The Field Feedback Formulation for Electromagnetic Scattering Computations

MICHAEL A. MORGAN, SENIOR MEMBER, IEEE, AND BRYANT E. WELCH

Abstract—A new procedure is described for the solution of electromagnetic scattering problems in unbounded regions. Using this technique a spatial region enclosing the scatterer is initially decoupled from the exterior region and the fields therein are considered as the solution of an interior Dirichlet boundary value problem. The interior region solution is then recoupled to the unbounded exterior region by use of the equivalence principle. Such a process can be symbolically represented as a simple feedback system. The formulation is demonstrated for the case of plane wave scattering by finite metallic circular cylinders. A finite element solution of the interior problem is utilized in this example in conjunction with a field representation using a special case of coupled azimuthal potentials.

I. INTRODUCTION

THE NUMERICAL computation of electromagnetic interaction with increasingly complex material structures has been a major research topic for over two decades. Surface integral equations have been utilized extensively in the solution of problems involving both metallic structures and layered penetrable materials. Formulations based upon differential equations have been employed for the most generalized inhomogeneous geometries, with numerical solutions effected through the *finite methods* (i.e., finite elements and finite differences) [1]. The numerical implementations of the finite methods usually result in the production of very sparse system matrices while full matrices are generally produced by integral equation formulations. The use of sparse matrix algorithms often results in significant reductions of needed memory and computation time when compared to the case of full matrix inversions if the same number of unknowns is to be considered. As a result of this, for given computational constraints of time and memory, the finite methods have the potential to solve larger volume electromagnetic problems (in wavelengths) than can be handled by volume integral equations.

Although the finite methods offer numerical efficiency they are, by necessity, formulated as boundary-value problems. As such, the solution of scattering and radiation problems in unbounded spatial regions requires a mechanism for coupling closed region solutions to the exterior space. This procedure must also ensure that the proper radiation conditions are

produced. The *unimoment* method, as developed by Mei [2], provided a self-consistent approach to coupling interior and exterior field problems through a separable surface interface. This method was employed by Stovall and Mei [3] and Morgan and Mei [4] to both radiation and scattering problems involving inhomogeneous axisymmetric dielectrics. Further applications, using the finite-element method, were made by Morgan [5], [6] to problems involving raindrop scattering and microwave energy deposition in the human head.

As developed in the unimoment method, the exterior region fields are represented by a functional expansion in one of the separable coordinate systems for the vector Helmholtz equation [7]. The spatial interface for coupling the interior numerical solution to the unbounded exterior region is thus a constant coordinate surface of the separable system employed in the outside expansion. Spherical interfaces were utilized in [3]–[6] due to the relative ease of generation of exterior region spherical harmonic field expansions. The numerical solution within a spherical interior region becomes very inefficient, however, for scattering shapes that occupy only a small portion of the enclosed volume (e.g. thin cylinders and flat disks). Although it is possible to utilize a separable nonspherical surface to increase the numerical efficiency of the interior region solution, this will be offset by additional requirements in both generating the special functions that are needed in the exterior expansion and in computing the required *moments* of these functions with the numerical solutions over the interface.

A new technique to circumvent the need for a separable boundary interface was developed by Morgan, Chen, Hill and Barber [8]. This hybrid procedure combines a finite-element solution of the interior region with the surface integral equation found in the extended boundary condition method. Such a procedure allows the use of a surface interface that conforms to the outer boundary of the scattering object. The technique has been shown to work well for T-matrix calculations involving moderately elongated lossy dielectric scatterers having multiwavelength dimensions. The method tends to have difficulties, however, with attaining convergence of the solution if the surface interface becomes extremely elongated (i.e., length to diameter ratios exceeding about ten). This failure occurs because of the generally *incomplete* nature of the exterior region spherical harmonic expansion which is employed to represent the field over the surface of the scattering body. Such a problem is related to the *Rayleigh hypothesis* [9].

Manuscript received January 17, 1986; revised May 15, 1986. This work was supported by the Foundation Research Program at the Naval Postgraduate School.

M. A. Morgan is with the Electrical and Computer Engineering Department, Naval Postgraduate School, Monterey, CA 93943.

B. E. Welch is with the U.S. Navy.
IEEE Log Number 8610044.

The field feedback formulation (F^3) has been proposed [10], [11], to mitigate the restrictions inherent in coupling interior and exterior region field solutions as are found in other techniques. Using F^3 , the interior boundary value problem is initially decoupled from the outside region. The interior problem may then be formulated and computed using the most expedient approach that can accurately accommodate the level of material complexity that is present. The exterior region field is represented in terms of modes generated from surface integrations involving equivalent currents obtained from the interior region solution. These modal fields, which satisfy the radiation conditions, do not rely upon the use of separable coordinate surfaces for their completeness. Another primary advantage of this method is its *modular* nature, where forward and feedback transfer matrices can be independently computed. The scattering solution may then be obtained by considering a simple feedback system that represents the scattering problem.

II. GENERAL FORMULATION

Consider the electromagnetic scattering problem depicted in Fig. 1. The scattering object, which may in general be composed of linear inhomogeneous anisotropic material, is enclosed within inner and outer geometric surfaces, S_i and S_o , respectively. The inner surface may optionally be made coincident with the surface of the object while the incident field (which need not be a plane wave) is assumed to be known.

We begin by considering the solution of the interior Dirichlet problem for the fields within S_o for specified tangential boundary conditions (BC's) on this surface. It can be shown that with the presence of any physical loss mechanism the specification of the tangential vector components of either $\vec{E}$ or $\vec{H}$ on the surface S_o are sufficient to guarantee uniqueness of the interior solution at all frequencies [12]. A *mixed* set of BC's may also be used, employing any one tangential vector component of $\vec{E}$ and the same component of $\vec{H}$.

To proceed, we define a two-element array Φ composed of tangential field components, ϕ_1 and ϕ_2 , whose selection is based upon the particular formulation that is being used to solve the interior problem within S_o ,

$$\Phi(\vec{r}_0) = \begin{bmatrix} \phi_1(\vec{r}_0) \\ \phi_2(\vec{r}_0) \end{bmatrix}. \quad (1)$$

The level of complexity of the scattering structure that can be accommodated by the F^3 procedure is determined by the formulation and numerical algorithm employed to solve the interior problem. For some geometries that display special symmetries, such as that of the example in the next section, only one field component may be needed both to formulate the interior problem and serve as boundary data on S_o .

The total tangential field on S_o is represented by the sum of the *known* incident field plus an appropriate vector basis function expansion for the unknown scattered field,

$$\Phi^{\text{tot}}(\vec{r}_0) = \Phi^{\text{inc}}(\vec{r}_0) + \sum_{n=1}^N C_n \Phi_n(\vec{r}_0). \quad (2)$$

Fig. 1. Generic electromagnetic scattering geometry.

Complex scalar C_n for independently expanding $\phi_1(\vec{r}_0)$ and $\phi_2(\vec{r}_0)$ may be generated by partitioning the N vector basis functions in (2) into two subsets: one subset having $\phi_1 = 0$ and the other having $\phi_2 = 0$. For Fourier type expansions, which employ basis functions having full-range support, the coefficients represent the *spectral* composition of the fields. An alternate procedure is to employ unit amplitude finite-support basis functions so that the coefficients become simply the pointwise values of the functions at the spatial *nodes* that have been defined in discretizing the problem. Thus far the procedure appears similar to that of the unimoment method, except that a separable surface need not be employed.

The application of the Φ^{inc} and the various Φ_n boundary conditions on S_o each produces interior field solutions within the enclosed volume. Using the equivalence principle [12], equivalent electric and magnetic surface currents, $\vec{J}_e = \hat{n}_i \times \vec{H}$ and $\vec{M}_e = \vec{E} \times \hat{n}_i$, can be computed from the tangential fields produced on S_i by each of the BC's applied on S_o , where $\hat{n}_i$ is the outbound unit normal vector on S_i . Defining a four-element array $\mathbf{K}$ which is composed of tangential components of $\vec{J}_e$ and $\vec{M}_e$ on S_i ,

$$\mathbf{K} = [J_{e1}, J_{e2}, M_{e1}, M_{e2}]^T \quad (3)$$

the total equivalent current can be expressed as the superposition of constituents generated by corresponding BC's on S_o , via (2),

$$\mathbf{K}^{\text{tot}}(\vec{r}_i) = \mathbf{K}^{\text{inc}}(\vec{r}_i) + \sum_{n=1}^N C_n \mathbf{K}_n(\vec{r}_i). \quad (4)$$

To enforce the conditions that are needed to solve for the unknown expansion coefficients in (2), the scattered field components that were used to define Φ^{scat} are computed using the electric and magnetic vector potentials obtained via the usual free space Green's function surface integrations on S_i [12]. In particular, the scattered field components so obtained

over the outer surface S_0 are compared to those that were originally postulated in (2) and which share the same unknown set of expansion coefficients C_n .

The details of the F^3 process can be succinctly described by considering the field feedback system representation, as shown in Fig. 2. In this diagram, the interior solution is expressed as a *forward* transfer linear operator $\mathbf{A}$ which yields the equivalent surface current array on S_i in terms of a specified boundary value array on S_0 .

$$\mathbf{K}(\bar{r}_i) = \mathbf{A} \otimes \Phi(\bar{r}_0). \quad (5)$$

The discretized form of the $\mathbf{A}$ -operator will be a matrix, while ordered nodal values of Φ and $\mathbf{K}$ will be used as the corresponding arrays in (5). The *feedback* operator β embodies Green's function surface integrations which relate the scattered tangential fields on S_0 to the equivalent surface currents on S_i .

$$\Phi^{\text{scat}}(\bar{r}_0) = \beta \otimes \mathbf{K}(\bar{r}_i). \quad (6)$$

Far-field integrals are employed in the Γ -operator, which yields the scattered radiation fields in terms of their equivalent current sources on S_i .

A simple analysis of the field feedback system in Fig. 2 provides an operator-solution for the tangential scattered field generated by a specified incident field boundary array on S_0 ,

$$\sum_{n=1}^N C_n \Phi_n = (\mathbf{I} - \beta \otimes \mathbf{A})^{-1} \otimes \beta \otimes \mathbf{A} \otimes \Phi^{\text{inc}} = \mathbf{D} \otimes \Phi^{\text{inc}} \quad (7)$$

where $\mathbf{I}$ is the identity operator and the negative exponent on the bracketed term represents an inverse. A weighted-residual procedure can be applied to (7) to solve for the unknown C_n

$$\sum_{n=1}^N C_n \langle \Phi_n \cdot \mathbf{W}_m \rangle = \langle \mathbf{D} \otimes \Phi^{\text{inc}} \cdot \mathbf{W}_m \rangle = V_m^{\text{inc}}, \quad m = 1, N. \quad (8)$$

The moment matrix formed by the bracketed inner products will be diagonal if the weighting functions $\mathbf{W}_m$ form an orthogonal co-basis set to the set of Φ_n . In that case, C_n will be directly proportional to the V_m^{inc} inner product in (8). Without this orthogonality, the linear system in (8) will need to be inverted.

After completing the solution for the C_n the equivalent surface currents can be constructed by way of (4) and the far-zone scattered fields can be found by using the Γ -matrix. If the field values within the interior region are desired, as would be the case in penetration and dosimetry type problems, the total tangential field solution Φ^{tot} can then be used as the known boundary values on S_0 . The interior field solution mechanism, as employed to generate the $\mathbf{A}$ -matrix, can then be used to produce the desired internal field distribution.

The procedure employed within the F^3 framework is conceptually related to the various formulations of surface integral equations. A typical integral equation derivation may be initiated by employing appropriate Green's function inte-

Fig. 2. Field feedback system representation.

Fig. 3. Cylindrical conductor scattering geometry.

grations over the surface of a homogeneous penetrable body to obtain expressions for interior surface total fields and exterior surface scattered fields. The integral equation formulation is completed by equating the total tangential fields on the interior and exterior surfaces as they are each brought into coincidence with the body surface. The feedback operator in (6) contains similar Green's function integrations for generating the scattered fields on the *exterior* surface S_0 . Fields on the *interior* surface S_i , (not interior to the body in this case) are produced by the boundary value problem solution in the F^3 method. The resultant forward operator in (5) replaces the Green's function integrations employed in the integral equation derivation. In the F^3 algorithm the scattering problem is solved by numerically enforcing the equality of the tangential fields over S_0 .

III. NUMERICAL EXAMPLE

To illustrate the F^3 method, the numerical solution of scattering by *thin* finite-length metallic cylinders will be considered. The problem geometry is shown in Fig. 3, where a perfectly conducting circular cylinder of length L and radius a is enclosed by an outer cylindrical surface S_0 . The inner surface S_i is coincident with the metal surface. As shown, the incident plane wave field has incident angle α and linear polarization with $\bar{H}^{\text{inc}}$ transverse to the z -axis.

The coupled azimuthal potential (CAP) formulation is used

to represent the interior fields within S_0 [13]. For the case of an electrically thin cylinder the scattered fields may be assumed to be axisymmetric. In this case the CAP formulation reduces to a much simplified form compared to that for more generalized field configurations. Using wavenumber normalized cylindrical coordinates, $(R, Z, \phi) = (k_0 \rho, k_0 z, \phi)$, the fields can be shown to be generated by a single scalar potential function $\psi(R, Z)$,

$$\bar{E}(R, Z) = \frac{j}{R\epsilon_r} \hat{\phi} \times \nabla \psi \quad (9a)$$

$$\bar{H}(R, Z) = \frac{1}{\eta_0 R} \psi(R, Z) \hat{\phi} \quad (9b)$$

where $\epsilon_r(R, Z)$ is the relative permittivity, $\eta_0 = 377$ ohms, $\hat{\phi}$ is the unit azimuthal vector and ∇ is a two-dimensional gradient operator in R and Z ,

$$\nabla = \hat{R} \frac{\partial}{\partial R} + \hat{Z} \frac{\partial}{\partial Z} \quad (10)$$

In addition to satisfying a second-order partial differential equation, the potential may also be found as the stationary value of a quadratic functional, with $\mu_r(R, Z)$ the relative permeability,

$$F = \iint_{(R,Z)} \frac{1}{R} \left(\mu_r \psi^2 - \frac{1}{\epsilon_r} \nabla \psi \cdot \nabla \psi \right) dR dZ \quad (11)$$

As described in [13], the functional F can be interpreted in terms of complex energy within the volume enclosed by S_0 .

The variational formulation in (11) is used to implement a finite element solution of the interior problem. The simple finite element mesh for this geometry is shown in the meridian plane of Fig. 4. There are N nodes on the exterior surface, S_0 , and M nodes on the metal surface S_i . Relating this to the F^3 terminology developed in Section II, we will form the Φ -vector from the N nodal values of H_ϕ on S_0 . The H_ϕ -field is related to the potential ψ by way of (9b). In addition, the nodal values of the total z -directed surface current on the thin perfect conductor can easily be found: $I(z) = 2\pi a H_\phi(a, z)$. In this instance, the forward matrix A will be used to generate the M -vectors formed from the nodal values of $I(z)$ in terms of multiplications with assumed N -vectors of nodal values of H_ϕ defined on S_0 .

The finite element solution for the forward matrix proceeds by expanding H_ϕ in a terms of piecewise linear basis functions, $u_k(R, Z)$, which have unit value at the k th node and are zero at all surrounding nodes. The nodal values of H_ϕ are denoted by h_k ,

$$H_\phi(R, Z) = \sum_{k=1}^M h_k u_k(R, Z) + \sum_{k=M+1}^{M+N} h_k u_k(R, Z) \quad (12)$$

The M nodal values of H_ϕ on the metal surface, which are to be determined, appear in the first subexpansion while the N known boundary nodal values of H_ϕ appear in the second summation. Substituting (12) into (11) results in a quadratic form in the h_k . The stationary solution for F is found by

Fig. 4. Special case finite element mesh.

formally differentiating this quadratic form with respect to each of the M unknown nodal values of H_ϕ . This procedure yields the system,

$$\sum_{k=1}^M h_k U_{k,l} = - \sum_{k=M+1}^{M+N} h_k U_{k,l}, \quad l=1, M \quad (13)$$

where

$$U_{k,l} = \iint_{(R,Z)} \frac{1}{R} \nabla(Ru_k) \cdot \nabla(Ru_l) - Ru_k u_l dR dZ \quad (14)$$

The R - Z surface integrations may be performed analytically, where these integrals are nonzero only when the k th and l th nodes share a common triangular element. By returning to the nodal vector notation, the linear system in (13) can be recast into the matrix form,

$$U_m \otimes \bar{H}_i = -U_N \otimes H_0 \quad (15)$$

where $\bar{H}_i$ and H_0 are respective M and N -dimensional nodal vectors on S_i and S_0 . The matrices are formed from the integral terms in (14) and by proper nodal ordering these matrices can be made tri-diagonal. The forward matrix A is obtained by first inverting the U_m -matrix to solve for $\bar{H}_i$ and then scaling by $2\pi a$ to give the M -vector of nodal currents,

$$I_z = -2\pi a U_m^{-1} \otimes U_N \otimes H_0 = A \otimes H_0 \quad (16)$$

Fig. 5. Induced wire current for $L = \lambda/2$, broadside incidence.

The elements of the feedback matrix β are generated by a weighted integration of the linear basis function expansion of the discretized thin-wire current $I(z)$,

$$H_{\phi}^{\text{scat}}(k_0 b, Z_n) = \sum_{m=1}^M I(Z_m) \int_Z u_m(k_0 a, Z) G(Z_n, Z) dZ. \quad (17)$$

The matrix elements $\beta_{m,n}$ can be identified by the integral terms within the expansion, with the Green's function given by

$$G(Z_n, Z) = \frac{k_0^2 b}{4\pi} \left(\frac{j}{R^2} + \frac{1}{R^3} \right) e^{-jR} \quad (18)$$

where

$$R(Z_n, Z) = [(k_0 b)^2 + (Z_n - Z)^2]^{1/2}. \quad (19)$$

Using the F^3 system representation, which is similar to that in Fig. 2, the M -vector of nodal currents can then be found from

$$\mathbf{I}_Z = (\mathbf{I} - \mathbf{A} \otimes \beta)^{-1} \otimes \mathbf{A} \otimes \mathbf{H}_0^{\text{inc}} \quad (20)$$

with $\mathbf{H}_0^{\text{inc}}$ the N -vector of nodal values on S_0 of the axisymmetric component of the incident H_{ϕ} -field. For incidence angle α as shown in Fig. 3, this is given by

$$H_{\phi}^{\text{inc}}(R, Z) = \frac{j}{\eta_0} e^{-jZ \cos \alpha} J'_0(R \sin \alpha) \quad (21)$$

where J'_0 is a derivative of an ordinary Bessel function and a unit-amplitude incident E -field is assumed.

The procedure just described was used to compute the induced complex phasor current for several test cases. Some representative results are displayed in Figs. 5, 6, and 7 where broadside incidence for $L = \lambda/2$ and canted incidence for $L = \lambda$ and $L = 2.0 \lambda$ are, respectively, considered. A

Fig. 6. Induced wire current for $L = \lambda$ with 45° incidence. Fig. 7. Induced wire current for $L = 2\lambda$ with 45° incidence.

comparison of magnitude and phase is made in each case with the results of a separate computation using a Hallen electric field integral equation [12]. For these cases the wire radius is 0.02λ and the mesh radius between the conductor surface S_i and the outer surface S_0 is 0.01λ . This very small mesh size was needed in this particular application due to the nature of the field behavior near the wire surface, as is discussed below. In general, the spacing between S_0 and S_i is determined by the mesh density that is employed in performing the interior field numerical solution. This density is mandated by the need to accurately replicate the interior field behavior via the basis functions used.

As is apparent from these displayed results, the F^3 -

computed phasor currents compare relatively well with those obtained by way of an integral equation. The correlation between these computations tends to decrease slightly with increasing length. One factor that contributed to this trend was the use of a small computer, which placed limitations on the number of nodes used in the computations, thus prohibiting full numerical convergence of the solution for the one and two-wavelength long cases. A second source of error resulted from the use of piecewise linear basis functions to represent $H_0(R, Z)$ in the immediate vicinity of the conductor in the finite element solution. The field behavior in the outward radial direction close to the thin conductor is characterized by a rapidly decreasing evanescence. The use of more specialized basis functions to track such a singularity would allow more natural convergence of the numerical solution.

IV. CONCLUSION

A novel numerical procedure has been described for allowing the systematic coupling of interior region solutions to the unbounded exterior domain in scattering problems. Such a formalism permits the use of sparse matrix finite methods for field solutions within complicated scatterers composed of inhomogeneous and anisotropic materials. Furthermore, a minimal volume conformal bounding surface may be employed to enclose the scatterer.

This new formulation casts the scattering problem into a vector-matrix equivalent of a simple feedback system, wherein the forward (interior problem) and feedback (radiated field) matrices may be computed independently and then combined to obtain a self-consistent scattering solution. In an earlier form of this type of procedure, interior finite-element solutions were coupled to a surface integral equation employed in the extended boundary condition method. Convergence problems for nonspherical boundary surfaces, as encountered in that initial work, led to the present effort which circumvents the need for any separable coordinate expansion.

The numerical example for thin-wire scattering was presented to illustrate the general formulation by way of a simple, verifiable computation. As discussed in Section III, the specific finite element procedure employed here for the thin-wire case was not optimal. It did, however, serve to demonstrate computational details and provided relatively good comparisons with results generated by an integral equation method.

The unique power of this method resides in extending the realm of electromagnetic computation to large and quite

complex scattering compositions where volume integral equation techniques are numerically *bottlenecked* by their resultant full system matrices. Current work is being directed at demonstrating this potential power. In addition, an effort is underway at reformulating the field feedback concept for direct time-domain computations.

REFERENCES

- [1] K. K. Mei, M. A. Morgan, and S. K. Chang, "Finite methods in electromagnetic scattering," in *Electromagnetic Scattering*, P. L. E. Ushlenghi, Ed. New York: Academic, 1978.
- [2] K. K. Mei, "Unimoment method of solving antenna and scattering problems," *IEEE Trans. Antennas Propagat.*, vol. AP-22, pp. 760-766, Nov. 1974.
- [3] R. E. Stovall and K. K. Mei, "Application of a unimoment technique to a biconical antenna with inhomogeneous dielectric loading," *IEEE Trans. Antennas Propagat.*, vol. AP-23, pp. 335-341, May 1975.
- [4] M. A. Morgan and K. K. Mei, "Finite element computation of scattering by inhomogeneous penetrable bodies of revolution," *IEEE Trans. Antennas Propagat.*, vol. AP-27, pp. 202-214, Mar. 1979.
- [5] M. A. Morgan, "Finite element computation of microwave scattering by raindrops," *Radio Sci.*, vol. 15, pp. 1109-1119, Nov. 1980.
- [6] —, "Finite element calculation of microwave absorption by the cranial structure," *IEEE Trans. Biomed. Eng.*, vol. BME-28, pp. 687-695, Oct. 1981.
- [7] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, vol. 1. New York: McGraw-Hill, 1953, sec. 5.1.
- [8] M. A. Morgan, C. H. Chen, S. C. Hill and P. W. Barber, "Finite element-boundary integral formulation for electromagnetic scattering," *J. Wave Motion*, vol. 6, pp. 91-103, Jan. 1984.
- [9] R. F. Miller, "Rayleigh hypothesis in scattering problems," *Electron. Lett.*, vol. 5, no. 17, pp. 416-418, 1969.
- [10] B. E. Welch, "Concept evaluation: Field feedback computation of electromagnetic scattering," M.S. thesis, Dept. Elec. Eng., Naval Postgrad. School, June 1980.
- [11] M. A. Morgan and B. E. Welch, "Field feedback formulation," Abstracts, 1981 Nat. Radio Sci. Meet., Boulder, CO, Jan. 1981.
- [12] R. F. Harrington, *Time-Harmonic Electromagnetic Fields*. New York: McGraw-Hill, 1961, ch. 3.
- [13] M. A. Morgan, K. K. Mei, and S. K. Chang, "Coupled azimuthal potentials for electromagnetic field problems in inhomogeneous axially-symmetric media," *IEEE Trans. Antennas Propagat.*, vol. AP-25, pp. 413-417, May 1977.

Michael A. Morgan (S'73-M'76-SM'86), for a photograph and biography please see page 473 of the May 1984 issue of this TRANSACTIONS.

Bryant E. Welch was born in Urbana, IL, on January 26, 1950. He received the B.S. degree in physics from the University of Illinois at Urbana in 1972 and the M.S. degree in electrical engineering from the Naval Postgraduate School, Monterey, CA, in 1980.

As a U.S. Naval Officer he has held numerous positions in the areas of electronics and communications engineering.