

Self-Patch Integrals in Transient Electromagnetic Scattering

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Abstract—The self-patch integrals that arise in the integral equations of electromagnetic scattering are evaluated analytically for general orthogonal coordinate systems to first order in the linear size of the patch. There are terms that contain spatial derivatives of the surface fields, and these terms may not be negligible. Although the formulas are derived for transient waves, the same integrals appear for monochromatic waves.

I. INTRODUCTION

THE ELECTROMAGNETIC fields scattered by a homogeneous body can be obtained from tangential fields defined on the surface of the scatterer by integrations. These surface fields obey integral equations which are often solved numerically. To perform the numerical integrations, the surface of the body is divided into patches, and a representative value of the field on the patch is used. The values of these fields are determined by the incident field and the properties of the scatterer.

In the case of a transient field that reaches the body at a time that can be chosen to be $t = 0$, the integral equation can be solved by a stepping-in-time procedure. This method takes advantage of causality and the finite speed of propagation of the electromagnetic waves to express the field at a time t in terms of values at earlier times that have already been calculated [1]–[5].

These integral equations are singular, that is, the kernel has a singularity in the region of integration. The patch where the kernel becomes singular is called the self-patch, and the contribution to the integral from this patch can be computed analytically by expanding the integrand in powers of the distance between the field point and the source point. The self-patch integral provides a correction term in an integral equation of the second kind, such as the magnetic field integral equation (MFIE) [2], and it is frequently neglected in actual computations.¹ On the other hand, this term is needed to compute the unknown field in an integral equation of the first kind, such as the electric field integral equation (EFIE) [2]. In Section II we determine the self-patch contributions to first order in the linear size of the patch for singular integrals that appear in representative integral equations for electromagnetic scattering. We find the self-patch integrals for arbitrary orthogonal coordinates on the surface, and we expand the fields about the point on the patch where the surface coordinates take their midvalue. Most of the integrals are absolutely integrable, with the exception of one whose highest order contribution vanishes by symmetry, in a manner similar to the principal-value integral of functions of a single variable. We also discuss the behavior of the self-patch integral when the incident wave has a discontinuous wavefront.

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¹ The self-patch contributions to singular integrals have been computed by Bennett [3] and Mieras [4]. They find general expressions for the self-patch integrals in a principal coordinate system. They approximate the self-patch by a circular patch of equal area, and they assume that the fields do not vary over the patch.

We apply the formulas to the simple case of patches on a sphere in Section III, and we determine the additional terms needed for the spherical caps at the poles. We also compute some numerical values to compare different terms with each other and with the circular-patch approximation.

We recall some of the formulas and definitions we need from the geometry of surfaces in the Appendix, where we also show how the integrals are calculated.

The same self-patch integrals occur in the steady-state formulation of electromagnetic scattering.

II. SELF-PATCH INTEGRALS

We now compute the self-patch integrals that arise in some typical integral equations found in electromagnetic scattering.

The MFIE for perfect conductors [2] is

$$\vec{J}_s(\vec{x}, t) = (2/\mu_0)\hat{n} \times \vec{B}^{\text{in}}(\vec{x}, t) - \frac{1}{2\pi} \hat{n} \times \oint_S dS' \vec{R} \times \left[\frac{1}{R^2 c} \frac{\partial \vec{J}_s(\vec{x}', \tau)}{\partial t'} + \frac{1}{R^3} \vec{J}_s(\vec{x}', \tau) \right], \quad (1)$$

where \vec{B}^{in} is the known magnetic induction of the incoming field, $\vec{J}_s(\vec{x}', t)$ is the unknown surface current density at the point \vec{x}' on the surface S of the perfect conductor at the time t' and $\partial \vec{J}_s(\vec{x}', \tau)/\partial t'$ stands for $\partial \vec{J}_s(\vec{x}', t')/\partial t'|_{t'=\tau}$, μ_0 and c are the constant free-space permeability and speed of propagation of light, respectively, and τ is the retarded time

$$\tau = t - R/c, \quad (2)$$

where R is the distance between the field point and the source point, that is, $R = |\vec{R}|$, $\vec{R} = \vec{x} - \vec{x}'$.

In most instances we assume that surface fields such as \vec{J}_s and $\partial \vec{J}_s/\partial t$ are constant over each patch, but here we need their expansion on the self-patch S_1 . The first terms of the Taylor series of the surface current density are given by²

$$\vec{J}_s(\vec{x}', \tau) = \vec{J}_s(\vec{x}, t) - \vec{R} \cdot \nabla_s \vec{J}_s(\vec{x}, t) - \frac{R}{c} \frac{\partial \vec{J}_s(\vec{x}, t)}{\partial t} + \dots \quad (3)$$

² Dyadics or tensors of rank two are convenient ways of handling quantities with two space indices or groupings of two vectors. Dyadics can be written as sums of dyads, which are dyadics formed by the tensor product of two vectors (which are just written next to each other). These vectors participate in scalar and vector products with other vectors either from the left or from the right. The gradient is a vector operator, and the gradient of a vector is a dyadic. The first scalar invariant of a dyadic is the trace of the three-by-three matrix of components of the dyadic or the scalar product of the vectors in a dyad. The vector invariant is the sum of the vector products of the vectors in the corresponding dyads. Further details about dyadics can be found, for instance, in [6].

The first term of the integrand in (1) is less singular than the second term, and it leads to no first-order contributions to the self-patch integral. The second term is not absolutely integrable and the integral can be computed as a kind of principal value given to first order by

$$\begin{aligned} \int_{S_1} dS' \frac{\vec{R}}{R^3} \times \vec{J}_s(\vec{x}', \tau) \\ \approx \vec{F} \times \vec{J}_s(\vec{x}, t) + \vec{x}_u \cdot [\nabla_s \vec{J}_s(\vec{x}, t)] \times \vec{x}_u C_2 \\ + \vec{x}_v \cdot [\nabla_s \vec{J}_s(\vec{x}, t)] \times \vec{x}_v C_3 \end{aligned} \quad (4)$$

where \vec{x}_u and \vec{x}_v are tangent vectors and \vec{F} , C_2 , and C_3 are given by (69), (57), and (58), respectively.

The self-patch correction to the computed current is then

$$\begin{aligned} \Delta \vec{J}_s = -\frac{1}{2\pi} \hat{n} \times \left[\vec{F} \times \vec{J}_s + \frac{\partial \vec{J}_s}{\partial u} \times \vec{x}_u C_2 + \frac{\partial \vec{J}_s}{\partial v} \times \vec{x}_v C_3 \right] \\ = -\frac{1}{4\pi} \left[(\hat{n} \cdot \vec{x}_{uu} C_2 + \hat{n} \cdot \vec{x}_{vv} C_3) \vec{J}_s \right. \\ \left. - 2\hat{n} \cdot \frac{\partial \vec{J}_s}{\partial u} \vec{x}_u C_2 - 2\hat{n} \cdot \frac{\partial \vec{J}_s}{\partial v} \vec{x}_v C_3 \right] \end{aligned} \quad (5)$$

where we have used (30) for the surface gradient.³ If the surface is flat, $\partial \vec{J}_s / \partial u$ and $\partial \vec{J}_s / \partial v$ are tangential vectors and the last two terms vanish. Otherwise, their contributions depend on the curvature of the surface, the rate of change of \vec{J}_s , and the size of the patch, and these terms may not be negligible.

The EFIE for perfect conductors [2] is

$$\begin{aligned} 0 = \hat{n} \times \vec{E}^{\text{in}}(\vec{x}, t) + \frac{1}{4\pi} \hat{n} \times \oint_S dS' \left[\frac{\vec{R}}{\epsilon_0 R^2 c} \frac{\partial \rho_s(\vec{x}', \tau)}{\partial t'} \right. \\ \left. + \frac{\vec{R}}{\epsilon_0 R^3} \rho_s(\vec{x}', \tau) - \frac{\mu_0}{R} \frac{\partial \vec{J}_s(\vec{x}', \tau)}{\partial t'} \right], \end{aligned} \quad (6)$$

where \vec{E}^{in} is the incident electric field, ρ_s is the surface charge density, and ϵ_0 is the free-space permittivity. We expand ρ_s in a manner similar to (3) and find the self-patch integral

$$\int_{S_1} dS' \frac{\vec{R}}{R^3} \rho_s(\vec{x}', \tau) \approx \vec{F} \rho_s(\vec{x}, t) + \vec{F} \cdot \nabla_s \rho_s(\vec{x}, t), \quad (7)$$

where the dyadic \vec{F} is given by (70). We also find

$$\int_{S_1} dS' \frac{1}{R} \frac{\partial \vec{J}_s(\vec{x}', \tau)}{\partial t'} \approx \vec{F} \frac{\partial \vec{J}_s(\vec{x}, t)}{\partial t}. \quad (8)$$

The field ρ_s is not an independent unknown; it is related to \vec{J}_s by conservation of charge,

$$\partial \rho_s / \partial t + \nabla_s \cdot \vec{J}_s = 0. \quad (9)$$

The contributions to the integrals in (1) and (6) from patches

³ In a principal coordinate system, the principal curvatures defined by (38) are $k_1 = \hat{n} \cdot \vec{x}_{uu} / \alpha^2$ and $k_2 = \hat{n} \cdot \vec{x}_{vv} / \beta^2$. The terms C_2 and C_3 are linear in Δu and Δv and positive, and the first two terms in (5) do not resemble those in [4, eqs. (59), (60), and (61)]. The reason for this discrepancy is the signs in [4, eqs. (55) and (56)]; according to the expression given for $\hat{n} \cdot \vec{R}$ after [4, eq. (48)], all terms should be negative. An integration over the circular patch gives $\vec{J}_s = -(\gamma/4)(k_1 + k_2)\vec{J}$, where γ is the radius of the patch, which is in better agreement with our result.

other than the self-patch involve fields at times earlier than t which have been previously computed; this is the basis of the stepping-in-time solution of these integral equations. The quantities F , \vec{F} , \vec{F} , C_2 , and C_3 are of first order in the linear size of the patch, so that the self-patch integral in the MFIE constitutes a small correction to the \vec{J}_s on the left side of (1). There is no such term on the left side of the EFIE, and the self-patch integrals in (6) have to be taken into account.

There is a further problem with the EFIE when the incident field has a discontinuous wavefront. At the time the wave first reaches the scatterer, only the self-patch contributes to the integral. The total contribution has to cancel the term $\hat{n} \times \vec{E}^{\text{in}}$, which is independent of the size of the patch. When the linear size of the patch decreases, the factors F , \vec{F} , and \vec{F} in (7) and (8) decrease proportionately, and we would have to increase the fields at the center of the patch for the products to remain constant. Physically, these fields cannot depend on the size of the patch. The reason for this apparent contradiction is that $\mu_0 \vec{J}_s$ jumps from zero to twice the magnetic field at the time the wave first reaches the patch; the time derivative is proportional to a delta function and the self-patch integral is really independent of the patch. The MFIE does not show this problem if the self-patch integral is ignored, but derivatives in the correction from the self-patch are also proportional to delta functions. It is best to stay away from discontinuous wavefronts.

Similar integral equations are obtained for electromagnetic scattering by a simple dielectric [1], [5]. A single integral equation can be expressed [5] in terms of operators \vec{L} , \vec{M} , and \vec{M}' , which are functionals of a tangential vector field designated by the dummy variable $\vec{\xi}_s$. The operators \vec{L} and \vec{M} are defined by

$$\vec{L}\{\vec{\xi}_s\} = \oint_S dS' \left[\frac{1}{c'} \frac{\partial \vec{\xi}_s(\vec{x}', \tau)}{\partial t'} + \frac{\vec{\xi}_s(\vec{x}', \tau)}{R} \right] \times \frac{\vec{R}}{4\pi R^2}, \quad (10)$$

$$\begin{aligned} \vec{M}\{\vec{\xi}_s\} = \oint_S dS' \left\{ \left[\frac{1}{c'} \frac{\partial \xi_s(\vec{x}', \tau)}{\partial t'} + \frac{\xi_s(\vec{x}', \tau)}{R} \right] \frac{R}{4\pi R^2} \right. \\ \left. - \frac{\partial \vec{\xi}_s(\vec{x}', \tau)}{\partial t'} \frac{1}{4\pi R} \right\}, \end{aligned} \quad (11)$$

where c' is the speed of light in the dielectric and ξ_s is a scalar function determined by $\vec{\xi}_s$ through the equation

$$\partial \xi_s / \partial t + c'^2 \nabla_s \cdot \vec{\xi}_s = 0. \quad (12)$$

The operator \vec{M}' is similar to \vec{M} . The self-patch integrals have the same form as those in (4), (7), and (8).

III. PATCHES ON A SPHERE

As an example, we consider an integration over a sphere. We divide the spherical surface into strips of equal size $\Delta\theta$ by circles of constant polar angle θ , and we divide each strip into patches of width $\Delta\phi$ by arcs of constant azimuthal angle ϕ ; we decrease the number of patches on a strip as we near the poles. Each of the two spherical caps can be left as one patch, or it can be subdivided into triangular patches.

The coordinates of a point on a sphere of radius a are given by

$$\vec{x}(\theta, \phi) = a(\hat{i} \sin \theta \cos \phi + \hat{j} \sin \theta \sin \phi + \hat{k} \cos \theta), \quad (13)$$

whence we can compute the derivatives of \vec{x} and also

$$dS = a^2 \sin \theta d\theta d\phi. \quad (14)$$

The other coefficients defined in the Appendix are

$$\alpha = a, \beta = a \sin \theta,$$

$$\Delta \rho = a[(\Delta \theta)^2 + \sin^2 \theta (\Delta \phi)^2]^{1/2} = a \Delta \gamma, \quad (15)$$

$$\kappa_1 = 0, \kappa_2 = 0, \kappa_3 = a^2 \sin \theta \cos \theta, \quad \kappa_4 = 0, \quad (16)$$

which permit us to write down the coefficients in the self-patch integrals, such as

$$F = a \Delta \theta \log \frac{\Delta \gamma + \sin \theta \Delta \phi}{\Delta \gamma - \sin \theta \Delta \phi} + a \sin \theta \Delta \phi \log \frac{\Delta \gamma + \Delta \theta}{\Delta \gamma - \Delta \theta}, \quad (17)$$

$$C_2 = \frac{\sin \theta \Delta \phi}{a} \log \frac{\Delta \gamma + \Delta \theta}{\Delta \gamma - \Delta \theta}. \quad (18)$$

The corresponding coefficients for the spherical caps have to be calculated separately, and in the Appendix we obtain

$$F' \approx 2\pi a \Delta \theta, \quad (19)$$

$$\vec{F}' \approx \pm \pi \Delta \theta \hat{k}, \quad (20)$$

$$\vec{F}' \approx \pi a \Delta \theta (\hat{i}i + \hat{j}j). \quad (21)$$

To get some idea of the size of the terms we have computed, we consider a patch located at $\theta = \pi/3, \phi = 0$ on a sphere of unit radius, and we choose the size of the patch by setting $\Delta \theta = \pi/20, \Delta \phi = \pi/15$.

The values of the integrals C_1 through C_6 as given by (54), (57), (58), (61), (62), and (63) are

$$C_1 = 0.594, C_2 = 0.284, C_3 = 0.413, C_4 = 0.205, \\ C_5 = 0.106, C_6 = 0.410. \quad (22)$$

Bennett approximates the integrals by replacing the patch by a circular patch of equal area ΔS ; the radius of the modified patch is

$$\gamma = \sqrt{(\Delta S/\pi)}. \quad (23)$$

By this method, the corresponding integrals are evaluated to be

$$C'_1 = 2\pi\gamma = 0.598, C'_2 = \pi\gamma = 0.299, C'_3 = \pi\gamma/\sin^2 \theta = 0.399, \\ C'_4 = 3\pi\gamma/(4 \sin^4 \theta) = 0.399, \\ C'_5 = \pi\gamma/(4 \sin^2 \theta) = 0.100, C'_6 = 3\pi\gamma/4 = 0.224, \quad (24)$$

and we see that the agreement varies from very good for C_1 to poor for C_4 and C_6 .

We now use the values in (22) to compute \vec{F} from (69) and obtain

$$\vec{F}/2\pi = 0.038\hat{i} + 0.029\hat{k}, \quad (25)$$

which is the vector that multiplies \vec{J}_s for the self-patch integral in (1). The magnitude of this term is not negligible compared to one except when the required accuracy is low. We can reduce the magnitude of this vector by increasing the number of patches, since it is effectively inversely proportional to the square root of the number of patches. The term $\vec{F} \times \vec{J}_s$ appears in (4) together with other terms proportional to the surface gradient of \vec{J}_s ; these terms will be negligible if \vec{J}_s varies slowly over the sphere, which is not necessarily the case.

IV. CONCLUSION

We have obtained general expressions (4), (7), and (8) for self-patch integrals that appear in the integral equations of elec-

tromagnetic scattering. We have used orthogonal coordinate systems on surfaces of three-dimensional scatterers, but similar results can be obtained for other coordinate systems and for fewer dimensions.

In particular, (4) and (7) show first-order terms that involve spatial derivatives of the surface fields. There is no *a priori* reason to neglect these terms other than a desire to simplify the numerical solution of the equations. Each such term should be examined in the context of the particular problem under consideration, which may indicate that the fields vary slowly enough across the scatterer to make these terms negligible.

We have kept all first-order contributions to the self-patch integrals. Some terms cancel for particular integral equations, as seen in (5), but not necessarily in other equations. The success of any method that involves the numerical integration of a singular integral depends on the correct computation of the correction terms, unless they are negligible. Terms that involve derivatives of the unknown fields are difficult to compute numerically and they interfere with the stepping-in-time procedure. It is possible to compute a first approximation to a field such as \vec{J}_s at time t ignoring the self-patch corrections, use these fields to compute the corrections, and then use these corrections to find the new values of the field. This procedure can be iterated. The incorporation of the self-patch corrections in a calculation can increase the accuracy of the results or, by allowing larger patches, decrease the time required for the computation.

We have also indicated how special care is needed to evaluate the self-patch integrals of the EFIE when the incident wave has a discontinuous front, in which case the derivative in (8) leads to a delta function. Similar contributions may arise from spatial derivatives such as those in (4) and (7).

APPENDIX

In this Appendix, we first recall some definitions and formulas from the geometry of surfaces, including the curvatures used in [3]. Then we evaluate a number of integrals for rectangular patches, and we also compute those needed for spherical caps.

We assume that the surface S is described by the parametric equations

$$\vec{x} = \vec{x}(u, v), \quad (26)$$

and we further assume that these coordinates are orthogonal, that is, that the tangent vectors $\vec{x}_u = \partial \vec{x}/\partial u$ and $\vec{x}_v = \partial \vec{x}/\partial v$ satisfy

$$\vec{x}_u \cdot \vec{x}_v = 0. \quad (27)$$

We designate their magnitudes by

$$\alpha = |\vec{x}_u|, \beta = |\vec{x}_v|, \quad (28)$$

and unit normal then is

$$\hat{n} = \vec{x}_u \times \vec{x}_v / (\alpha\beta). \quad (29)$$

In these orthogonal coordinates, the surface gradient operator [6] is

$$\nabla_s = \alpha^{-2} \vec{x}_u \partial/\partial u + \beta^{-2} \vec{x}_v \partial/\partial v. \quad (30)$$

To relate these derivatives to the curvature of the surface [3], we observe that $|\hat{n}|^2 = 1$ implies that the derivatives \hat{n}_u and \hat{n}_v are tangent to the surface, and from

$$\hat{n} \cdot \vec{x}_u = \hat{n} \cdot \vec{x}_v = 0 \quad (31)$$

we find

$$\vec{n}_v \cdot \vec{x}_u = \vec{n}_u \cdot \vec{x}_v = -\hat{n} \cdot \vec{x}_{uv}, \quad (32)$$

$$\vec{n}_u \cdot \vec{x}_u + \hat{n} \cdot \vec{x}_{uu} = \vec{n}_v \cdot \vec{x}_v + \hat{n} \cdot \vec{x}_{vv} = 0, \quad (33)$$

and the dyadic surface gradient² of the unit normal can be expressed as

$$\begin{aligned} \nabla_s \hat{n} = & -\hat{n} \cdot \vec{x}_{uu} \vec{x}_u \vec{x}_u / \alpha^4 - \hat{n} \cdot \vec{x}_{vv} \vec{x}_v \vec{x}_v / \beta^4 \\ & - \hat{n} \cdot \vec{x}_{uv} (\vec{x}_u \vec{x}_v + \vec{x}_v \vec{x}_u) / (\alpha^2 \beta^2). \end{aligned} \quad (34)$$

(The last term vanishes for a principal coordinate system as used in [4].) We also have

$$\nabla_s \times \hat{n} = 0, \quad (35)$$

$$\nabla_s \cdot \hat{n} = -\hat{n} \cdot \vec{x}_{uu} / \alpha^2 - \hat{n} \cdot \vec{x}_{vv} / \beta^2 = -J, \quad (36)$$

where J is the mean curvature of the surface. The total curvature is the second scalar invariant of the dyadic (34),

$$K = [\hat{n} \cdot \vec{x}_{uu} \hat{n} \cdot \vec{x}_{vv} - (\hat{n} \cdot \vec{x}_{uv})^2] / (\alpha^2 \beta^2), \quad (37)$$

and the principal curvatures k_1 and k_2 are the roots of the equation

$$k^2 - Jk + K = 0. \quad (38)$$

To do the self-patch integrals, we assume that the field point is fixed at the center of the patch with coordinates u_0 and v_0 . The source point has coordinates u' and v' , and we define the (small) patch variables \bar{u} and \bar{v} by setting

$$u' = u_0 + \bar{u}, \quad v' = v_0 + \bar{v}. \quad (39)$$

The surface element is

$$dS' = \alpha' \beta' du' dv' = \alpha' \beta' d\bar{u} d\bar{v}, \quad (40)$$

where α' and β' are the values of α and β at the point \vec{x}' . The vector $\vec{R} = \vec{x} - \vec{x}'$ is expanded in the power series

$$\vec{R} = -[\vec{x}_u \bar{u} + \vec{x}_v \bar{v} + \frac{1}{2} (\vec{x}_{uu} \bar{u}^2 + 2\vec{x}_{uv} \bar{u} \bar{v} + \vec{x}_{vv} \bar{v}^2) + \dots], \quad (41)$$

whence the square of its magnitude is

$$R^2 = \alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2 + \kappa_1 \bar{u}^3 + \kappa_2 \bar{u}^2 \bar{v} + \kappa_3 \bar{u} \bar{v}^2 + \kappa_4 \bar{v}^3 + \dots, \quad (42)$$

where, using the derivatives of (27), we have

$$\begin{aligned} \kappa_1 = \vec{x}_u \cdot \vec{x}_{uu}, \quad \kappa_2 = \vec{x}_u \cdot \vec{x}_{uv}, \quad \kappa_3 = \vec{x}_v \cdot \vec{x}_{uv}, \\ \kappa_4 = \vec{x}_v \cdot \vec{x}_{vv}. \end{aligned} \quad (43)$$

To find other powers of R , we write

$$R^2 = (\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2) [1 + (\kappa_1 \bar{u}^3 + \kappa_2 \bar{u}^2 \bar{v} + \kappa_3 \bar{u} \bar{v}^2 + \kappa_4 \bar{v}^3) / (\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2) + \dots], \quad (44)$$

$$R^\nu = (\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{\nu/2} [1 + (\nu/2)(\kappa_1 \bar{u}^3 + \kappa_2 \bar{u}^2 \bar{v} + \kappa_3 \bar{u} \bar{v}^2 + \kappa_4 \bar{v}^3) / (\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2) + \dots]. \quad (45)$$

We also note that $\alpha^2 = \vec{x}_u \cdot \vec{x}_u$ and $\beta^2 = \vec{x}_v \cdot \vec{x}_v$ imply that

$$\begin{aligned} \partial \alpha / \partial u = \kappa_1 / \alpha, \quad \partial \alpha / \partial v = \kappa_2 / \alpha, \quad \partial \beta / \partial u = \kappa_3 / \beta, \\ \partial \beta / \partial v = \kappa_4 / \beta. \end{aligned} \quad (46)$$

We now compute exactly a number of integrals of functions

of the patch variables \bar{u} and \bar{v} . We first consider

$$C_1 = \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\alpha \beta}{(\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{1/2}}, \quad (47)$$

where α and β are the constant values of α' and β' at \vec{x}_0 , and

$$u_{\pm} = u_0 \pm \frac{1}{2} \Delta u, \quad v_{\pm} = v_0 \pm \frac{1}{2} \Delta v. \quad (48)$$

We change variables to

$$\bar{u} = \alpha(u' - u_0) = \alpha \bar{u}, \quad \bar{v} = \beta(v' - v_0) = \beta \bar{v}, \quad (49)$$

which reduces the integral to

$$C_1 = \int_{-a}^a d\bar{u} \int_{-b}^b d\bar{v} \frac{1}{(\bar{u}^2 + \bar{v}^2)^{1/2}}, \quad (50)$$

where

$$a = \frac{1}{2} \alpha \Delta u, \quad b = \frac{1}{2} \beta \Delta v. \quad (51)$$

We change to polar coordinates and take advantage of the symmetry with respect to the axes to obtain

$$\begin{aligned} C_1 = 4 \int_0^{\phi_0} d\phi \int_0^{a/\cos \phi} d\rho + 4 \int_{\phi_0}^{\pi/2} d\phi \int_0^{b/\sin \phi} d\rho \\ = 2a \log \frac{1 + \sin \phi_0}{1 - \sin \phi_0} + 2b \log \frac{1 + \cos \phi_0}{1 - \cos \phi_0}, \end{aligned} \quad (52)$$

where $\phi_0 = \arctan(b/a)$. We define

$$\Delta \rho = [(\alpha \Delta u)^2 + (\beta \Delta v)^2]^{1/2} \quad (53)$$

and find

$$C_1 = \alpha \Delta u \log \frac{\Delta \rho + \beta \Delta v}{\Delta \rho - \beta \Delta v} + \beta \Delta v \log \frac{\Delta \rho + \alpha \Delta u}{\Delta \rho - \alpha \Delta u}. \quad (54)$$

We now compute

$$C_2 = \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\alpha \beta \bar{u}^2}{(\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{3/2}} \quad (55)$$

either as above or by noting that

$$C_2 = (C_1 - \alpha \partial C_1 / \partial \alpha) / \alpha^2. \quad (56)$$

The result is

$$C_2 = \frac{\beta \Delta v}{\alpha^2} \log \frac{\Delta \rho + \alpha \Delta u}{\Delta \rho - \alpha \Delta u}. \quad (57)$$

Similarly,

$$\begin{aligned} C_3 = \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\alpha \beta \bar{v}^2}{(\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{3/2}} \\ = \frac{\alpha \Delta u}{\beta^2} \log \frac{\Delta \rho + \beta \Delta v}{\Delta \rho - \beta \Delta v}. \end{aligned} \quad (58)$$

For

$$C_4 = \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\alpha \beta \bar{u}^4}{(\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{5/2}}, \quad (59)$$

we have

$$C_4 = (C_2 - \alpha \partial C_2 / \partial \alpha) / (3\alpha^2), \quad (60)$$

whence

$$C_4 = \frac{\beta \Delta v}{\alpha^4} \log \frac{\Delta \rho + \alpha \Delta u}{\Delta \rho - \alpha \Delta u} - \frac{2\beta \Delta u \Delta v}{3\alpha^3 \Delta \rho}. \quad (61)$$

Similarly,

$$C_5 = \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\alpha \beta \bar{u}^2 \bar{v}^2}{(\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{5/2}} \\ = (C_2 - \beta \partial C_2 / \partial \beta) / (3\beta^2) = \frac{2\Delta u \Delta v}{3\alpha \beta \Delta \rho}, \quad (62)$$

$$C_6 = \int_{u_-}^{u_+} du' \int_{v_-}^{v_+} dv' \frac{\alpha \beta \bar{v}^4}{(\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{5/2}} \\ = \frac{\alpha \Delta u}{\beta^4} \log \frac{\Delta \rho + \beta \Delta v}{\Delta \rho - \beta \Delta v} - \frac{2\alpha \Delta u \Delta v}{3\beta^3 \Delta \rho}. \quad (63)$$

These integrals satisfy the relations

$$\alpha^2 C_2 + \beta^2 C_3 = C_1, \quad (64)$$

$$\alpha^2 C_4 + \beta^2 C_5 = C_2, \quad (65)$$

$$\alpha^2 C_5 + \beta^2 C_6 = C_3. \quad (66)$$

We now compute certain integrals over the self-patch S_1 in terms of C_1 to C_6 , that is, up to terms linear in Δu or Δv . We have

$$F = \int_{S_1} \frac{dS'}{R} \approx \iint \frac{\alpha \beta du' dv'}{(\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{1/2}} = C_1. \quad (67)$$

The vector integral

$$\vec{F} = \int_{S_1} \frac{dS' \vec{R}}{R^3} \quad (68)$$

is more singular than F and not absolutely integrable, but the integrals in the expansion with the highest powers of \bar{u} and \bar{v} are odd in those variables and vanish by symmetry; the remaining terms give a kind of principal value of this integral. We use the higher order expansions of α' , β' , \vec{R} and R^{-3} and retain only the terms with even powers of \bar{u} and \bar{v} to find, with the help of (41), (45), and (46),

$$\vec{F} \approx -\frac{1}{2} \iint \alpha \beta du' dv' (\vec{x}_{uu} \bar{u}^2 + \vec{x}_{vv} \bar{v}^2) (\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{-3/2} \\ - \frac{1}{2} \iint \alpha \beta du' dv' \{ \vec{x}_u [(\kappa_1 + 4\kappa_3 \alpha^2 / \beta^2) \bar{u}^4 + (4\kappa_1 \beta^2 / \alpha^2 \\ + \kappa_3) \bar{u}^2 \bar{v}^2] + \vec{x}_v [(\kappa_2 + 4\kappa_4 \alpha^2 / \beta^2) \bar{u}^2 \bar{v}^2 \\ + (4\kappa_2 \beta^2 / \alpha^2 + \kappa_4) \bar{v}^4] \} (\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{-5/2} \\ = -\frac{1}{2} \vec{x}_{uu} C_2 - \frac{1}{2} \vec{x}_{vv} C_3 - \frac{1}{2} \vec{x}_u [(\kappa_1 + 4\kappa_3 \alpha^2 / \beta^2) C_4$$

$$+ (4\kappa_1 \beta^2 / \alpha^2 + \kappa_3) C_5] - \frac{1}{2} \vec{x}_v [(\kappa_2 + 4\kappa_4 \alpha^2 / \beta^2) C_5 \\ + (4\kappa_2 \beta^2 / \alpha^2 + \kappa_4) C_6]. \quad (69)$$

We also compute the dyadic

$$\vec{F} = \int_{S_1} dS' \frac{\vec{R} \vec{R}}{R^3} \approx \iint \frac{\alpha \beta du' dv' (\vec{x}_u \vec{x}_u \bar{u}^2 + \vec{x}_v \vec{x}_v \bar{v}^2)}{(\alpha^2 \bar{u}^2 + \beta^2 \bar{v}^2)^{3/2}} \\ = \vec{x}_u \vec{x}_u C_2 + \vec{x}_v \vec{x}_v C_3. \quad (70)$$

Other integrals, such as $\int dS' \vec{R} / R^2$, are of higher order in Δu and Δv .

To compute the corresponding integrals on a spherical cap about $\theta = 0$, we note that

$$\vec{R} = a[-\hat{i} \sin \theta' \cos \phi' - \hat{j} \sin \theta' \sin \phi' + \hat{k}(1 - \cos \theta')], \quad (71)$$

$$R^2 = 2a^2(1 - \cos \theta') \approx a^2(\theta'^2 - \theta'^4/12), \quad (72)$$

$$F' = \int_{S_1} dS' \frac{1}{R} \approx \int_0^{\Delta \theta} d\theta' \int_0^{2\pi} d\phi' \frac{a^2 \theta'}{a\theta'} = 2\pi a \Delta \theta, \quad (73)$$

$$\vec{F}' = \int_{S_1} dS' \frac{\vec{R}}{R^3} \approx \int_0^{\Delta \theta} d\theta' \int_0^{2\pi} d\phi' \\ \frac{a^3 \theta' \left(-\hat{i} \theta' \cos \phi' - \hat{j} \theta' \sin \phi' + \frac{1}{2} \hat{k} \theta'^2 \right)}{a^3 \theta'^3} = \pi \Delta \theta \hat{k}, \quad (74)$$

$$\vec{F}' = \int_{S_1} dS' \frac{\vec{R} \vec{R}}{R^3} \approx \int_0^{\Delta \theta} d\theta' \int_0^{2\pi} d\phi' \\ \frac{a^4 \theta' (-\hat{i} \theta' \cos \phi' - \hat{j} \theta' \sin \phi') (-\hat{i} \theta' \cos \phi' - \hat{j} \theta' \sin \phi')}{a^3 \theta'^3} \\ = \pi a \Delta \theta (\hat{i} \hat{i} + \hat{j} \hat{j}). \quad (75)$$

For the cap at $\theta = \pi$, it suffices to change the sign of \vec{F}' .

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Egon Marx, for a photograph and biography please see page 172 of the February 1984 issue of this TRANSACTIONS.