Synthesis of Antenna Patterns with Prescribed Nulls

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Abstract—**The least mean square pattern synthesis method is extended to include constraints such as pattern nulls or pattern-derivative nulls at a given set of angles. The problem is formulated as a constrained approximation problem which is solved exactly, and a clear geometrical interpretation of the solution in a multidimensional vector space is given. The relation of the present method to those of constrained gain maximization and signal-to-noise ratio (SNR) maximization is discussed and conditions for their equivalence stated. For a linear uniform N-element array it is shown that, when M single nulls are imposed on a given "quiescent" pattern, the optimum solution for the constrained pattern is the initial pattern and a set of M-weighted (sin Nx)/sin x-beams. Each beam is centered exactly at the corresponding pattern null, irrespective of its relative location. For the case of higher order nulls, the nth pattern derivative is similarly canceled by the nth derivative of a (sin Nx)/sin x-beam. In addition, simple quantitative expressions are derived for the pattern change and gain cost associated with the forced pattern nulls. Several illustrative examples are included.**

I. INTRODUCTION

THE PROBLEM of forming nulls in the radiation pattern of an antenna, in order to suppress interference from certain directions, presently receives much attention. Most work is in the area of adaptive nulling systems, as discussed by Applebaum [1] where a performance index such as the signal-to-noise ratio (SNR) is maximized. In the case where jammers are the dominant noise source this process automatically places pattern nulls in the directions of the jammers. A seemingly different approach is that of Drane–McIlvenna [2] where another index, antenna gain, is maximized, subject to a set of null constraints on the pattern. In both methods the performance index is the quantity of prime interest, whereas the role of the antenna pattern is not too clear, which to an antenna engineer is unsatisfactory. The purpose of this paper is to show that the problem can be formulated as a direct pattern synthesis problem which includes the pattern nulls.

A narrow-band interference source can be suppressed by imposing a single null in the antenna pattern at the proper angle. A wide-band jammer, however, appears to cover an angular sector of the pattern because of the frequency dependence of the antenna, and therefore it may be required to null an entire sector. This can be done either by imposing single pattern nulls on a set of closely spaced points over the sector or by imposing nulls in the pattern and its derivatives at the center point of the sector. In circuit theory this is analogous to the Chebyshev and Butterworth filter alternatives. We will consider the synthesis problem for both cases and also derive expressions for the effects of the imposed nulls, in terms of pattern change and gain cost.

The synthesis method is based on least mean square or Gaussian approximation [3], which allows an attractive geometrical interpretation in a multidimensional vector space. It will be shown that under certain conditions the present approach yields the same result as the methods of constrained gain maximization and as SNR maximization. The least mean square error criterion with single-null constraints has been lucidly discussed [4] in very general terms and as applied to satellite multiple-beam antennas. In contrast we will study the classic problem of pattern synthesis for a linear array of isotropic elements, which leads to a slightly different formulation and some complementary viewpoints and results. The use of the derivative to broaden a pattern null was first suggested in [2] and recently applied in the context of pattern synthesis in [5] and adaptive systems in [6].

II. FORMULATION OF THE PROBLEM

We consider a situation where an array antenna is being illuminated by desired signals and also by highly dominant interference signals from certain discrete directions. The optimum antenna pattern for this case is reasonably defined as the desired pattern in the absence of the jammers, the so called quiescent pattern, suitably modified so as to form pattern nulls in the interference directions. The degrees of freedom available in the antenna pattern are thus used in first place to form the pattern nulls, with remaining degrees of freedom being used for approximation of the quiescent pattern.

The corresponding antenna pattern synthesis problem consists of determining the closest approximation p_a to a given quiescent pattern p_0, subject to a set of null constraints. The solution of this problem requires a definition of "distance" between two patterns and this will be defined in Gauss's sense as the mean square difference between the patterns. This particular metric provides an overall measure of approximation, and in contrast to, for instance, the Chebyshev approximation, places no explicit bound on the maximum deviation from the desired function at any particular point. However it is the only metric that allows the approximation problem to be solved with any sense of generality.

For simplicity we consider a linear array of N isotropic antenna elements with uniform half-wavelength spacing. Setting $u = \sin\theta$ where θ is defined in Fig. 1, the antenna far-field pattern is described by the array factor

$$p(u) = \sum_{1}^{N} x_n e^{-in\pi u}, \tag{1}$$

where x_n denotes the complex excitation of the nth array element.

The synthesis problem can now be stated mathematically: find the pattern $p_a(u)$, such that the mean square difference

$$\epsilon(p_a) = \frac{1}{2}\int_{-1}^{1} |p_0(u) - p_a(u)|^2\, du = \min, \tag{2a}$$

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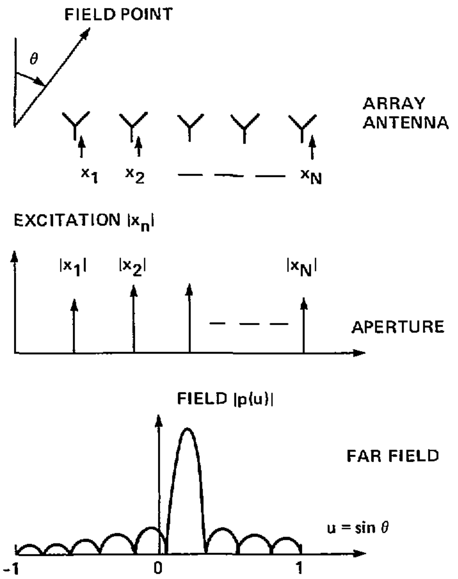

Fig. 1. Array antenna: its aperture and far field.

subject to the constraints

$$\frac{d^\nu}{du^\nu} p_a(u_\mu) = 0, \quad \mu = 1, \cdots, M_1, \ \nu = 0, \cdots, M_2, \tag{2b}$$

where $\{u_\mu\}_1^{M_1}$ denotes the angular location of the M_1 interference sources. Since the total number of constraints must not exceed the number of free variables it is required that $M_1(M_2 + 1) \leq N$.

We assume that the desired quiescent pattern is given as a sum of N harmonics, as represented by (1). For the general case where $p_0(u)$ has any functional form, p_0 may be simply approximated by the first N terms of its Fourier-series expansion. Although the synthesis procedure then involves two subsequent approximations, it is easily shown to lead to the correct least mean square approximation of the initial pattern [7].

III. METHOD OF SOLUTION

The synthesis problem posed above is most conveniently formulated in a multidimensional vector space, where each point represents one array excitation, which allows a clear geometrical interpretation of the approximations involved. From the solution for the array excitation the desired pattern is then simply given by (1). The underlying principle for this equivalence between array excitation and radiation pattern is of course Parseval's theorem.

We introduce an N-dimensional excitation space X, in which the array excitation $\{x_n\}_1^N$ is represented by the vector $\bar{x} = (x_1, \cdots, x_N)$. The inner product of two vectors we define as $(\bar{x}, \bar{y}) = \Sigma x_n y_n^*$, where the asterisk denotes complex conjugate, and the norm $\|\bar{x}\| = (\bar{x}, \bar{x})^{1/2}$.

In order to express the mean square error ϵ in terms of the array excitation we substitute (1) in (2a) and obtain after integration

$$\epsilon = \frac{1}{2} \int_{-1}^{1} |p_0 - p_a|^2 \, du = \sum_{n=1}^{N} |x_{0n} - x_{an}|^2$$

$$= \|\bar{x}_0 - \bar{x}_a\|^2, \tag{3}$$

where $\bar{x}_0$ and $\bar{x}_a$ are the excitation vectors corresponding to the patterns p_0 and p_a, respectively. Likewise the pattern constraints (2b) can be expressed as constraints on the array excitation. The mathematical expressions simplify somewhat if we first multiply the pattern function p by a phase factor $\exp(i\psi u)$, where

$$\psi = \frac{N+1}{2}, \tag{4}$$

which shifts the phase center of the pattern to the array center. Substituting this new function in (2b), we find in view of (1) that a single or zero-order null at $u = u_\mu$ requires

$$\sum_1^N x_n e^{i(\psi - n\pi)u_\mu} = 0, \tag{5}$$

and in general, for a νth order null

$$\sum_1^N x_n e^{i(\psi - n\pi)u_\mu} = \cdots$$

$$= \sum_1^N x_n [i(\psi - n\pi)]^\nu e^{i(\psi - n\pi)u_\mu} = 0. \tag{6}$$

Defining constraint vectors $\bar{y}_\mu^{(\nu)}$ by

$$\bar{y}_\mu^{(\nu)} = ([i(\psi - \pi)]^\nu e^{i(\psi - \pi)u_\mu}, \cdots,$$
$$[i(\psi - N\pi)]^\nu e^{i(\psi - N\pi)u_\mu})^* \tag{7}$$

finally lets us write (6) as orthogonality conditions on the array excitation

$$(\bar{x}, \bar{y}_\mu^{(0)}) = \cdots = (\bar{x}, \bar{y}_\mu^{(\nu)}) = 0. \tag{8}$$

Note that we now have characterized each combination of jammer direction u_μ and pattern derivation order ν by one constraint vector. Since we have a total of $M = M_1(1 + M_2)$ such combinations, see (2b), we have M different constraint vectors. In the following we will suppress the superscript ν and denote the constraint vectors just by $\bar{y}_m$, where m runs from 1 to M.

In view of (3) and (8) the synthesis problem, as expressed by (2), now becomes

$$\epsilon = \|\bar{x}_0 - \bar{x}_a\|^2 = \min \tag{9a}$$

$$(\bar{x}_a, \bar{y}_m) = 0, \qquad m = 1, \cdots, M, \tag{9b}$$

where $\bar{x}_0$ and $\bar{x}_a$ denote the unconstrained and constrained array excitation, respectively.

Equation (9) shows that the desired solution $\bar{x}_a$ is orthogonal to the constraint vectors $\{\bar{y}_m\}_1^M$. A geometrical interpretation of this relation is obtained if the excitation space X is divided into an M-dimensional subspace Y, spanned by the vectors $\{\bar{y}_m\}_1^M$ and its $(N-M)$-dimensional orthogonal complement Z. Any vector $\bar{x}$ now has a unique decomposition [8]

$$\bar{x} = \bar{y} + \bar{z}, \tag{10}$$

where $\bar{y} \in Y$, $\bar{z} \in Z$, $\bar{z} \perp Y$, and because of this orthogonality

$$\|\bar{x}\|^2 = \|\bar{y}\|^2 + \|\bar{z}\|^2. \tag{11}$$

Using this decomposition for $\bar{x}_0$ and $\bar{x}_a$ we get from (9), (10), and (11)

$$\epsilon = \|\bar{y}_0 - \bar{y}_a\|^2 + \|\bar{z}_0 - \bar{z}_a\|^2 = \min, \tag{12a}$$

$$(\bar{x}_a, \bar{y}_m) = (\bar{y}_a, \bar{y}_m) = 0, \qquad m = 1, \cdots, M. \tag{12b}$$

Equation (12b) yields

$$\bar{y}_a \equiv 0, \tag{13}$$

and therefore ϵ in (12a) is minimized by setting $\bar{z}_a = \bar{z}_0$ leading to the sought constrained excitation

$$\bar{x}_a = \bar{z}_0 = \bar{x}_0 - \bar{y}_0, \tag{14}$$

and the least mean square error

$$\epsilon_{\min} = \|\bar{x}_0 - \bar{x}_a\|^2 = \|\bar{y}_0\|^2. \tag{15}$$

Equations (14) and (15) constitute the mathematical solution to the posed problem. Its properties will now be discussed from various points of view.

The method of solution is illustrated in Fig. 2. The excitation $\bar{x}_0$, which is to be approximated, has the projections $\bar{y}_0$ and $\bar{z}_0$ in subspaces Y and Z. Equation (9b) implies that the approximation $\bar{x}_a$ is orthogonal to the constraint vector set $\{\bar{y}_m\}_1^M$ which spans Y, and therefore $\bar{x}_a$ is confined to the subspace Z. Under these circumstances the best approximation to $\bar{x}_0$ is obtained by setting $\bar{x}_a = \bar{z}_0$, since of all elements $\bar{z} \in Z$ this point is closest to $\bar{x}_0$.

Returning to the solution for the constrained excitation as given by (14) we note that $\bar{y}_0$ is a linear combination of the vectors $\bar{y}_m$ and therefore $\bar{x}_a$ may be written

$$\bar{x}_a = \bar{x}_0 - \sum_1^M \alpha_m \bar{y}_m, \tag{16}$$

where the coefficients α_m will be determined later. Presently we infer from (16) that the sought excitation $\bar{x}_a$ is composed of the quiescent excitation $\bar{x}_0$ and a weighted sum of the vectors $\bar{y}_m$. Note the dual role of these vectors: initially they characterized a constraint, now they represent an array excitation.

As for the resultant antenna pattern, it follows from (16) and the linear relation between the array excitation and the pattern, that the constrained pattern $p_a(u)$ will be the quiescent pattern $p_0(u)$ with M beams superimposed. The beam corresponding to the excitation $\bar{y}_m$ we call a cancellation beam, denoted by $q_m(u)$, and it is easily shown by using (7) in (1) that

$$q_m(u) = \frac{d^\nu}{du^\nu} \frac{\sin N\pi(u - u_\mu)/2}{\sin \pi(u - u_\mu)/2}, \tag{17}$$

where μ and ν are determined by the index m. For the case of M single nulls in the pattern ($\nu = 0$) the constrained pattern

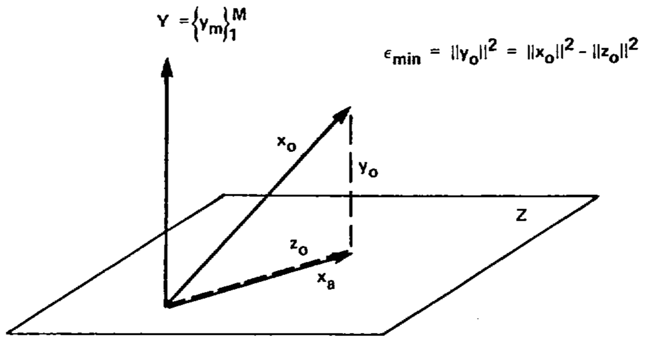

Fig. 2. Geometrical illustration of approximation problem: desired point = x_0, closest approximation confined to subspace Z is $x_a = z_0$.

becomes

$$p_a(u) = p_0(u) - \sum_1^M \alpha_m \frac{\sin N\pi(u - u_m)/2}{\sin \pi(u - u_m)/2}. \tag{18}$$

When N is large, (18) can be approximated as

$$p_a(u) \cong p_0(u) - N \sum_1^M \alpha_m \operatorname{sinc}[N\pi(u - u_m)/2]. \tag{19}$$

Thus the pattern p_0 is simply given by the quiescent pattern p_0 and M superimposed sinc-beams. This result agrees with the single-jammer case considered in [1] and the general conclusion in [4].

The M cancellation beams represent M degrees of freedom, and clearly it should be possible to realize M pattern nulls with these. However it is noteworthy that each of these beams is centered exactly on the corresponding null, irrespective of their relative location and that the beam shape, given by $\sin((N\pi u/2)/\sin(\pi u/2)$, is fixed regardless of how much the individual beams overlap. Similar observations hold for the cancellation beams corresponding to higher order pattern derivatives. These properties are consequences of the isotropic array elements and the least mean square approximation we have adopted.

The present synthesis method with single-null constraints will yield the same pattern as does SNR maximization [1] in the limiting case, where the jammers become infinitely strong. This latter condition forces the optimum SNR pattern to maintain true nulls, rather than shallow dips, in the jammer directions and then the two methods are comparable, as shown in the appendix. Further, we also find equivalence with constrained gain maximization [2] in the special case where p_0 is a maximum gain pattern, on which a set of single nulls is imposed. Minimizing the pattern change ϵ then simultaneously minimizes the gain cost, as shown below, in conjunction with (26), and thus the constrained least mean square pattern coincides with the constrained maximum gain pattern.

Compared to these methods, however, the pattern synthesis method is a more direct and therefore conceptually more appealing approach, which provides valuable insight into fundamental pattern properties.

IV. THE SYNTHESIZED PATTERN

The pattern p_a, which satisfies the desired null constraints is given by (18) where, however, the coefficients α_m so far are unknown. They may be determined from (16) and (12b) which leads to the following system of equations:

$$\begin{pmatrix} (\bar{y}_1, \bar{x}_0) \\ \vdots \\ (\bar{y}_M, \bar{x}_0) \end{pmatrix} = \begin{pmatrix} (\bar{y}_1, \bar{y}_1) & (\bar{y}_1, \bar{y}_2) & \cdots & (\bar{y}_1, \bar{y}_M) \\ (\bar{y}_2, \bar{y}_1) & (\bar{y}_2, \bar{y}_2) & & \\ & & \cdots & \\ (\bar{y}_M, \bar{y}_1) & (\bar{y}_M, \bar{y}_2) & & (\bar{y}_M, \bar{y}_M) \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_M \end{pmatrix}^* . \tag{20}$$

Applying Cramer's rule and substituting into (18) yields

$$p_a(u) = p_0(u) - \frac{1}{G} \sum_1^M D_m{}^* q_m(u), \tag{21}$$

where the Gram determinant $G = G(\bar{y}_1, \cdots, \bar{y}_M)$ is the coefficient matrix in (20), see [8], and D_m is the determinant of the same matrix with the mth column replaced by the column vector $((\bar{y}_1, \bar{x}_0), \cdots, (\bar{y}_M, x_0))$. Note that there are only as many equations as there are constraints and usually therefore (20) will represent a small system of equations, which will be easy to invert.

To illustrate the synthesis method we have programmed (21) on a digital computer and calculated a few actual patterns. We considered "sinc-patterns" defined by the function $\sin(N\pi u/2)/N \sin(\pi u/2)$ and Chebyshev patterns, since they are in a sense complementary—the former have sidelobes of constant width and varying height, the latter have sidelobes of varying width but constant height. In the first examples we chose the original pattern p_0 to be a sinc-pattern and imposed, respectively, a zero-, a first-, and a second-order null in the pattern at the rather arbitrary value $u = 0.22$. Fig. 3 shows that the null width indeed increases as expected and also that the changes in the pattern are rather local.

As a comparison to the latter case, which uses three degrees of freedom for sidelobe cancellation, we also calculated the pattern with three single nulls located at $u_1 = 0.21$, $u_2 = 0.22$, $u_3 = 0.23$. Fig. 4 shows that in this case we do get somewhat less sidelobe cancellation but over a wider sector. In contrast to derivative nulling the placement of single nulls thus provides efficient control over the trade-off between cancellation ratio and sector width.

In the next two examples the unconstrained pattern is a 40-dB Chebyshev pattern in which we place four single nulls over a narrow sector (0.22, 0.28) and eight single nulls over a wider sector (0.22, 0.36), respectively. In both cases the nulls are equally densely spaced $\Delta u = 0.02$ apart. The resultant patterns, given in Figs. 5 and 6, show a sidelobe cancellation of 30 dB and 51 dB, respectively, over the sectors. This is a surprising fact. Intuition would lead us to expect less cancellation for the wider sector, which contains a larger number of nulls, i.e., a larger number of superimposed sinc-beams, whose uncontrolled sidelobes we might expect to add up to a relatively higher average sidelobe level between the nulls. A related problem of practical interest concerns the number of nulls required to suppress a jammer over a given bandwidth, which will be addressed in the future.

Finally, we show a sinc-pattern and a 20-dB Chebyshev pattern in Figs. 7 and 8, again with four single nulls equispaced over the interval (0.22, 0.28). The sidelobe cancellation in this

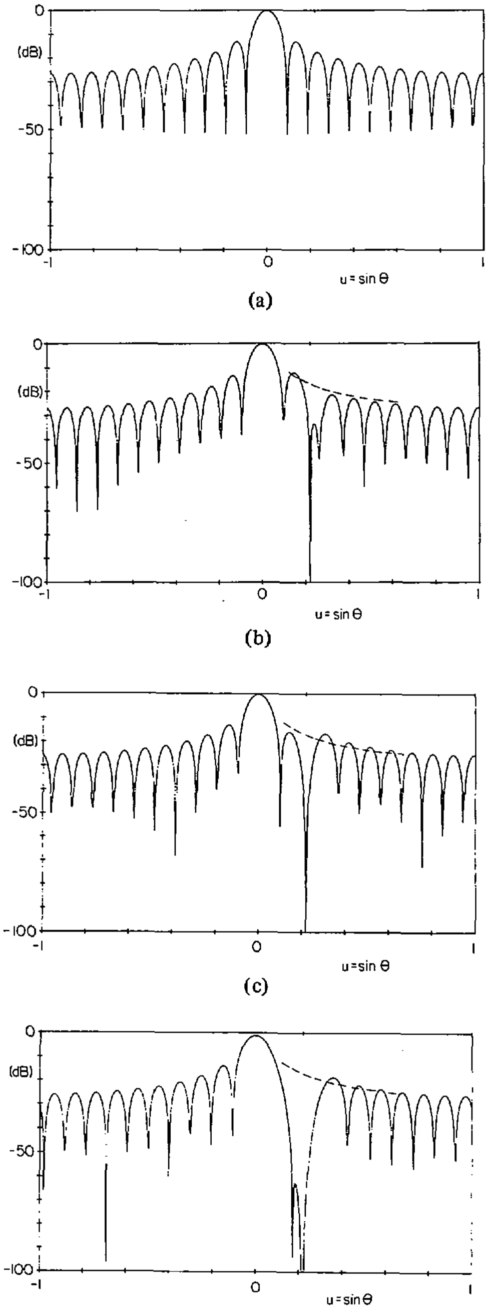

Fig. 3. (a) Initial sinc-pattern. (b) Pattern with a null of zero order imposed at $u = 0.22$. (c) With null of first order. (d) With null of second order. Pattern change = 0.01, 0.03, 0.13, and gain cost = 0.06, 0.11, 0.59 dB, respectively. Twenty-one array elements.

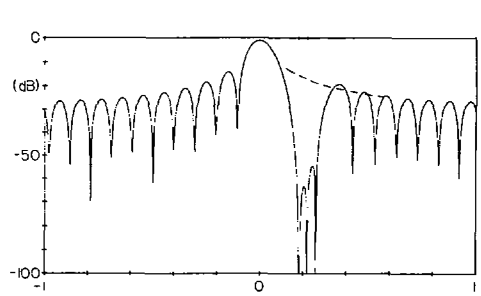

Fig. 4. Sinc-pattern with three nulls equispaced over the sector (0.18, 0.26). Sidelobe cancellation = 36 dB, pattern change = 0.12, gain cost = 0.55 dB. Twenty-one array elements.

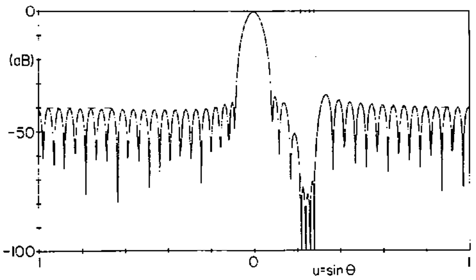

Fig. 5. Initial 40-dB Chebyshev pattern with four nulls equispaced over the sector (0.22, 0.28). Sidelobe cancellation = 30 dB, pattern change = 0.001, gain cost = 0.04 dB. Forty-one array elements.

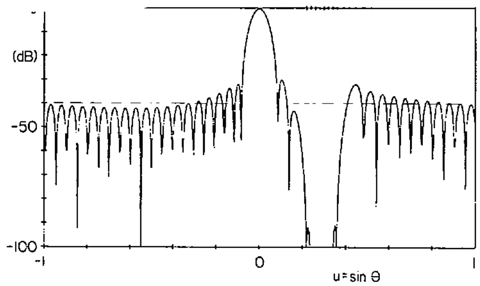

Fig. 6. Initial 40-dB Chebyshev pattern with eight nulls equispaced over the sector (0.22, 0.36). Sidelobe cancellation = 51 dB, pattern change = 0.004, gain cost = 0.15 dB. Forty-one array elements.

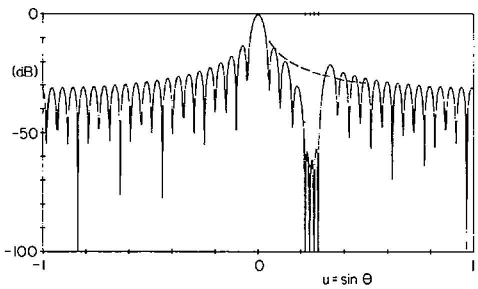

Fig. 7. Initial sinc-pattern with four nulls equispaced over the sector (0.22, 0.28). Sidelobe cancellation = 34 dB, pattern change = 0.03, gain cost = 0.13 dB. Forty-one array elements.

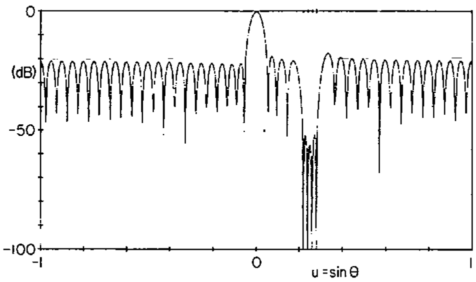

Fig. 8. Initial 20-dB Chebyshev pattern with four nulls equispaced over the sector (0.22, 0.28). Sidelobe cancellation = 32 dB, pattern change = 0.04, gain cost = 0.03 dB. Forty-one array elements.

case is 34 dB and 32 dB respectively, which is of roughly the same magnitude as the 30 dB obtained for the 40-dB Chebyshev pattern above. This indicates that the cancellation ratio is rather independent of the actual sidelobe level. It thus takes as many degrees of freedom to suppress the sidelobe level, for example, from 20 dB to 60 dB, as from 40 dB to 80 dB.

V. THE EFFECTS OF NULL CONSTRAINTS ON THE PATTERN

It is clear that forcing the antenna pattern to zero at certain directions does affect the pattern over the entire angular region, and the extent of these effects is a matter of practical as well as theoretical interest. The following two measures for the difference between the quiescent and the constrained pattern seem natural.

1) Pattern Change $\epsilon = 1/2 \int |p_0 - p_a|^2\, du$: This overall measure is relevant for shaped beam patterns or for patterns described over the entire angular sector.

2) Gain Cost $\epsilon_g = G_o(u_l) - G_a(u_l)$: This is the reduction in maximum directivity in the look direction u_l and is of interest particularly for pencil beams.

The pattern change which by definition is minimized by the pattern p_a is given by (11), (14), and (15) as

$$\epsilon_{\min} = \|\bar{x}_0\|^2 - \|\bar{x}_a\|^2. \tag{22}$$

This form is convenient when the constrained excitation x_a is explicitly known.

A simple estimate for $\epsilon_{\min}$ is obtained for the case of single nulls when the constraint vectors y_m are nearly orthogonal, so that

$$(\bar{y}_m, \bar{y}_n) \ll \|\bar{y}_m\|\,\|\bar{y}_n\|, \qquad m \neq n. \tag{23}$$

This condition is satisfied as soon as the jammers are more than a beamwidth ($2/N$) apart. In this case the vectors $\{\bar{y}_m\}_1^M$ after normalization by $\|\bar{y}_m\| = \sqrt{N}$ form an approximately orthonormal basis for the subspace Y, and thus we find

$$\epsilon_{\min} = \|\bar{y}_0\|^2 \cong \frac{1}{N} \sum_{i}^{M} |(\bar{x}_0, \bar{y}_m)|^2$$

$$= \frac{1}{N} \sum_{1}^{M} |p_0(u_m)|^2. \tag{24}$$

Equation (24) for $\epsilon_{\min}$ seems reasonable since 1) the cancellation beams, which are superimposed on p_0 to produce the nulls, are proportional to $p_0(u_m)$, and 2) the beamwidth of the cancellation beams, and therefore, their power content is inversely proportional to N.

The quantity $\epsilon_{\min}$, normalized to the total radiation power $\|\bar{x}_0\|^2$, is given in the text accompanying Figs. 3–8. Clearly the pattern change is insignificant for all cases considered.

The gain cost ϵ_g evaluated at $u = u_l$ is

$$\epsilon_g(u_l) = G_0(u_l) \left[1 - \frac{|p_a(u_l)|^2}{\int |p_a|^2\, du} \Big/ \frac{|p_0(u_l)|^2}{\int |p_0|^2\, du} \right]. \tag{25}$$

Equation (25) can be expressed in terms of the array excitation if we, for the look direction u_l, define a vector $\bar{x}_l =$

$(e^{i\pi u_l}, \cdots, e^{iN\pi u_l})$, which leads to

$$\epsilon_g(u_l) = G_0(u_l)\left[1 - \frac{\|\bar{x}_0\|^2}{\|\bar{x}_a\|^2}\frac{|(\bar{x}_a, \bar{x}_l)|^2}{|(\bar{x}_0, \bar{x}_l)|^2}\right]. \tag{26}$$

For the particular case where the quiescent pattern p_0 is a sinc-pattern directed at $u = u_l$, it is found that the corresponding excitation $\bar{x}_0 = \bar{x}_l$ and $\epsilon_g(u_l) = 2(\|\bar{x}_0\|^2 - \|\bar{x}_a\|^2) = 2\epsilon$. In this case therefore minimizing the pattern change ϵ simultaneously minimizes the gain cost, as mentioned earlier.

An estimate for ϵ_g can be obtained when the constraint vectors are approximately orthogonal again. Assuming single pattern nulls, setting for simplicity $u_l = 0$ and neglecting higher order terms of $\|x_0 - x_a\|/\|x_0\|$, it is easily shown from (26) that

$$\begin{aligned}\epsilon_g &\cong 2G_0(0)\frac{|(\bar{x}_0 - \bar{x}_a, \bar{x}_l)|}{(\bar{x}_0, \bar{x}_a)|} \\ &\cong \frac{2G_0(0)}{Np_0(0)}\left|\sum_1^M p_0(u_m)q_m(0)\right| \\ &\leqslant 2G_0(0)\sum_1^M\left|\frac{p_0(u_m)}{p_0(0)}\operatorname{sinc}(N\pi u_m/2)\right|. \end{aligned} \tag{27}$$

For the examples discussed earlier, the gain cost, measured in decibels, is given in the captions of Figs. 3–8. Apparently the cost, which ranges between 0.03 and 0.6 dB, is not severe although in some cases we have suppressed the sidelobes substantially.

VI. SUMMARY AND CONCLUSION

We have extended the general method of least mean square pattern synthesis [3] to include null constraints on the pattern and its derivatives. The problem has been posed as a constrained approximation problem and an exact solution has been obtained. The relation to other methods to achieve pattern nulls under mathematically well-defined conditions has been discussed. For a linear uniform array we have shown that, when M single nulls are imposed on a pattern, the constrained pattern is the sum of the original pattern and M weighted sinc-beams. Each beam is centered on the corresponding null, irrespective of how closely spaced they are or how much the beams overlap. In the general case of higher order nulls, the mth pattern derivative is canceled with a beam given by the mth derivative of a sinc-beam. In addition we have derived simple quantitative expressions for the pattern change and the gain cost associated with the forced pattern nulls.

Several illustrative examples of patterns with single nulls and higher order nulls are given. These indicate that when we impose a set of closely spaced single nulls on a pattern sector the sidelobes are reduced by a rather constant factor, which is independent of the actual sidelobe level. The relation between the number of nulls and the sidelobe cancellation is more complex. For instance, doubling simultaneously the sector width and the number of nulls produces, contrary to intuition, an increased cancellation ratio for the wider sector. The gain costs seems rather insignificant for all examples considered.

Finally, it is worth noting that, although we have formulated the constrained synthesis method for a linear array with isotropic half-wavelength spaced elements, it is not limited to these cases. It can readily be formulated in more general terms, in which case any desired linear passive beamforming network may be included in the antenna. It is hoped that this approach can contribute to an understanding of the fundamental properties and limitations of an adaptive antenna.

APPENDIX

We consider Applebaum's approach in which we seek a set of array element weights, denoted by the row vector $\bar{w} = (w_1, \cdots, w_N)$ that maximizes the generalized signal-to-noise (S/N) ratio

$$(S/N) = \frac{|(\bar{w}, \bar{t}^*)|^2}{(\bar{w}^*M, w^*)}. \tag{28}$$

Here $\bar{t}$ is a generalized signal vector and M the noise-covariance matrix. The latter consists of two parts:

$$M = A + B, \tag{29}$$

where A represents the quiescent environment (receiver noise only) and B represents the statistically independent, external interference sources (jammers). For the present comparison with pattern synthesis, we can assume all array elements to contribute uncorrelated noise of equal power $|V_0|^2$, which leads to

$$A = |V_0|^2 I, \tag{30}$$

where I is the identity matrix.

The matrix B is derived as per Applebaum [1]. Assuming M jammers located in the directions $\{u_m\}_1^M$, the total interference signal at the kth element is

$$v_k = \sum_1^M V_m e^{ik\pi u_m}, \tag{31}$$

where V_m denotes the complex amplitude of the individual jammer. The terms u_{kl} of the matrix B are given by

$$u_{kl} = E(v_k^* v_l), \tag{32}$$

where E denotes "expected value," and in view of the fact that the jammers are statistically independent,

$$u_{kl} = \sum_1^M |V_m|^2 e^{-i(k-l)\pi u_m}. \tag{33}$$

A pleasant consequence of (33) is that B can be written as the sum of the covariance matrices of the individual jammers; therefore,

$$B = \sum_1^M |V_m|^2 B_m \tag{34}$$

with $(B_m)_{pq} = \exp[ik(q-p)u_m]$. Substitution of (30) and (34) in (28) yields

$$(\text{S/N}) = \frac{|(\bar{w}, \bar{t}^*)|^2}{(\bar{w}^*[V_0^2 + V_1^2 B_1 + \cdots + V_M^2 B_M], \bar{w}^*)}. \tag{35}$$

In the limit of infinitely strong jammers, a necessary condition for a nontrivial result is

$$(\overline{w}^*B_m, \overline{w}^*) = 0, \qquad m = 1, \cdots, M. \tag{36}$$

Noting that B_m can be rewritten as the outer product of the vectors $\overline{y}_m{}^\dagger$ and $\overline{y}_m$, where $\overline{y}_m$ is given by (7) and the dagger denotes the complex conjugate transpose, it is easily shown that (36) is equivalent to

$$(\overline{w}^*, \overline{y}_m) = 0, \qquad m = 1, \cdots, M. \tag{37}$$

Summarizing (35), (36), and (37) we thus arrive at the problem to seek

$$\left|\left(\frac{\overline{w}^*}{\|\overline{w}^*\|}, \bar{t}\right)\right|^2 = \max, \tag{38a}$$

subject to the constraints

$$(\overline{w}^*, \overline{y}_m) = 0, \qquad m = 1, \cdots, M. \tag{38b}$$

As before we decompose the vectors $\overline{w}$ and $\bar{t}$ into their components in the subspace Y spanned by $\{\overline{y}_m\}_1{}^M$ and its orthogonal complement Z to obtain $\overline{w} = \overline{w}_y + \overline{w}_z$, $\bar{t} = \bar{t}_y + \bar{t}_z$. In view of (38b) the component $\overline{w}_y = 0$ and (38a) become

$$\left|\left(\frac{\overline{w}_z^*}{\|\overline{w}_z^*\|}, \bar{t}_z\right)\right|^2 = \max, \tag{39}$$

which leads to $\overline{w}_z^* = \lambda \bar{t}_z$, where λ is an arbitrary proportionality constant. To compare this result to that of pattern synthesis (14) we set $\lambda = 1$, note that $\bar{t} = \overline{x}_0$, and invoke the phase conjugacy between a weight distribution in the receive mode and an aperture distribution in the transmit mode to obtain

$$\overline{x}_a = \overline{w}^* = 0 + \overline{w}_z^* = (\overline{x}_0)_z = \overline{z}_0.$$

Thus in the case of infinitely strong jammers, the pattern that maximizes the generalized SNR is the least mean square approximation to the desired quiescent pattern with single-null constraints.

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