The difference in the two cases is the admittance factor. The interpretation of z and d should be the separation distance in the dissipative medium from the source point to the observation point.

A numerical example is helpful in appreciating the value of an induced wire current due to an HABEMP. Notice that Z_T in (52) is proportional to $\sqrt{t_0}$. Therefore, dropping the second term in (51) results only in about 10 percent accuracy. Consider an earth conductivity of $\sigma = 10^{-3}$ S/m.

Thus, $t_e = \epsilon_0 / \sigma \sim 10^{-8}$ s.

$$Z_T(t_f, t) = \frac{16 \sqrt{\pi \times 10^{-8} \times 2.5 \times 10^{-7}}}{4\pi \times 10^{-7} \times 16} = 0.07.$$

In obtaining the above number, the interface effect has been included. Also, $A \sim 10^{-2}$, and from Fig. 2, the peak value of $g_{t_f}(d, t) \sim 0.5$.

When accounting for a multiplicative constant of $E_0 = 5 \times 10^4$ V/m in (44) for an HABEMP and the error introduced by using Fig. 2, the peak value of the current is approximately given by 2 kA.

VIII. CONCLUSION

This paper describes the transient solutions of an infinite cylindrical antenna in a dissipative medium. First, an exact solution in the form of definite integrals for the antenna current ((18) and (19)) is obtained; then this exact solution is converted to a numerically efficient integral ((18) and (22)). Second, a new and general asymptotic method is used to evaluate the exact solution to give a simple, numerically accurate formula ((23)). Third, use of this simple formula is

illustrated for a double exponential input voltage ((31)). The diffusion limit is carried out and shown to give rise to Sunde's classic result ((34)). Numerical comparisons of the present theory and the diffusion limit are discussed. Fourth, although theoretically trivial, the inclusion of an air/ground interface can cause the current to differ by a factor of two. A simple image argument is applied to derive a time-domain correction factor ((38)). Fifth, a numerical example, which is directly applicable to the coupling of the SBEMP to a buried power line, is given (Fig. 5). Sixth, the current of a buried wire under HABEMP excitation is also obtained ((50)). When a diffusion approximation is used, an error of a factor of two is found in the literature [6], [7]. Seventh, the clarification of an inconsistency in the early-time solution in the literature is noteworthy (Appendix III). Eighth, a new time-harmonic formula for an infinite wire in a dissipative medium is generated.

Finally, three research problems are identified below. First, the asymptotic technique, as discussed in Appendix II, can be used for evaluating many complicated integrals. Second, although the alternative integral formula ((29)) can be used routinely to obtain transient solutions of infinite wires, it is important to consider the termination problem of a buried wire on a metallic structure. Third, the air/ground interface effect on the transient solution for an infinite wire still needs to be treated rigorously.

APPENDIX I

DERIVATION OF AN ALTERNATIVE FORMULA FOR NUMERICAL INTEGRATION

The expression for I_2 is the contribution of the integral (19) along the axis. This Appendix gives an alternative formula with K_0 in the numerator and I_0 and K_0 in the denominator to avoid the oscillatory behavior of J_0 and Y_0 via contour integration. This formula has the real arguments in the modified Bessel function. Using the identities [12], [13],

$$J_0(\eta) = I_0(\eta e^{-1/2\pi i}),$$

$$H_0^{(1)}(\eta) = -\frac{i2}{\pi} K_0(\eta e^{-1/2\pi i}) \quad (53)$$

in I_2 , as given in (19), leads to

$$I_2 = \frac{4}{\pi \zeta_0} e^{-(\sigma t / 2\epsilon)} R_e \left[\frac{i\pi}{2} \int_{\alpha}^{\infty} \frac{d\eta}{\eta} \cdot \{ I_0(\eta e^{-(\pi i/2)}) K_0(\eta e^{-(\pi i/2)}) \}^{-1} I_0(\tau e^{-(\pi i/2)} \sqrt{\eta^2 - \alpha^2}) \right]. \quad (54)$$

Again, the identity [12]

$$i\pi I_0(\tau e^{-(\pi i/2)} \sqrt{\eta^2 - \alpha^2}) = K_0(\tau e^{-(\pi i/2)} \sqrt{\eta^2 - \alpha^2}) - K_0(\tau e^{+(\pi i/2)} \sqrt{\eta^2 - \alpha^2})$$

converts I_2 into

$$I_2 = \frac{2}{\pi \zeta_0} e^{-(\sigma t/2\epsilon)} R_e \left[\int_{\alpha}^{\infty} \frac{d\eta}{\eta} \{ I_0(\eta e^{-(\pi i/2)}) K_0(\eta e^{-(\pi i/2)}) \}^{-1} \right. \\ \cdot K_0(\tau e^{-(\pi i/2)} \sqrt{\eta^2 - \alpha^2}) - \int_{\alpha}^{\infty} \frac{d\eta}{\eta} \{ I_0(\eta e^{-(\pi i/2)}) \\ \cdot K_0(\eta e^{-(\pi i/2)}) \}^{-1} K_0(\tau e^{(\pi i/2)} \sqrt{\eta^2 - \alpha^2}) \left. \right]. \quad (55)$$

Deforming the contour for the first integral in (55) to the solid line in the upper half-plane (Fig. 6), the first integral can be shown to be

$$\frac{i2}{\pi} \int_{\alpha_0}^{\alpha} \frac{d\eta}{\eta} \{ J_0(\eta) H_0^{(1)}(\eta) \}^{-1} K_0(\tau \sqrt{\alpha^2 - \eta^2}) \\ - \frac{2}{\pi} \int_0^{\pi/2} d\theta \{ J_0(\alpha_0 e^{i\theta}) H_0^{(1)}(\alpha_0 e^{i\theta}) \}^{-1} \\ \cdot K_0(\tau \sqrt{\alpha^2 - \alpha_0^2} e^{i2\theta}) \\ + \int_{\alpha_0}^{\infty} \frac{d\eta}{\eta} \{ J_0(\eta) K_0(\eta) \}^{-1} K_0(\tau \sqrt{\alpha^2 + \eta^2}). \quad (56)$$

Notice that α_0 cannot be zero, because if it is, the first term in (56) diverges. This is the same integral as the infinite antenna impedance at the gap.

The second integral in (55) is integrated with the contour shown as solid lines in the lower half-plane. It is approximated by

$$-\frac{i2}{\pi} \int_{\alpha_0}^{\alpha} \frac{d\eta}{\eta} \{ J_0(\eta) H_0^{(1)}(\eta) \}^{-1} K_0(\tau \sqrt{\alpha^2 - \eta^2}) \\ + \frac{2}{\pi} \int_0^{\pi/2} d\theta \{ J_0(\alpha_0 e^{i\theta}) H_0^{(1)}(\alpha_0 e^{i\theta}) \}^{-1} \\ \cdot K_0(\tau \sqrt{\alpha^2 - \alpha_0^2} e^{i2\theta}) \\ - \int_{\alpha_0}^{\infty} \frac{d\eta}{\eta} \{ I_0(\eta) [K_0(\eta) + i\pi I_0(\eta)] \}^{-1} K_0(\tau \sqrt{\alpha^2 + \eta^2}). \quad (57)$$

Substitution of (56) and (57) into (55) gives

$$I_2 = \frac{2}{\pi \zeta_0} e^{-(\sigma t/2\epsilon)} \left\{ 2R_e \int_0^{\pi/2} d\theta [J_0(\alpha_0 e^{i\theta}) H_0^{(1)}(\alpha_0 e^{i\theta})]^{-1} \right. \\ \cdot K_0(\tau \sqrt{\alpha^2 - \alpha_0^2} e^{i2\theta}) + \pi^2 \int_{\alpha_0}^{\infty} \frac{d\eta}{\eta} \frac{I_0(\eta)}{K_0(\eta)} \\ \cdot \left. \frac{K_0(\tau \sqrt{\alpha^2 + \eta^2})}{\{ [K_0(\eta)]^2 + \pi^2 [I_0(\eta)]^2 \}} \right\}. \quad (58)$$

For $\alpha_0 \ll \alpha$, I_2 can be approximated by (22).

Fig. 6. Contour deformation in the η -plane.

APPENDIX II

ASYMPTOTIC EVALUATION OF $I(z, t)$

This Appendix evaluates two definite integrals (59), (60) and then uses these results asymptotically to evaluate $I(z, t)$ ((18) and (19)).

$$\int_0^{\alpha} \frac{\eta I_1(\tau \sqrt{\alpha^2 - \eta^2})}{\sqrt{\alpha^2 - \eta^2}} \ln \eta d\eta \quad (59)$$

and

$$\int_{\alpha}^{\infty} \frac{\eta I_1(\tau \sqrt{\alpha^2 - \eta^2})}{\sqrt{\alpha^2 - \eta^2}} \ln \eta d\eta. \quad (60)$$

The first integral can be evaluated term by term when I_1 is expanded in a series form to give

$$\sum_{m=0}^{\infty} \frac{\left(\frac{\tau}{2}\right)^{1+2m}}{m!(m+1)!} \int_0^{\alpha} \eta (\alpha^2 - \eta^2)^m \ln \eta d\eta \\ = \frac{1}{4} \sum_{m=0}^{\infty} \frac{\left(\frac{\tau}{2}\right)^{1+2m}}{m!(m+1)!} \int_0^{\alpha^2} y^m \ln(\alpha^2 - y) dy \\ = \frac{1}{4} \sum_{m=0}^{\infty} \frac{\left(\frac{\tau}{2}\right)^{1+2m}}{[(m+1)!]^2} \left[\ln(\alpha^2) - \sum_{k=1}^{m+1} \frac{1}{m-k+2} \right] \alpha^{2m+2} \\ = \frac{1}{2\tau} \left\{ [I_0(\alpha\tau) - 1] [\ln(\alpha^2)] \right. \\ \left. - K_0(\alpha\tau) - I_0(\alpha\tau) \left[\ln\left(\frac{\alpha\tau}{2}\right) + \gamma \right] \right\} \\ = \frac{I_0(\alpha\tau)}{\tau} \left\{ \frac{1}{2} \left[\ln\left(\frac{\alpha}{\tau}\right) - \ln 2 - \gamma - \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} \right] \right\} - \frac{\ln \alpha}{\tau}. \quad (62)$$

The step from (61) to (62) is the identification of modified Bessel functions.

The second integral of interest (60) can be deduced as follows:

$$\begin{aligned} & \int_{\alpha}^{\infty} \frac{\eta J_1(\tau\sqrt{\eta^2-\alpha^2})}{\sqrt{\eta^2-\alpha^2}} \ln \eta \, d\eta \\ &= \frac{1}{2} \int_0^{\infty} J_1(\tau\eta_1) \ln(\eta_1^2 + \alpha^2) \, d\eta_1 \\ &= \frac{\ln \alpha}{\tau} + \frac{1}{\tau} \int_0^{\infty} \frac{J_0(\tau\eta_1)\eta_1 \, d\eta_1}{\eta_1^2 + \alpha^2} \end{aligned} \quad (63)$$

$$= \frac{\ln \alpha}{\tau} + \frac{K_0(\alpha\tau)}{\tau}. \quad (64)$$

Integration by parts is used in obtaining (63), and (64) follows from an identity in [12].

Finally, combining (62) and (64) gives

$$\begin{aligned} & \int_0^{\infty} \frac{\eta J_1(\tau\sqrt{\eta^2-\alpha^2})}{\sqrt{\eta^2-\alpha^2}} \ln \eta \, d\eta \\ &= \frac{I_0(\alpha\tau)}{\tau} \left\{ \frac{1}{2} \left[\ln \left(\frac{\alpha}{\tau} \right) + \ln 2 - \gamma + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} \right] \right\}. \end{aligned} \quad (65)$$

Next, consider the definite integral (65). It is possible to write it as

$$\begin{aligned} & \int_0^{\infty} \frac{\eta J_1(\tau\sqrt{\eta^2-\alpha^2})}{\sqrt{\eta^2-\alpha^2}} \ln \eta \, d\eta \\ &= \langle \ln \eta \rangle_{\eta=0} \int_0^{\infty} \frac{\eta J_1(\tau\sqrt{\eta^2-\alpha^2})}{\sqrt{\eta^2-\alpha^2}} \, d\eta \\ &= \langle \ln \eta \rangle_{\eta=0} \frac{I_0(\alpha\tau)}{\tau} \end{aligned} \quad (66)$$

where

$$\langle \ln \eta \rangle_{\eta=0} = \frac{1}{2} \left[\ln \left(\frac{\alpha}{\tau} \right) + \ln 2 - \gamma + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} \right]. \quad (67)$$

To evaluate the sum of (18) and (19), proceed as follows:

$$\begin{aligned} I &= \frac{4}{\pi \xi_0} e^{-(\sigma t/2\epsilon)} \int_0^{\infty} J_0(\tau\sqrt{\eta^2-\alpha^2}) \{ [J_0(\eta)]^2 + [Y_0(\eta)]^2 \}^{-1} \frac{d\eta}{\eta} \\ &\sim \frac{2}{\xi_0} e^{-(\sigma t/2\epsilon)} \tau \int_0^{\infty} \arctan \left[\frac{-\pi/2}{\ln \left(\frac{\eta}{2} \right) + \gamma} \right] \\ &\quad \cdot \frac{J_1(\tau\sqrt{\eta^2-\alpha^2})}{\sqrt{\eta^2-\alpha^2}} \eta \, d\eta. \end{aligned} \quad (68)$$

Replacing $\ln \eta$ in (68) by (67) gives

$$\begin{aligned} I &\sim \frac{2}{\xi_0} e^{-(\sigma t/2\epsilon)} \tau \arctan \left[\frac{-\pi/2}{\langle \ln \eta \rangle_{\eta=0} - \ln 2 + \gamma} \right] \\ &\quad \cdot \int_0^{\infty} \frac{J_1(\tau\sqrt{\eta^2-\alpha^2})}{\sqrt{\eta^2-\alpha^2}} \eta \, d\eta \\ &= \frac{2}{\xi_0} e^{-(\sigma t/2\epsilon)} I_0(\alpha\tau) \arctan \left[\frac{-\pi}{\ln \left(\frac{\alpha}{\tau} \right) + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} - \ln 2 + \gamma} \right]. \end{aligned} \quad (69)$$

APPENDIX III

AN EARLY-TIME SOLUTION

Previous investigations of the early-time behavior for free space are described in [1] and [3]. To study the early-time behavior of $I(z, t)$, use (18) and (22) to write $I(z, t)$ as

$$\begin{aligned} I(z, t) &= I_1 + I_2 \\ &\sim \frac{2\pi}{\xi_0} e^{-(\sigma t/2\epsilon)} \int_{\tau^{-\beta}}^{\infty} \frac{d\eta}{\eta} \frac{I_0(\eta)}{K_0(\eta)} \frac{K_0(\tau\sqrt{a^2+\eta^2})}{([K_0(\eta)]^2 + \pi^2[I_0(\eta)]^2)}. \end{aligned} \quad (70)$$

When $\tau \ll 1$, $I(z, t)$ is contributed mostly from the last term of (70). Furthermore, β is chosen to satisfy $0 < \beta < 1$, so that as $\tau \rightarrow 0$ ($\tau^{-\beta} \rightarrow \infty$), $I_0(\eta)$ and $K_0(\eta)$ can be replaced by the asymptotic expansions.

As a result, for $\tau \ll 1$,

$$\begin{aligned} I(z, t) &\sim \frac{4}{\xi_0 \pi} e^{-(\sigma t/2\epsilon)} \int_{\tau^{-\beta}}^{\infty} K_0(\tau\eta) \, d\eta = \frac{4}{\xi_0 \pi \tau} e^{-(\sigma t/2\epsilon)} \\ &\quad \int_{\tau^{1-\beta}}^{\infty} K_0(x) \, dx \sim \frac{2}{\xi_0 \tau} e^{-(\sigma t/2\epsilon)}. \end{aligned} \quad (71)$$

Notice that in (71), the lower limit $\tau^{1-\beta}$ in the last integration can be replaced by zero. When $\sigma \rightarrow 0$, (71) reduces to the lossless case studied by Wu [1] but differs from Lee and Latham [3], [14], whose early-time current is a factor of $\sqrt{2\pi}$ too large.

Alternatively, (71) could be obtained by using (18) and (19) as follows:

$$\begin{aligned} I(z, t) &\sim \dots + \frac{4}{\pi \xi_0} e^{-(\sigma t/2\epsilon)} \int_{\tau^{-\beta}}^{\infty} \frac{J_0(\tau\sqrt{\eta^2-\alpha^2})}{([J_0(\eta)]^2 + [Y_0(\eta)]^2)} \frac{d\eta}{\eta} \\ &\sim \frac{4}{\xi_0 \pi \tau} e^{-(\sigma t/2\epsilon)} \int_{\tau^{1-\beta}}^{\infty} J_0(x) \, dx = \frac{2}{\xi_0 \tau} e^{-(\sigma t/2\epsilon)}. \end{aligned}$$

APPENDIX IV

RIGOROUS TRANSMISSION LINE THEORY

The diffusion limit of (23) and a rigorous transmission line solution are considered below. Equation (23) is expanded for

large $\alpha\tau$ to give diffusion limit of $t \gg z/c$. The

$$\frac{\sigma}{2\epsilon} t - \alpha\tau = \frac{\sigma}{2\epsilon} (t - \sqrt{t^2 - z^2/c^2}) \sim \frac{\sigma\mu}{4t} z^2 = \frac{z^2}{2\delta^2}$$

$$\delta = \sqrt{\frac{2t}{\sigma\mu}}$$

and

$$\ln \frac{\tau}{\alpha} \sim 2 \ln \frac{\delta}{a} \gg 1.$$

As a result,

$$I(z, t) \sim 2\epsilon \left(\frac{\pi}{\mu\sigma t} \right)^{1/2} \frac{e^{-z^2/2\delta^2}}{2 \ln \left(\frac{\delta}{a} \right) + \ln 2 - \gamma}. \quad (72)$$

Proceeding with the simplified Maxwell's equations gives

$$\mu \left(\sigma E_\rho + \epsilon \frac{\partial E_\rho}{\partial t} \right) = \frac{\partial B_\theta}{\partial z} \quad (73)$$

$$\mu \left(\sigma E_z + \epsilon \frac{\partial E_z}{\partial t} \right) = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\theta) \quad (74)$$

$$\frac{\partial B_\theta}{\partial t} = \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}. \quad (75)$$

Near the wire $E_z \sim 0$; therefore, (74) is omitted from further discussion. Furthermore, B_θ can be given in the diffusion limit as

$$B_\theta(\rho, z, t) = \frac{\mu \mathcal{G}(z, t)}{2\pi\rho} e^{-\rho^2/2\delta^2}. \quad (76)$$

Substituting (76) and (15) with $V \equiv 1$ into (73) gives

$$E_\rho(\rho, z, t) = \frac{\partial I}{2\pi\epsilon\rho} e^{-\rho^2/2\delta^2}. \quad (35)$$

The line voltage can be defined as

$$\begin{aligned} \mathcal{V} &= \frac{-1}{2\pi\epsilon} \frac{\partial I(z, t)}{\partial z} E_1 \left(\frac{a^2}{2\delta^2} \right) \\ &\sim \frac{1}{2\pi\epsilon} \frac{\partial I(z, t)}{\partial z} \left[2 \ln \frac{\delta}{a} + \ln 2 - \gamma \right] \end{aligned} \quad (77)$$

where $E_1(a^2/2\delta^2)$ is the exponential integral.

Equation (77) can be rewritten as

$$-\frac{\partial \mathcal{G}}{\partial z}(z, t) = \left(\frac{\partial}{\partial t} + \frac{\sigma}{\epsilon} \right) \frac{2\pi\epsilon}{\left(2 \ln \frac{\delta}{a} + \ln 2 - \gamma \right)} \mathcal{V}(z, t). \quad (78)$$

Let

$$C = \frac{2\pi\epsilon}{2 \ln \frac{\delta}{a} + \ln 2 - \gamma} \quad \text{and} \quad G = \frac{2\pi\sigma}{2 \ln \frac{\delta}{a} + \ln 2 - \gamma}. \quad (79)$$

The first of the transmission line equations is obtained as

$$\left(C \frac{\partial}{\partial t} + G \right) \mathcal{V} = -\frac{\partial \mathcal{G}}{\partial z}. \quad (80)$$

The second equation can easily be shown from (75) to be

$$L \frac{\partial \mathcal{G}}{\partial t} = -\frac{\partial \mathcal{V}}{\partial z} \quad (81)$$

with

$$L = \frac{2 \ln \frac{\delta}{a} + \ln 2 - \gamma}{2\pi} \mu. \quad (82)$$

Explicit expressions for $\mathcal{V}(z, t)$ and $\mathcal{G}(z, t)$ for unit impulse input voltage at the gap are

$$\mathcal{V}(z, t) \sim \frac{1}{\sqrt{2\pi}} \left(\frac{z}{\delta t} \right) e^{-z^2/2\delta^2} \quad (83)$$

and

$$\mathcal{G}(z, t) = \frac{1}{\sqrt{2\pi}} \left[\frac{G\delta}{t} + C \left(-\frac{\delta}{2t^2} + \frac{z^2}{2\delta t^2} \right) \right] e^{-z^2/2\delta^2}. \quad (84)$$

APPENDIX V

A TIME-HARMONIC SOLUTION

To obtain a time-harmonic solution for the wire current in a dissipative medium, it is necessary to consider the following integral [15]:

$$\begin{aligned} &\int_1^\infty (x^2 - 1)^{\nu/2} e^{-px} I_\nu(q\sqrt{x^2 - 1}) dx \\ &= \sqrt{\frac{2}{\pi}} q^\nu (p^2 - q^2)^{-1/2\nu - 1/4} K_{\nu+1/2}(\sqrt{p^2 - q^2}). \end{aligned} \quad (85)$$

Differentiate (85) with respect to ν and then set $\nu = 0$. The following equation emerges:

$$\begin{aligned} &\frac{1}{2} \int_1^\infty \ln(x^2 - 1) e^{-px} I_0(q\sqrt{x^2 - 1}) dx \\ &= \int_1^\infty e^{-px} K_0(q\sqrt{x^2 - 1}) dx \\ &+ \sqrt{\frac{2}{\pi}} \ln \frac{q}{\sqrt{p^2 - q^2}} \frac{K_{1/2}(\sqrt{p^2 - q^2})}{(p^2 - q^2)^{1/4}} \\ &+ \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial \nu} K_{\nu+1/2}(\sqrt{p^2 - q^2})|_{\nu=0} \frac{1}{(p^2 - q^2)^{1/4}}. \end{aligned} \quad (86)$$

The right side of (86) can be evaluated, and an averaging for $1/2 \ln(x^2 - 1)$ can be defined as

$$\begin{aligned} & \left\langle \frac{1}{2} \ln(x^2 - 1) \right\rangle \\ &= \frac{\text{equation (86)}}{\int_1^\infty e^{-px} I_0(q\sqrt{x^2 - 1}) dx} \\ &= \ln \frac{p + \sqrt{p^2 - q^2}}{q} - Ei(-2\sqrt{p^2 - q^2}) e^2 \sqrt{p^2 - q^2} \\ & \quad + \sqrt{\frac{2}{\pi}} \ln \frac{q}{\sqrt{p^2 - q^2}} K_{1/2}(\sqrt{p^2 - q^2}) (p^2 - q^2)^{1/4} e^{\sqrt{p^2 - q^2}} \\ & \sim \ln \frac{p + \sqrt{p^2 - q^2}}{\sqrt{p^2 - q^2}}. \end{aligned} \quad (87)$$

The time-harmonic wire current is related to the forward Fourier transform of (23) as

$$\begin{aligned} \tilde{I}(z, \omega) &= \frac{2}{\xi_0} \int_0^\infty dt e^{i\omega t - \sigma t/2\epsilon} I_0 \left(\frac{\sigma}{2\epsilon} \sqrt{t^2 - z^2/c^2} \right) \\ & \quad \cdot \arctan \left[\frac{-\pi}{\ln \frac{\alpha}{\tau} + \frac{K_0(\alpha\tau)}{I_0(\alpha\tau)} - \ln 2 + \gamma} \right]. \end{aligned} \quad (88)$$

To evaluate (88), let

$$\begin{aligned} p &= \left(-i\omega + \frac{\sigma}{2\epsilon} \right) \frac{z}{c} \\ q &= \frac{\sigma z}{2\epsilon c}. \end{aligned}$$

Equation (88) can be approximated by replacing τ by its averaging. Note that

$$\frac{1}{2} \langle \ln(x^2 - 1) \rangle \sim \ln \frac{\frac{\sigma}{2\epsilon}}{\sqrt{(-i\omega) \left(-i\omega + \frac{\sigma}{\epsilon} \right)}} = \ln \left(\frac{\sigma}{-i2\epsilon kc} \right)$$

and

$$\langle \tau \rangle \sim \frac{z\sigma}{-i2a\epsilon kc}, \quad -ikc = \sqrt{(-i\omega) \left(-i\omega + \frac{\sigma}{\epsilon} \right)}. \quad (89)$$

Use of (89) in (88) gives

$$\begin{aligned} \tilde{I}(z, \omega) &\sim \frac{2}{\xi_0} \frac{e^{-ikz}}{(-ik)} \\ & \quad \cdot \arctan \left[\frac{-\pi}{\ln \frac{\alpha}{\langle \tau \rangle} + \frac{K_0(\alpha\langle \tau \rangle)}{I_0(\alpha\langle \tau \rangle)} - \ln 2 + \gamma} \right]. \end{aligned} \quad (90)$$

When $\alpha\langle \tau \rangle \gg 1$, (90) reduces to

$$\begin{aligned} \tilde{I}(z, \omega) &\sim \frac{-1}{\xi_0} \frac{e^{-ikz}}{kc} \\ & \quad \cdot \ln \left[1 - \frac{i2\pi}{\ln(2kz) - 2 \ln(ka) - \gamma + i \frac{3\pi}{2}} \right]. \end{aligned} \quad (91)$$

If (15) is used, the time-harmonic solution is given by

$$\mathcal{J}(z, \omega) \sim \left(-i\omega + \frac{\sigma}{\epsilon} \right) \tilde{I}(z, \omega). \quad (92)$$

Equations (91) and (92) agree with Wu's result [16]. Notice that the time-harmonic solution for an infinite wire in free space is given by (92) with $k = \omega/c$ and $\sigma = 0$.

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