Array Grating Lobes due to Periodic Phase, Amplitude, and Time Delay Quantization

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Abstract—**Simple approximate formulas and generalized curves to evaluate the grating lobe levels of various kinds of contiguous subarrays with uniform illumination within each subarray are presented. The cases considered include phase quantization due to discrete phase shifters, amplitude taper at subarray input ports and time delay at the subarray input ports. In addition, the work is generalized to include certain cases of quantized phase or time delay at the input ports in addition to amplitude taper at subarray input ports.**

I. INTRODUCTION

There are a number of reasons for choosing to design large phase arrays with uniformly illuminated contiguous subarrays. Aside from the obvious simplicity of constructing an array of large, similar clusters of elements, contiguous subarray fabrication affords an economical means of introducing amplitude taper or time delay across an array. In addition, the use of phase shifters with discrete phase states leads to the formation of contiguous subarrays with a constant phase state or a phase progression across each subarray different from that for the commanded array scan angle. In each of these occurrences, the partitioning of an array into subarrays gives rise to grating lobes caused by the periodic errors in the aperture illumination function.

This communication presents several simple approximate formulas for estimating the grating lobe levels of arrays with continuous, uniformly illuminated subarrays. The formulas are sufficiently general to be applicable to most array factor illumination functions, and so the results are presented in terms of array beam broadening factors to retain this generality. Although the primary areas of concern are the grating lobes caused by step-wise approximation of amplitude taper (Fig. 1(b)), and the use of time delay steering in combination with phase steered subarrays (Fig. 1(c)), the communication begins with a brief treatment of the grating lobes caused by phase quantization effects in a uniformally illuminated array (Fig. 1(a)), a problem that has received more attention in prior literature [1].

Shown to the right of Figs. 1(a), 1(b), and 1(c) are representative patterns for 64-element arrays, with $\lambda_0/2$ spacing between elements, and eight-elements for subarray. In each case the grating lobes due to subarray level pattern distortion are evident. The horizontal bars denoted at each grating lobe are the approximate grating lobe levels computed using the analysis to follow.

II. CHARACTERISTICS OF AN ARRAY OF UNIFORMLY ILLUMINATED CONTIGUOUS SUBARRAYS

Fig. 1(d) shows the schematic organization of an array of contiguous, uniformly illuminated subarrays. In the most general cases treated, there will be time delay devices and amplitude taper applied at the subarray input ports, but there will at most be phase shift $\Delta\phi$ between the elements within each subarray. The phase increment between elements required to scan the subarray to an angle θ_0 is

$$\Delta\phi = 2\frac{\pi d}{\lambda_0}u_0 \tag{1}$$

where $u_0 = \sin\theta_0$, d is the element spacing, and λ_0 is the wavelength at the design frequency.

The normalized far-field pattern of this array (neglecting the array element patterns) is given below for an array of m subarrays of M elements each.

$$F(u) = \frac{1}{m}\sum_{q=-(\frac{m-1}{2})}^{\frac{m-1}{2}} \omega_q e^{jqMZ}\left[\frac{1}{M}\sum_{i=-(\frac{m-1}{2})}^{\frac{M-1}{2}} e^{jiz}\right] \tag{2}$$

where

$$\sum_q |\omega_q| = m.$$

In this convention the sums over i and q are over integers for M or m odd, and over half-integers for M or m even. The variables Z and z are given by

$$Z = \frac{2\pi d}{\lambda}(u - u_0); \qquad u = \sin\theta \qquad u_0 = \sin\theta_\theta$$

$$z = \frac{2\pi d}{\lambda}u - \Delta\phi = 2\pi d\left(\frac{u}{\lambda} - \frac{u_s}{\lambda_0}\right) \tag{3}$$

for the array with subarrays steered to u_s by phase shifters ($u_s \neq u_0$ in general), and with time delay between the subarray ports. Since all subarrays are the same size, with constant illumination, the pattern is written as the product of an array factor $A(Z)$ and a subarray pattern $f(z)$ or

$$F(u) = A(Z)f(z) \tag{4}$$

with the subarray pattern the bracketed expression of (2) summed to

$$f(z) = \frac{\sin Mz/2}{M\sin z/2} \tag{5}$$

and the array factor the remaining part of (2).

Since the subarrays will in general be several wavelengths across, grating lobes of the array factor $A(Z)$ will occur at the direction cosines

$$u_p = u_0 + \frac{p\lambda}{Md}, \quad p = (\pm 1, \pm 2 \cdots) \tag{6}$$

for all p corresponding to real space ($|u_p| < 1$).

Very simple and convenient estimates of the residual grating lobe levels are obtained by approximating the array factor and the subarray pattern in the vicinity of any "pth" grating lobe be defining the variable δu such that

$$u = u_0 + \frac{p\lambda}{Md} + \delta u. \tag{7}$$

With this substitution the parameters Z and z become

$$Z = 2\pi\left(\frac{d}{\lambda}\delta u + \frac{p}{M}\right) \tag{8}$$

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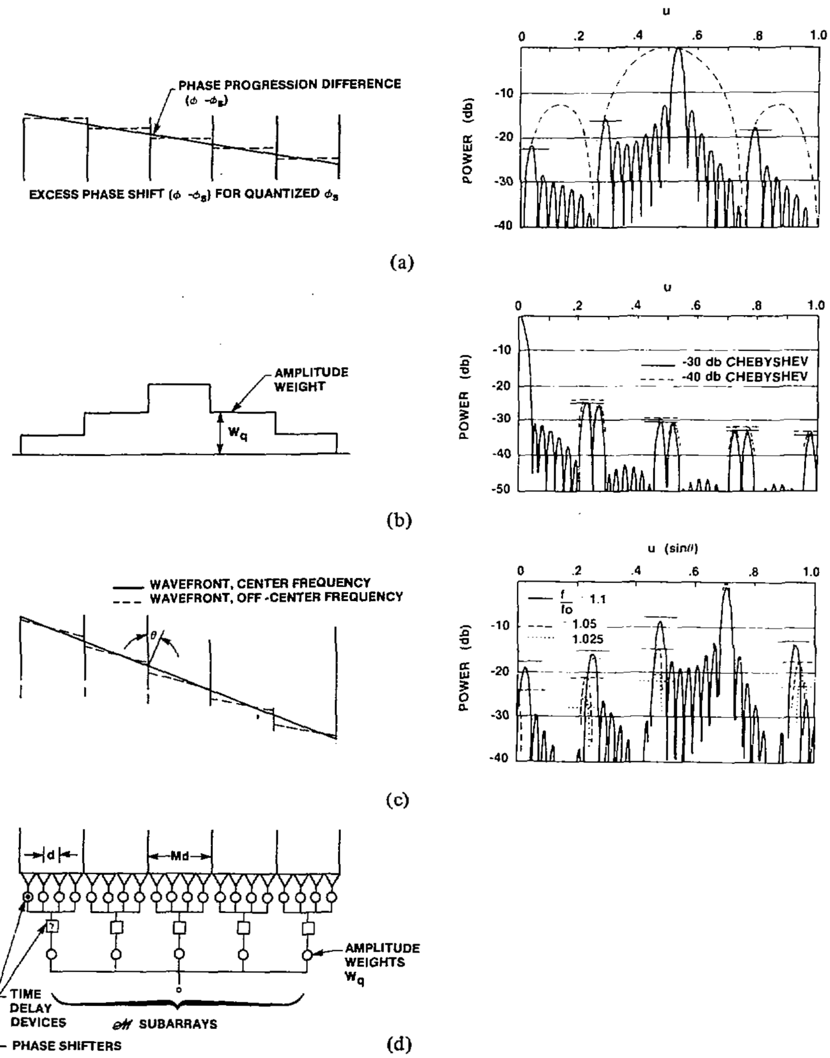

Fig. 1. Three types of contiguous subarrays and their radiation patterns. (a) Case 1: Discrete phase shifter states (equal amplitude weights). (b) Case 2: Taper at subarray input ports (ideal phase progression). (c) Case 3: Time delay at subarray ports (equal amplitude weights). (d) Array of contiguous subarrays. Data are for 64 element arrays with $\lambda/2$ spacing between elements and eight elements per subarray.

$$z = 2\pi\left[\frac{d}{\lambda}\delta u + \frac{p}{M} + \frac{u_0 d}{\lambda_0}\left(\frac{\Delta f}{f_0}\right) + \frac{d}{\lambda_0}(u_0 - u_s)\right] \tag{9}$$

where $\Delta f = f - f_0$ is the difference between the frequency and the design frequency f_0.

The coordinate δu is thus the angular displacement (in direction cosine space) from the p'th grating lobe. Note that for integer p the array factor is exactly replicated in the vicinity of each lobe, or (for $|u| < 1$)

$$A(Z) = A\left(\frac{2\pi d}{\lambda}\,\delta u\right).$$

The subarray pattern for this generalized case is given in terms of the localized coordinate δu as

$$f(z) = (-1)^p \cdot \frac{\sin\left[\pi M d \dfrac{u_0}{\lambda_0}\dfrac{\Delta f}{f_0} + \dfrac{\pi d}{\lambda} M\delta u + \dfrac{\pi M d}{\lambda_0}(u_0 - u_s)\right]}{M \sin\left[\dfrac{\pi u_0 d}{\lambda_0}\dfrac{\Delta f}{f_0} + \dfrac{\pi p}{M} + \dfrac{\pi d}{\lambda}\delta u + \dfrac{\pi d}{\lambda_0}(u_0 - u_s)\right]}. \tag{10}$$

The expressions (2) and (10) will be used to investigate the three special cases that are the subject of this communication.

In each of these cases, the methodology will be to use the above expression for the subarray pattern, and seek local expansions of the numerator, when appropriate, that describe behavior near grating lobe peaks. The accuracy of this analysis is greatest for cases in which the array factor beam is very narrow compared to the subarray beamwidth, and in practice this means that the results are applicable to arrays with six or more subarrays.

Case 1. Phase Quantization in a Uniformly Illuminated Array

Although most evaluations of phase quantization effects are based upon integrals of continuous phase distributions, an array excited by digitally quantized phase shifters is one example that can be readily analyzed as a subarraying problem, with results that are more general than obtained in previous studies. In this case (Fig. 1(a)) the desired linear phase progression cannot be exactly maintained by phase shifters with phase progression having the least significant bit

$$\phi_0 = \frac{2\pi}{2^N}. \tag{11}$$

To form a beam at u_0, the required incremental phase shift between elements is (at $\lambda = \lambda_0$):

$$\Delta\phi = \frac{2\pi d}{\lambda_0} \sin\theta_0 = \frac{2\pi d}{\lambda_0} u_0. \tag{12}$$

Since the least significant bit is ϕ_0, the array can only be perfectly collimated at angles with required phase progression some multiple of ϕ_0. At any other scan angle a phase progression that is some multiple of ϕ_0 is applied to the array, so that sections of the array have phase increment, $(2\pi d/\lambda)u_s$ and point to some angle with direction cosine u_s. The remaining error in phase increment is $(2\pi d/\lambda)\cdot(u_s - u_0)$, which leads to an increasing error that can be corrected at various places across the array by inserting an added phase shift of the least significant bit. (See Fig. 1(a).) The resulting pattern distortion takes its most serious form when the error buildup and its correction are entirely periodic, for then the array is divided into virtual subarrays, each subarray having the same phase error gradient. In this case the distance between subarray phase centers is such that the total phase error increment between subarrays is equal to the phase of the least significant bit, or

$$|M\Delta\phi - M\Delta\phi_s| = \frac{2\pi d}{\lambda} M|(u_0 - u_s)| = \frac{2\pi}{2^N}. \tag{12}$$

This expression is used to determine the subarray size Md for specific $(u_0 - u_s)$ conditions.

The grating lobe level for this type of distortion is approximately the value of the subarray pattern at the grating lobe peaks ($\phi_0 = 0$). At center frequency that is

$$|f(z)| = \left| \frac{\sin\left[M\frac{\pi d}{\lambda_0}(u_0 - u_s)\right]}{M\sin\left[\frac{\pi d}{\lambda_0}(u_0 - u_s) + \frac{p\pi}{M}\right]} \right| = \left| \frac{\sin[\pi/2^N]}{M\sin\left[\frac{\pi}{M2^N} + \frac{p\pi}{M}\right]} \right|. \tag{13}$$

The power of this grating lobe is written approximately

$$P_{gl} = |f|^2 = \left[\frac{\pi/2^N}{M\sin\left(\frac{p'\pi}{M}\right)} \right]^2 \tag{14}$$

where

$$p' = p + \frac{1}{2^N}.$$

The factor $[M\sin(p'\pi/M)]^{-2}$ is the envelope of the subarray pattern peak power sampled at the grating lobe point; this factor occurs in another expression given later, and so is plotted (in dB) in Fig. 2 for values of $M > |p'|$ (the near grating lobes). The general expression for power at the peak of the p'th grating lobe (14) is thus obtained using this envelope function as

$$GL = 10\log P_{gl} = \text{envelope (dB)} + 20\log\pi - 20N\log 2$$
$$= \text{envelope (dB)} + 9.94 - 6.02N. \tag{15}$$

Fig. 1(a) shows a typical case of a uniformally illuminated 64-element array with $\lambda/2$ spacing. With three-bit phase shifters (least bit 45°) the array forms a perfectly collimated beam at $u_s = u_0 = 0.5(\theta_0 = 30°)$, but at 32.1° ($u_0 = 0.53125$) with interelement phase shifters set to the least significant bit phase gradient $(2\pi d/\lambda)u_s$, there is an excess phase shift of 45° across each set of eight elements. The pattern shows that grating lobes at various levels between −16 and −23 dB result from this periodic phase error. The horizontal lines in the figure show the approximate grating lobe levels computed using (15). This figure also shows the subarray pattern of the eight-element subarray scanned to $u_s = 0.5$, and indicates how the product of subarray pattern and array factor limits the grating lobe heights.

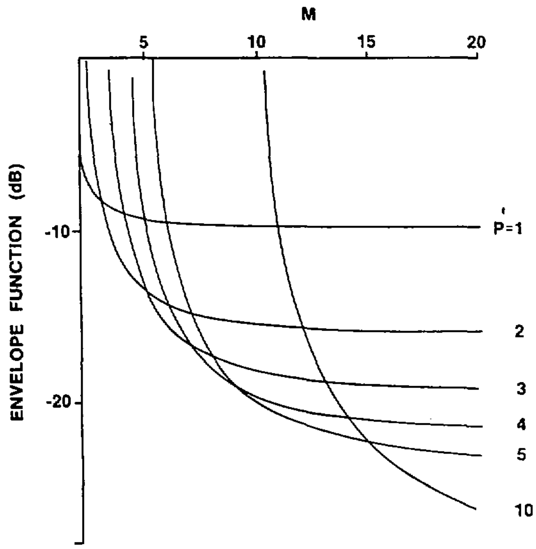

Fig. 2. Envelope factor (dB) versus number of elements in subarray.

Depending upon which scan angles are required, other size subarrays are formed at different scan angles. For example, at $u_0 = 0.5156$, $\theta = 31.04°$ the excess of 45° phase shift spans 16-element subarrays. In each case, (12) is used to give the subarray size "$M\cdot d$."

In the limiting case of M large, the envelope curves tend to an asymptote and the grating lobe power is:

$$P_{gl} \doteqdot \left(\frac{1}{p'2^N}\right)^2 \tag{16}$$

of which a special case for $p = 1$ is the $P_{gl} = 1/2^{2N}$ result obtained by Miller [1], using a different approach, and which is quoted in many standard texts. It is significant to note, however, that Miller's result underestimates the size of this maximum grating lobe.

Case 2. Amplitude Taper at Subarray Input Ports

A phase steered array, organized into equally spaced, uniformally illuminated subarrays, has its grating lobes located at the null points of the subarray pattern. If the array is uniformally excited, its beamwidth is narrow and the subarray nulls completely remove the grating lobes. When the excitation amplitude at the subarray input ports is weighted for array factor sidelobe reduction, the beamwidth broadens, and at the grating lobe angles there occur split (monopulse-like) beams as shown in Fig. 1(b).

To evaluate the power level of these split grating lobes, it is convenient to use a fairly general representation of the array factor (2) in the vicinity of each p'th grating lobe. At center frequency, and with each subarray scanned to θ_0 $\Delta f = 0$; $\theta_s = \theta_0$) the array factor grating lobe is centered on the subarray pattern null, and in the localized region from the beam peak out beyond the −3 dB point can be approximated at each p'th grating lobe by the expression at

$$A(Z) = \frac{B \sin\left[\frac{Mm\pi d}{B\lambda_0}\ \delta u\right]}{Mm \frac{\pi d}{\lambda_0}\ \delta u} \tag{17}$$

which represents a broadened beam with beam broadening factor B, that has been normalized to unity.

The beam broadening factor B is defined with reference to the half power beamwidth of a uniformally illuminated array of mM elements, and so that the beamwidth is given approximately by 0.886 $\lambda_0 B/Mmd$, with B the ratio of the beamwidth of the tapered array to that of the uniformally illuminated array.

The subarray pattern for this case is given by (10) with $u_s = u_0$ and $\Delta f = 0$. For arrays with more than four subarrays, the grating lobe is so narrow that the sine in the numerator of ten is approximated by its argument. Equation (10) thus reduces to the following:

$$f(z) = \frac{(-1)^p \left[\frac{M\pi d}{\lambda_0}\ \delta u\right]}{M \sin\left(\frac{p\pi}{M}\right)}. \tag{18}$$

Near the p'th grating lobe, the product of the subarray pattern and the array factor is thus approximately

$$A(z)f(z) = \frac{(-1)^p\, B \sin\left[\frac{mM\pi}{B}\ \frac{d}{\lambda_0}\ \delta u\right]}{mM \sin\left(\frac{p\pi}{M}\right)}. \tag{19}$$

This expression has the proper zero at $\delta u = 0$ to produce the characteristic split lobe centered on the p'th grating lobe.

The normalized power at this grating lobe has the particularly simple form:

$$P_{gl} = \frac{B^2}{m^2 M^2 \sin^2\left(\frac{\pi p}{M}\right)} \tag{20}$$

indicating that the lobe value increases in proportion to the square of the array beamwidth, but is restricted in size by the envelope of the subarray power pattern,

The grating lobe level can again be obtained from the envelope pattern (Fig. 2) using

$$\text{GL} = 10 \log P_{gl} = \text{envelope (dB)} + 20 \log B - 20 \log m. \tag{21}$$

Fig. 1(b) shows an example of a 64-element array with $\lambda/2$ spacing grouped into eight contiguous subarrays, and illuminated at the subarray input ports by −30 and −40 Chebyshev tapers. The figure reveals the split grating lobe characteristic mentioned earlier, but shows the lobes as clearly dominating the far sidelobe pattern. Based on beam broadening factors of 1.29 and 1.43 for the Chebyshev patterns, evaluation of (19) shows that the −40 dB pattern should have grating lobes about 0.9 dB higher than the −30 dB pattern. The approximate levels (21) of Fig. 1(b) obviously agree very closely with the computed pattern.

Equation (21) is a good approximation for all beam broadening values, even the limiting case of uniform illumination ($B = 1$) where the grating lobe itself disappears, in which case (21) will give the approximate sidelobe envelope at the grating lobe locations even though there are no grating lobes.

Case 3. Time Delay at Subarray Ports

In the limit of very small frequency excursion for a large array, it may still be advantageous to use time delay at the subarray input ports. In this case, the grating lobe peak is not split, and the peak grating lobe values are given directly by the subarray pattern envelope, as in case 1.

Using a small angle expansion for the numerator of ten, the normalized power in the p'th lobe is

$$P_s = \frac{\pi^2 X^2}{\sin^2 \pi[X + p/M]}, \qquad \text{where } X = \frac{u_0 d}{\lambda_0}\ \frac{\Delta f}{f_0}. \tag{22}$$

Note that $|X| < 1/M$, in order that the main beam not "squint" out to a grating lobe location. This insures that P_s never becomes singular. A plot of grating lobe level vs the variable X is given in Fig. 3 for various P/M ratios.

Fig. 1(c) shows an example of a uniformally illuminated array with time delay steering at the subarray level. The results of (22) are plotted as horizontal lines, and are clearly quite good representations of the computed grating lobe levels for various f/f_0 ratios.

CONCLUSION AND COMMENTS ABOUT GENERALITY

Fig. 4 shows the grating lobe structure of a 64-element array with a −40 dB Chebyshev illumination at the input ports of eight-element subarrays. The array is scanned using time delay at the subarray input ports, and phase shifters within the subarrays. The solid horizontal lines show the grating lobe levels computed using (22) (or Fig. 3) with $f/f_0 = 1.05$. The figure clearly indicates that the results for case 3 can be extended to include a situation when there is a pattern distortion due to quantized amplitude taper in addition to time delay. The reason for this more general result is that (33) was derived on the basis of the subarray pattern envelope and, since the subarray pattern null does not fall at the grating lobe angle, the lobes are not split, and the beam broadening factor argument used in case 2 does not apply. So if the grating lobes that result from case 1 and case 3 type distortion are large, then the grating lobes are sampling subarray pattern envelopes quite far from the nulls, and the quantized amplitude taper has little effect on the validity of the approximations. The case 1 and case 3 results can thus be applied in most situations even when the amplitude taper is quantized.

The results of this brief analysis are simple approximate formulas and graphs to predict the levels of grating lobes due to various subarray excitations for arrays of contiguous subarrays. The scale of the graphs has been kept as general as possible in order to cover the widest possible range of array excitations.

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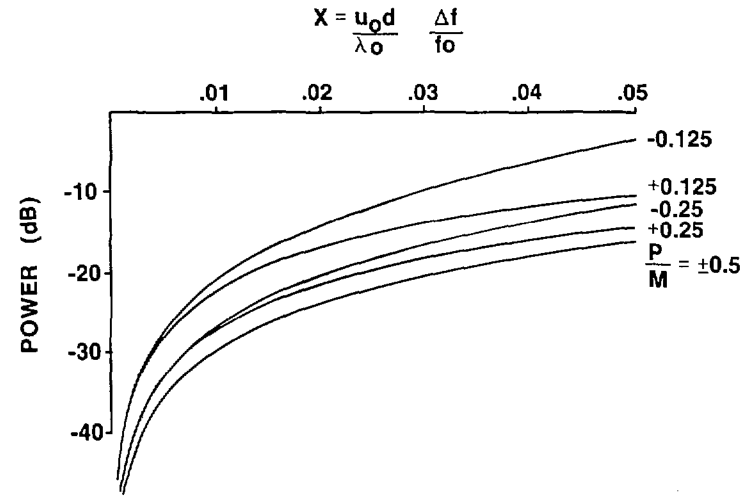

Fig. 3. Grating lobe power for array with time delay at subarray ports.

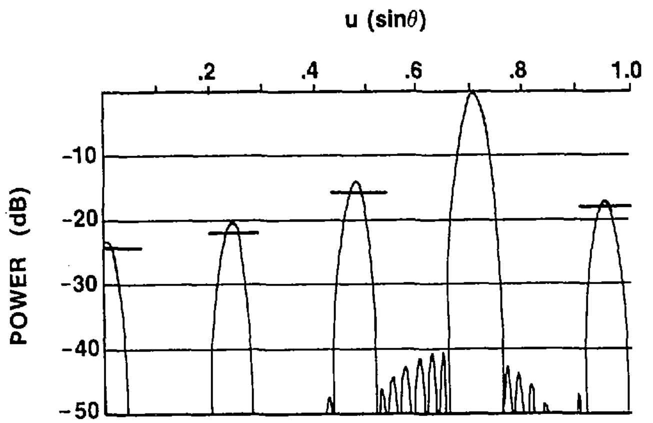

Fig. 4. Power pattern for array with time delay at subarray ports and a 40 dB Chebyshev taper $f/f_0 = 1.05$.

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Improvement of a Rain Attenuation Prediction Method for Terrestrial Microwave Links

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Abstract—**An empirical model for predicting rain induced attenuation on terrestrial paths using effective path length is proposed. Comparison with the recently prediction method proposed by the International Radio Consultative Committee (CCIR) is made. It appears that our prediction method reveals good agreement with experimental data from different locations in the world.**

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I. INTRODUCTION

Above a certain threshold of frequency, the excess attenuation due to rainfall becomes one of the most important limits of the performance of line-of-sight (LOS) microwave links. In temperate climates this frequency threshold is about 10 GHz, while in tropical climates in general and in equatorial climate particularly, since raindrops are larger than in temperate climates, the incidence of rainfall on radio links becomes important for frequencies as low as about 7 GHz [1]-[3]. Estimates of rain attenuation are usually derived from the available information on rain rates observed in the geographical area considered. Most of the many methods proposed for predicting rain induced attenuation make use of the rainfall cumulative distribution measured at a point. Certain authors have used the concept of equivalent path-averaged rain rate [5] which is obtained by multiplying the point rain rate for the time percentage of interest by a reduction factor, while other authors use an effective path length the value of which is obtained in multiplying the actual path length by a reduction coefficient [6]. This effective path length is the hypothetical length of a path along which the attenuation for a given time percentage results from a point rainfall rate that occurs for the same time percentage. Rain intensity over this effective path length is assumed to be constant [4]. In this communication, a prediction method using effective path length is proposed and compared with the prediction method recently adopted by the International Radio Consultative Committee (CCIR) during its XVth Plenary Assembly held in 1982.

II. DESCRIPTION OF THE PROPOSED METHOD

The rain induced attenuation on a line-of-sight (LOS) path can be expressed as

$$A(\text{dB}) = kR^{\alpha}L_{\text{eff}} \tag{1}$$

with

$$L_{\text{eff}} = rl \tag{2}$$