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Closed-Form Solution for Determining Emitter Location Using Time Difference of Arrival Measurements

A direct and short derivation of an algorithm based on the closed-form solution of the nonlinear equations for emitter location using time difference of arrival (TDOA) measurements from $N+1$ receivers, $N \geq 3$, is given.

I. INTRODUCTION

The solution of the problem of locating a signal source using time difference of arrival (TDOA) measurements has numerous applications in navigation, surveillance, and geophysics. Using an array of multiple sensors, the TDOAs of the received signal are measured. The TDOAs are proportional to the differences in sensor-source range, called range differences (RDs). Conventionally, the source location is estimated from the intersection of a set of hyperboloids defined by the RD measurements and the known sensor locations. The approaches to the solution of the emitter location problem include iterative least-squares (ILS) [1, 2] and maximum likelihood (ML) estimation [3]. Closed-form solutions have been given in [4, 5], using "spherical-intersection" and "spherical-interpolation" methods, respectively. Closed-form solutions are usually less computationally burdensome than iterative, nonlinear minimization, or the ML method, and achieve good accuracy.

In this correspondence a simple derivation of a closed-form solution similar to [4] is given. In [4], Schau and Robinson give a closed-form solution for calculating the source location in three dimensions using 4 sensors, viz., 3 TDOA measurements. This correspondence provides a direct and short derivation along the lines of [6] of the closed form solution-based emitter location algorithm, and extends this approach to the use of N TDOA measurements ($N \geq 3$). The existence of a solution and noise effects are also addressed.

II. CLOSED-FORM SOLUTION

To obtain a 3D source location solution, at least four sensors at known locations are needed. One of the sensors is used as a reference for the RD measurements. Also, without loss of generality, we assume that the designated reference sensor is located at the origin of our coordinates frame. We assume that $N+1$ sensors are used, $N \geq 3$.

Since a TDOA-based positioning system does not measure absolute time, but instead measures the time difference that a signal arrives at the TDOA sensors with respect to the reference TDOA sensor, the N TDOAs are expressed as

$$\Delta t_i = t_i - t_o, \qquad i = 1,\ldots,N \tag{1}$$

where t_o is the absolute time of arrival to the reference sensor, t_i is the absolute time that the signal arrives at the ith sensor, and N is the number of sensors, excluding the reference sensor (sensor 0). The TDOAs are converted to RDs by multiplying by c, the speed of light:

$$d_i = c\Delta t_i = c(t_i - t_o), \qquad i = 1,\ldots,N. \tag{2}$$

Thus, d_i is the RD between sensor i and the reference sensor.

The spatial coordinate vectors of the $N+1$ sensor are

$$\mathbf{x}_o \triangleq \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

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and

$$\mathbf{x}_i \triangleq \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}, \qquad i = 0, 1, \ldots, N \tag{4}$$

where $\mathbf{x}_o$ is the reference sensor position and $\mathbf{x}_i$ is the ith sensor position. The unknown signal source position is

$$\mathbf{x} \triangleq \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix}. \tag{5}$$

The Euclidian distance between the source and sensor i is given by

$$R_{is} = \|\mathbf{x}_i - \mathbf{x}\| = \sqrt{(x_i - x_s)^2 + (y_i - y_s)^2 + (z_i - z_s)^2}, \qquad i = 1, \ldots, N \tag{6}$$

and the distance between the reference sensor and the source is

$$R_s = \|\mathbf{x}\| = \sqrt{x_s^2 + y_s^2 + z_s^2}. \tag{7}$$

We use the RD notation d_i of (2). The RDs satisfy the basic relationships:

$$d_i = R_{is} - R_s, \qquad i = 1, \ldots, N \tag{8}$$

which can be rewritten using (6) and (7) as

$$d_i = \sqrt{(x_i - x_s)^2 + (y_i - y_s)^2 + (z_i - z_s)^2} - \sqrt{x_s^2 + y_s^2 + z_s^2}, \qquad i = 1, \ldots, N. \tag{9}$$

After algebraic manipulation, (9) yields

$$x_i x_s + y_i y_s + z_i z_s + d_i \sqrt{x_s^2 + y_s^2 + z_s^2} = \tfrac{1}{2}(x_i^2 + y_i^2 + z_i^2 - d_i^2), \qquad i = 1, \ldots, N. \tag{10}$$

For the general case of $N + 1$ sensors, define the regressor matrix

$$\mathbf{S} \triangleq \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & z_N \end{bmatrix}_{N\times 3} \tag{11}$$

and the vectors

$$\mathbf{z} \triangleq \frac{1}{2} \begin{bmatrix} x_1^2 + y_1^2 + z_1^2 - d_1^2 \\ \vdots \\ x_N^2 + y_N^2 + z_N^2 - d_N^2 \end{bmatrix}_{N\times 1} \tag{12}$$

and

$$\mathbf{d} \triangleq \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix}_{N\times 1}. \tag{13}$$

Hence, the data is encapsulated in the matrix **S** and in the vectors **z** and **d**. Moreover, the possibly noise corrupted measurements exclusively reside in the **z** and **d** vectors, whereas the regressor matrix **S** is "clean."

In matrix notation, (10) for multiple sensors becomes

$$\mathbf{S}\mathbf{x} = \mathbf{z} - \mathbf{d}R_s. \tag{14}$$

Equation (14) represent a linear system in the four unknowns: x_s, y_s, z_s, and R_s. Hence, solving for source position **x** we obtain the preliminary emitter position estimate

$$\begin{aligned} \mathbf{x} &= (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T(\mathbf{z} - \mathbf{d}R_s) \\ &= (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T z - (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{d}R_s. \end{aligned} \tag{15}$$

When not all RDs are measured to the same accuracy, a weighting matrix $R_{N\times N}$ is in order, in which case the preliminary emitter position estimate is

$$\hat{\mathbf{x}} = (\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{R}^{-1} z - (\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{R}^{-1}\mathbf{d}R_s. \tag{16}$$

Defining the new vectors

$$\mathbf{a} \triangleq (\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{R}^{-1}\mathbf{z} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{17}$$

and

$$\mathbf{b} \triangleq (\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{R}^{-1}\mathbf{d} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{18}$$

(16) becomes

$$\hat{\mathbf{x}} = \mathbf{a} - \mathbf{b}R_s. \tag{19}$$

Using the definitions (17) and (18), and (19), the following relationship holds:

$$\mathbf{x} = \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = \begin{bmatrix} a_1 - b_1 R_s \\ a_2 - b_2 R_s \\ a_3 - b_3 R_s \end{bmatrix}. \tag{20}$$

Inserting (20) into (7) and applying algebraic manipulations, yields the following quadratic equation in R_s:

$$\begin{aligned} &(b_1^2 + b_2^2 + b_3^2 - 1)R_s^2 - 2R_s(a_1 b_1 + a_2 b_2 + a_3 b_3) \\ &\quad + a_1^2 + a_2^2 + a_3^2 = 0 \end{aligned} \tag{21}$$

which has two solutions given by

$$\hat{R}_s = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3 \pm \sqrt{(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 - (b_1^2 + b_2^2 + b_3^2 - 1)(a_1^2 + a_2^2 + a_3^2)}}{b_1^2 + b_2^2 + b_3^2 - 1}. \tag{22}$$

The solutions to (22) are used as the solution to the estimated source-to-reference-sensor distance $\hat{R}_s$.

Substituting this value for $\hat{R}_s$ into (19) finally yields the estimated signal source location $\hat{\mathbf{x}}$.

III. EXISTENCE OF A SOLUTION

In 3D space, if at least four sensors are not coplanar and there is a subset of three sensors which are not collinear, then the matrix **S** has full rank. It should be noted that there remains the possibility that (22) has imaginary roots, in which case the solution to $\hat{R}_s$, and thus $\hat{\mathbf{x}}_s$, cannot be determined.

Consider the case of $N+1$ sensors.

ASSUMPTION 1

In 2D: $N \geq 2$, and at least 3 sensors are not collinear.

In 3D: $N \geq 3$, and at least 4 sensors are not collinear.

Assumption 1 assures the regressor matrix $S_{N\times 2}$ (in 2D) and $S_{N\times 3}$ (in 3D) is full rank. The measurement information is contained in the vectors $\mathbf{d},\mathbf{z} \in \Re^N$. The solutions of the quadratic equation are explicitly given by

$$\hat{R}_s = \frac{\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{z} \pm \sqrt{[\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{z}]^2 + \mathbf{z}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{z}\cdot[1-\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{d}]}}{\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{d}-1}. \tag{23}$$

PROPOSITION 1 *Assume that Assumption 1 holds. If*

$$\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{d} < 1 \tag{24}$$

then there exists a unique, real, positive solution to the quadratic equation (23) and the minus sign applies.

PROPOSITION 2 *Assume that Assumption 1 holds. If*

$$\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{d} > 1 \tag{25}$$

and

$$[\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{z}]^2 \geq \mathbf{z}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{z}\cdot[\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{d}-1] \tag{26}$$

then

$$\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{z} < 0 \tag{27}$$

implies that there exists a unique, real, positive solution to the quadratic equation (23) and the plus sign applies. If, however,

$$\mathbf{d}^T\mathbf{s}(\mathbf{s}^T\mathbf{s})^{-2}\mathbf{s}^T\mathbf{z} > 0 \tag{28}$$

then there exists two real, positive solutions to the quadratic equation (23) because both the plus and the minus signs apply. In this case, at least 4 sensors in 2D and 5 sensors in 3D are needed to resolve the ambiguity. If condition (26) does not hold, then a real solution does not exist. Such a case is an artifact of measurement noise and/or modeling error.

Note that this derivation is deterministic. Any errors in the TDOA measurements, whether random or bias-like, would propagate into the solution, and further development would need to be done to characterize this problem from a stochastic point of view.

IV. CONCLUSION

A direct and short derivation of an algorithm based on the closed-form solution of the nonlinear equations for emitter location using TDOA measurements from $N+1$ sensors, $N \geq 3$, was given. The existence of a solution and noise effects were also addressed.

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