for his programming and data reductions skills as well as to Karen Melvin for typing this communication.

REFERENCES

- [1] J. Goldhirsh, "Predictive methods for rain attenuation using radar and *in situ* measurements tested against the 28 GHz COMSTAR beacon signal," *IEEE Trans. Antennas Propagat.*, vol. AP-27, no. 3, pp. 398-406, 1979.
- [2] —, "Slant path fade and rain-rate statistics associated with the COMSTAR beacon at 28.56 GHz for Wallops Island, Virginia over a three-year period," *IEEE Trans. Antennas Propagat.*, vol. AP-30, no. 2, pp. 191-198, Mar. 1982.
- [3] —, "Radar prediction of absolute rain fade distributions for earth-satellite paths and general methods for extrapolation of fade statistics to other locations," *IEEE Trans. Antennas Propagat.*, vol. AP-30, no. 6, pp. 1128-1134, Nov. 1982.
- [4] —, "Space diversity performance prediction for earth-satellite paths using radar modeling techniques," *Radio Sci.*, vol. 17, no. 6, pp. 1400-1410, Nov.-Dec. 1982.
- [5] —, "A review on the application of nonattenuation frequency radars for estimating rain attenuation and space diversity performance," *IEEE Trans. Geosci. Electron.*, vol. GE-17, no. 4, pp. 218-239, Oct. 1979.
- [6] CCIR, "Recommendations and reports of the CCIR, 1978," in XIVth Plenary Assembly, vol. V, Propagation in Non-Ionized Media, Kyoto, Japan, 1978.
- [7] R. K. Crane, "Prediction of attenuation by rain," *IEEE Trans. Commun.*, vol. COM-28, no. 9, pp. 1717-1733, Sept. 1980.
- [8] J. Goldhirsh and J. R. Rowland, "A tutorial assessment of atmospheric height uncertainties for high precision satellite altimeter missions to monitor ocean currents," *IEEE Trans. Geosci. Remote Sensing*, vol. GE-20, no. 4, pp. 418-434, Oct. 1982.
- [9] W. L. Jones, L. C. Schroeder, D. H. Boggs, E. M. Bracelente, R. A. Brown, G. J. Dome, W. J. Pierson, and F. Wentz, "The Seasat-A satellite scatterometer: The geophysical evaluation of remotely sensed wind vectors over the ocean," *J. Geophys. Res.*, vol. 87, no. C5, pp. 3297-3317, Apr. 30, 1982.
- [10] J. Goldhirsh and W. J. Walsh, "Rain measurements from space using a modified Seasat-type radar altimeter," *IEEE Trans. Antennas Propagat.*, vol. AP-30, no. 4, pp. 726-733, July 1982.

On the Accuracy of a Numerical Integration Procedure for Computing Diffraction Fields Using Table Lookup

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Abstract—In this communication we have analyzed the errors introduced by table lookup when an electromagnetic diffraction integral is performed numerically. We have shown that in order to set the 3σ error of both the real and imaginary components equal to ϵ , the required table should have at least K entries uniformly distributed within a 2π range, where $K \geq 3.85/\epsilon\sqrt{N}$ and N is the number of terms used for numerical integration. The maximum deviation of the amplitude ($\bar{R}$) of this integral in this case is also less than ϵ and the maximum phase deviation is $\tan^{-1}(\epsilon/\bar{R})$. A numerical example is presented which confirms this prediction.

I. INTRODUCTION

In a classical electromagnetic problem it is often required to compute the radiation field either radiated directly from a known current source or scattered from a secondary source. In the latter case, the incident field on the scatterer is known or can be determined with the known boundary conditions. The vector potential function of such an electromagnetic field is evaluated

by integrating the current source or current distribution on the surface of the scatterer. The radiation field in space can then be determined from this vector potential function. In general this leads to the evaluation of the following integral (1), (2),

$$\vec{E}(\theta, \Phi) = k \int_S \vec{J}(\xi, \eta) \exp[jG(\xi, \eta, \Phi, \theta)] d\xi d\eta, \quad (1)$$

where $\vec{J}$ is the current density on the scatterer surface excited by the source and ξ, η are the coordinates of the current source; Φ and θ are the coordinates of a field point. It can be shown that the above vector integration can be converted into a scalar integration function such that

$$F(\theta, \Phi) = k \int f(\xi, \eta) \exp[jG(\xi, \eta, \Phi, \theta)] d\xi d\eta. \quad (2)$$

In general the current distribution and the geometry of the source or scatterers are very complicated. Except for a very few simple cases, it is impossible to integrate the above equation into a closed form. This integration is usually performed numerically on a digital computer. Although it is straightforward to program such an integration, the required computation time may sometimes be extremely lengthy.

There are several ways to combat this problem. For example, (2) is a generalized Fourier transformation. Therefore, the fast Fourier transformation (FFT) may be applied. Unfortunately, the exponential function $G(\theta, \Phi, \eta, \xi)$ in general has no linear relation between source coordinates and field point coordinates. In general, a prerequisite of the FFT is that the phase function G computed for any field point (θ, ϕ) must be contained in a set which is finite and the whole set is generated at a certain θ, Φ point. This requirement sometimes puts a constraint on the choice of the integration mesh points that cannot be achieved.

There are other ways. One example is found in the computation of reflector antenna patterns. In this case Galindo and Mitra [3] proposed to expand the integral with a infinite series as a function of the field point coordinates. The coefficients of such a series are the Fourier transformation of the η, ξ coordinates. To compute such coefficients, similar integration of the phase function is required. Furthermore, this series expansion method applies only to this special case.

The third way proposed by Ludwig [4] is to approximate the $f(\xi, \eta)$ and $\exp(jG)$ with a linear function at each integration point and then integrated piecewise at each of these points. The required number of points for numerical integration can be greatly reduced. However, at each point, the cosine and sine functions are still required to be computed repeatedly.

The fourth way [5], perhaps the simplest approach, is direct table lookup. In this approach, at the beginning of the computer program cosine and sine functions with a 2π period are computed and stored, using an adequate number of sampling points. In later computations, the required cosine and sine functions are found from the table, thus avoiding the repeated computation of such functions. However, unless the arguments of these cosine and sine functions coincide with the sampling points in the table, errors will be introduced. The amount of error is a function of the number of sampling points in this cosine and sine table. Therefore, one must decide how much error one can tolerate and then one can determine how big a table one must have. In this communication, we analyze this problem and present the results.

Manuscript received September 17, 1983; revised April 9, 1983.

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II. MATHEMATICAL FORMULATION

For a numerical integration, (2) can be transformed into the following form

$$F(\theta, \Phi) = \sum_n A_n \exp(jG_n), \quad (3)$$

where

$$A_n = f(\eta_n, \xi_n) \geq 0$$

$$G_n = G(\eta_n, \xi_n, \Theta, \Phi).$$

For simplification, the double summation of η and ξ in (2) are converted into a single summation index. The function $F(\theta, \Phi)$ is a complex function having both real and imaginary components, thus

$$F(\theta, \Phi) = x + jy.$$

When table lookup is used to determine the phase of the exponential term, errors are introduced, and (2) becomes

$$F(\theta, \Phi) = \sum_n A_n \exp(jG_n) \exp(j\delta_n) \quad (4)$$

where the δ_n are independent random variables. Therefore, $A_n \exp(jG_n) \exp(j\delta_n)$ are also independent random variables, and the function $F(\theta, \Phi)$ is the sum of many independent random variables. According to the central limit theorem [6], the probability density function of $F(\theta, \Phi)$ is asymptotically normal. Since a normal distribution is completely determined by the first and second moments, our next task is to find the mean and the variance of the random complex function $F(\theta, \Phi)$.

It can be shown that the mean of $F(\theta, \Phi)$ is

$$\bar{F}(\theta, \Phi) = \phi(1) \sum_n A_n \exp(jG_n), \quad (5)$$

where $\phi(1)$ is the characteristic function of the random variable δ_n , which is defined

$$\phi(k) = \int p(\delta) \exp(jk\delta) d\delta \quad (6)$$

and $p(\delta)$ is the probability density function of the random variable δ . If the entries of the cosine and sine table are sampled uniformly within a 2π range having a total number of sample points K , then

$$p(\delta) = \frac{1}{\alpha}, \quad n\alpha < \delta \leq (n+1)\alpha \quad (7)$$

$$= 0, \quad \text{otherwise}$$

and $\alpha = 2\pi/K$.

Two ways of using this table are considered. The first is to choose the cosine and sine function of the next index if the argument of the phase angle is larger than the angle $n\alpha$. The second method is to choose the function of the angle $n\alpha$ if the argument is less than $(n+1/2)\alpha$. For the former case the characteristic function is

$$\phi(k) = \exp(-j\frac{1}{2}k\alpha) \frac{\sin(\frac{1}{2}k\alpha)}{\frac{1}{2}k\alpha} \quad (8)$$

while for the latter case it is

$$\phi(k) = \frac{\sin \frac{1}{2} k \alpha}{\frac{1}{2} k \alpha} \quad (9)$$

One may see that the difference between (8) and (9) involves a constant phase $1/2 k \alpha$ which in general is not important, because the reference phase of function $F(\theta, \Phi)$ is arbitrary. However, in the actual programming, the first method is somewhat easier. One has

$$\frac{\sin \frac{1}{2} k \alpha}{\frac{1}{2} k \alpha} \approx 1 - \frac{1}{4} \frac{(k\alpha)^2}{6} \quad (10)$$

Since α is usually very small, the high-order terms are neglected. Using this result, (5) becomes

$$\bar{F}(\theta, \Phi) = (1 - 0.042\alpha^2) \sum_n A_n \exp(jG_n). \quad (11)$$

It can be shown that the variances of the real and imaginary components of $F(\theta, \Phi)$ are, respectively,

$$\sigma_x^2 = \frac{1}{2} (1 - |\phi(1)|^2) \sum_n A_n^2 + \frac{1}{2} [\phi(2) - \phi^2(1)] \cdot \sum_n A_n^2 \cos(2G_n) \quad (12c)$$

$$\sigma_y^2 = \frac{1}{2} (1 - |\phi(1)|^2) \sum_n A_n^2 - \frac{1}{2} [\phi(2) - \phi^2(1)] \cdot \sum_n A_n^2 \cos(2G_n) \quad (12b)$$

while the covariance of x and y is

$$\sigma_{xy} = \frac{1}{2} [\phi(2) - \phi^2(1)] \sum_n A_n^2 \sin(2G_n).$$

The joint probability density function of the real and imaginary components x and y is then

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp - \left\{ \frac{1}{2(1-r^2)} \left[\frac{(x-\bar{x})^2}{\sigma_x^2} - \frac{2r(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} + \frac{(y-\bar{y})^2}{\sigma_y^2} \right] \right\} \quad (13)$$

Where $\bar{x}$ and $\bar{y}$ are, respectively, the real and imaginary components of $\bar{F}(\theta, \Phi)$, and

$$\sigma_{xy} = r\sigma_x\sigma_y. \quad (14)$$

Our goal is to find the required number of samples, or how big a cosine and sine table is required to achieve a certain prescribed accuracy for both real and imaginary components x and y . The probability density function of (13) is too complicated to estimate this relation. However, certain approximations can be made.

Since $\sum_n A_n^2 \cos 2G_n$ and $\sum_n A_n^2 \sin 2G_n$ are in general much smaller than $\sum_n A_n^2$. For a first-order estimation, one may assume

that

$$\sigma^2 = \sigma_x^2 = \sigma_y^2 = \frac{1}{2} (1 - |\phi(1)|^2) \sum_n A_n^2 \quad (15)$$

$$\sigma_{xy} \approx 0.$$

The probability density then becomes

$$p(x, y) = \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{1}{2\sigma^2} [(x - \bar{x})^2 + (y - \bar{y})^2] \right\}. \quad (16)$$

This equation implies that the real and imaginary components of $F(\theta, \Phi)$ are two independent variables and have the same variance.

From (11),

$$\bar{x} = (1.0 - 0.042\sigma^2) \operatorname{Re} \sum_n A_n \exp(G_n) \quad (17a)$$

$$\bar{y} = (1.0 - 0.042\sigma^2) \operatorname{Im} \sum_n A_n \exp(G_n). \quad (17b)$$

If the table has enough sampling points and σ is small, then the $0.042\sigma^2$ term can be ignored. Both $\bar{x}$ and $\bar{y}$ are then equal to the actual value. The probability density function for either x or y can be written separately

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(x - \bar{x})^2}{2\sigma^2} \right] \quad (18a)$$

$$p(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(y - \bar{y})^2}{2\sigma^2} \right].$$

III. TABLE LOOKUP ERROR

From (10) and (15), it can be shown that

$$\sigma \approx 0.2\alpha \sqrt{\sum_n A_n^2} \quad (19)$$

where $\alpha = 2\pi/K$, and K is the number of entries in the table, one thus has

$$\sigma = 1.28/K \sqrt{\sum_n A_n^2}. \quad (20)$$

Equation (20) relates the table size K to the standard deviation σ . Since $|x - \bar{x}| \leq 3\sigma$ with probability higher than 0.99, if one should require that x or y values should not deviate from the actual value by an amount ϵ with a probability of at least 0.99, the number of entries K in the table should be

$$K \geq \frac{3.85}{\epsilon} \sqrt{\sum_n A_n^2}. \quad (21)$$

One notes that $\epsilon = |x - \bar{x}|$ or $|y - \bar{y}|$ is the measurement of the deviation of x or y from the exact value. Since $\bar{x}$ and $\bar{y}$ are functions of θ and Φ , their values can be very small or very large, depending on θ and Φ . For example, if ϵ is set to be 0.01, when $\bar{x}$ or $\bar{y}$ is equal to one, the actual value x or y deviates from $\bar{x}$ or $\bar{y}$ no more than one percent. However, if $\bar{x}$ or $\bar{y}$ has a value of 0.01, the deviation can be 100 percent of the actual value. In general, when numerical computation is performed, some limitation on accuracy always exists. For example, for an antenna computation, a -60 to -80 dB accuracy probably is all one can expect. Thus one can use this limitation to establish criterion.

The values of A_n in (21) are the amplitudes of the current distribution. These A_n are not known in general before hand. However, if one asserts that all A_n are equal and that they are normalized such that

$$\sum_n A_n = 1$$

then

$$\sum_n A_n^2 = \frac{1}{N} \quad (22)$$

for a first-order estimation. The number N is the total number of terms summed in (4). Equation (21) thus becomes

$$K \geq \frac{3.85}{\epsilon} \sqrt{1/N}. \quad (23)$$

One may notice that $\sum_n A_n$ represents the maximum value of x and y . When it is normalized to be unity, the desired error ϵ in (23) can be set in reference to this level. For example, if one requires that a computed antenna pattern to be accurate to -60 dB, one must set $\epsilon = 10^{-3}$.

IV. ESTIMATION OF AMPLITUDE AND PHASE ERRORS

In the previous section we showed the relationship between the table size K and the achievable accuracy of the real and imaginary parts of the scattering integral, (4). However, we would also like to know what are the amplitude and phase deviations. The probability density function of the amplitude of a complex random variable with equal σ for real and imaginary parts is known. However, we are not particularly interested in its probability distribution. We want to know how much it can deviate from the exact value with nearly unity probability. A somewhat approximate method will be used here to estimate this error.

The statistical distribution of the function $F(\theta, \phi)$ is shown in Fig. 1. Point 0 represents the mean values $\bar{x}$ and $\bar{y}$, which are approximately equal to the true value of the real and imaginary components of $F(\theta, \phi)$ when phase error is small. With a probability of 0.99, all points of $R = |F(\theta, \phi)|$ lies within a circle with a radius of 3σ and centered at point 0 ($\bar{x}, \bar{y}$). It is evident that the maximum R is

$$R_{\max} = \bar{R} + 3\sigma \quad (24)$$

and minimum R is

$$R_{\min} = \bar{R} - 3\sigma \quad (25)$$

where

$$\bar{R} = \sqrt{\bar{x}^2 + \bar{y}^2} \quad (26)$$

and σ is defined in (20). The maximum deviation of $|F(\theta, \phi)|$ is similar to that of its real and imaginary components. Therefore, the same error estimation shown in (23) can be used in this case.

The phase deviation angle Δ is

$$\Delta = \tan^{-1} \frac{3\sigma}{\bar{R}} \quad (27)$$

and

$$\phi_{\max} = \bar{\phi} + \Delta \quad (28)$$

$$\phi_{\min} = \bar{\phi} - \Delta \quad (29)$$

Fig. 1. Statistic distribution of $F(\theta, \phi)$, the diffraction integral.

Fig. 2. (a) Theoretical antenna pattern compared with the pattern computed by table lookup $K = 32$. (b) Theoretical antenna pattern compared with the pattern computed by table lookup $K = 128$.

where

$$\bar{\phi} = \tan^{-1} \frac{\bar{y}}{\bar{x}} \quad (30)$$

The phase error is measured as the ratio of $\bar{R}$ and 3σ . If $\bar{R}$ is large, the phase deviation is very small. However, at point where $\bar{R}$ is very small, phased error introduced by table lookup can be significant.

V. NUMERICAL EXAMPLE

A numerical example is presented here. In this example, the antenna pattern of a uniformly excited line source is plotted. This antenna has a length of 30 wavelengths. Since this pattern function is a $\sin x/x$ function, this exact pattern function is shown in both Figs. 1 and 2 as reference. Next this same pattern is com-

puted by numerical integration. A total of 100 points is used in the integration. Fig. 1 shows the pattern plotted with a table of 32 entries. According to (23), the amplitude error should be in the order of 0.012, which corresponds to an error of about -38 dB. In Fig. 1 the maximum error occurs at an angle of about 80° . The correct pattern has a -36 dB sidelobe level while the approximate pattern has a peak of -31 dB at that point. At -36 dB the radiation amplitude is 0.0158 while at -31 dB it is 0.0282. The deviation of the amplitude is about 0.0124 which is very close to what (23) predicts. Fig. 2 shows the same antenna pattern, except that the number of table entries is increased to 128. The maximum error in this figure is about 0.0038 while (23) predicts 0.003, corresponding to an error level of about -50 dB.

VI. CONCLUSION

In this communication we have analyzed the error introduced by table look-up for the numerical integration of an electromagnetic diffraction integral. We have shown that in order to limit the error in both real and imaginary components to less than ϵ , the required table should have at least K entries uniformly distributed within a 2π range, where $k \geq 3.85/\epsilon\sqrt{N}$ and N is the number of terms used for numerical integration. The maximum deviation of the amplitude $\bar{R}$ of this integral in this case is also less than ϵ and the maximum phase deviation is equal to $\tan^{-1}(\epsilon/\bar{R})$.

REFERENCES

- [1] *Microwave Antenna Theory and Design*. Samuel Silver, Ed. Radiation Lab. Series, vol. 12, New York: McGraw-Hill, 1949.
- [2] *Antenna Theory*. R. E. Collin, and F. J. Zucker, Eds. New York: McGraw-Hill, 1969.
- [3] V. Galindo-Israel and R. Mittra, "A new series representation for the radiation integral and application to reflector antenna," *IEEE Trans. Antennas Propagat.*, vol. AP-25, Sept. 1977.
- [4] A. C. Ludwig, "Computation of radiation pattern involving numerical double integration," *IEEE Trans. Antennas Propagat.*, vol. AP-16, Nov. 1968.
- [5] J. K. Hsiao, "On the accuracy of a numerical integration procedure for computing diffraction field using table-look-up," NRL Rep. 8628, Sept. 1982.
- [6] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*. New York: McGraw-Hill, 1965.

Perturbational and High Frequency Profile Inversion

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Abstract—Although exact profile inversion may be difficult to perform with limited scattering or reflection data, perturbational and/or high frequency techniques do provide useful, though restricted, information regarding unknown profiles. Since high frequency reflections are small, perturbation methods can be used to examine this asymptotic regime. Here we examine the applications and limitations of these methods. Of particular interest is the approximate reconstruction of a double peaked profile.

Manuscript received March 25, 1982; revised July 22, 1982. This work was supported in part by the Moore School of Electrical Engineering, University of Pennsylvania, and in part by the University of Pennsylvania Committee on Faculty Grants and Awards.

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