A Comparison of Three Direct Sampling Techniques

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ABSTRACT

We describe numerical techniques that might be used as alternatives to heterodyning for the processing of ultra wideband radar data. Also discussed are techniques for the measurement of the central or "carrier" frequency of an ultra wideband signal waveform.

1. *Introduction.* Let $s(t)$ be a given real-valued signal waveform. We seek representations of the type

$$s(t) = p(t) \cos(\omega t) - q(t) \sin(\omega t) \quad (1)$$

where the functions p and q are presumed to be slowly varying narrow band signals, and the frequency ω is presumed to have a large value lying outside the spectral bands of p and q . The functions p and q are called the in-phase and quadrature components of s , and ω is called the carrier frequency. These quantities are not uniquely defined because for any given data function s and for any value of ω one can obtain a function pair p, q satisfying (1) by setting

$$\left. \begin{aligned} p(t) &= s(t) \cos(\omega t) + r(t) \sin(\omega t) \\ q(t) &= -s(t) \sin(\omega t) + r(t) \cos(\omega t) \end{aligned} \right\} \quad (2)$$

where $r=r(t)$ is any arbitrarily given function. Conversely, for any solution pair p, q satisfying (1) there is a unique function r for which p and q are given by (2), viz.,

$$r(t) = p(t) \sin(\omega t) + q(t) \cos(\omega t) \quad (3)$$

The function r will be called the auxiliary function for the function pair (p, q) , and the corresponding complex "envelope" function $z = z(t)$ is given by

$$z(t) := p(t) + iq(t) = (s(t) + ir(t)) e^{-i\omega t} \quad (4)$$

In the sections below we shall discuss techniques for the conceptual definition and practical measurement of p, q , and ω .

2. *The Variational Approach.* The variational approach is based on the idea that a reasonable representation is one for which p and q have the smallest possible amount of functional variation, or "jiggle". A convenient measure for the joint variation of p and q commonly used in the modern calculus of variations is the squared Sobolev space norm $\mathcal{V}(p, q)$ defined by [1]

$$\mathcal{V}(p, q) := \int_{-\infty}^{\infty} \{ \dot{p}^2(t) + \dot{q}^2(t) \} dt \quad (5)$$

where the dots denote time derivatives. Hence, for a given value of ω and data function $s(t)$, the variationally defined components are obtained by minimizing $\mathcal{V}(p, q)$, where p and q are varied subject to the constraint (1). It turns out that the variational components are provided by the auxiliary function $r(t)$ whose Fourier Transform (FT) $\hat{r}(\nu)$ is given by [5]

$$\hat{r}(\nu) = -i \frac{2\omega\nu}{\omega^2 + \nu^2} \hat{s}(\nu) \quad (\text{Variational Solution}) \quad (6)$$

where, by convention, a general function $f(t)$ and its FT $\hat{f}(\nu)$ are related by

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\nu) e^{i\nu t} d\nu \quad (7)$$

The minimizing function pair will sometimes be denoted by (p_ω, q_ω) to indicate their dependence on ω , and we define the carrier frequency ω_V to be the (positive) value of ω at which $\mathcal{V}(p_\omega, q_\omega)$ attains its minimum value. It turns out that [5]

$$\mathcal{V}(p_\omega, q_\omega) = 4\pi \int_0^{\infty} \frac{(\nu^2 - \omega^2)^2}{(\nu^2 + \omega^2)} |\hat{s}(\nu)|^2 d\nu \quad (8)$$

Hence ω_V can be calculated directly from the power spectrum of $s(t)$ in a manner that does not require the calculation of p_ω and q_ω for each value of ω . More specifically, $\mathcal{V}(p_\omega, q_\omega)$ is a convex function of ω^2 , $\mathcal{V}(p_\omega, q_\omega) = \mathcal{V}(\omega^2)$, and ω_V is therefore the unique positive zero to the equation $[\mathcal{V}'(\omega^2) = 0]$. We have found that Newton's Algorithm converges very rapidly to the zero of $\mathcal{V}'(\omega^2)$, with no more than five iterations being required to obtain convergence to one part in 10^{-10} . This has been true in both computer simulations and in applications to discretely sampled real world data for which the right-hand side of (8) must be approximated with a discrete sum analogue [5].

To Summarize : To obtain the variational representation for a given signal waveform $s(t)$, we first calculate ω_V using Newton's Algorithm to solve $[\mathcal{V}'(\omega_V^2) = 0]$. We then obtain $\hat{r}(\nu)$ by setting $\omega = \omega_V$ in (6), and we obtain $r(t)$ by applying an inverse FT to $\hat{r}(\nu)$. The components $p(t)$ and $q(t)$ are then obtained by setting $\omega = \omega_V$ in (2). For discretely sampled data, all integrals and FT's are approximated by their discrete sum analogues.

3. *The Gabor Solution.* The Gabor solution is the standard theoretical representation. It corresponds to the choice $r(t) = \mathcal{H}s(t)$, where $\mathcal{H}$ denotes the Hilbert transform [4,8]. In terms of FT's,

$$\hat{r}(\nu) = -i \operatorname{sgn}(\nu) \hat{s}(\nu) \quad (\text{Gabor Solution}). \quad (9)$$

With this choice the complex-valued function $A(t) = s(t) + ir(t)$ is called the *analytic signal* or *pre-envelope* [4,8]. From (9) it follows that all the energy of $A(t)$ is contained in the positive frequency domain; therefore if $s(t)$ is a narrow band waveform whose spectrum is confined to two narrow bands centered around the central frequencies $\omega = \pm\omega_0$, it follows that the energy of the corresponding envelope function $z(t) = [p(t) + iq(t)]$ (obtained by setting $\omega = \omega_0$ in (4)), will have its energy confined to a narrow band centered around the zero frequency. The most common choice for the "central" or "carrier" frequency is the *mean frequency* $\bar{\omega}$ defined by

$$\bar{\omega} = \frac{\int_0^\infty \nu |\hat{s}(\nu)|^2 d\nu}{\int_0^\infty |\hat{s}(\nu)|^2 d\nu}. \quad (10)$$

Another possible choice is the "mode" ω_m , which is the positive frequency at which the power spectrum $|\hat{s}(\nu)|^2$ attains its peak value. In the general case one cannot expect that $\bar{\omega}$, ω_m , and ω_V will have the same value, as will be illustrated below by examples.

4. *The D-Transform.* Both the variational and Gabor solutions require the use of a FT to calculate $\hat{r}(\nu)$ from $\hat{s}(\nu)$, and an inverse FT to calculate $r(t)$ from $\hat{r}(\nu)$. We shall now discuss a method which does not require the use of FT's. This method will be called the *differential* or "*D*" *method* or *transform* because it is obtained by taking the formal time derivative of (1) and throwing out terms involving $\dot{p}$ and $\dot{q}$. One then obtains expressions for $p(t)$ and $q(t)$ in terms of $s(t)$, $\dot{s}(t)$, and ω . It turns out that this solution corresponds to the choice

$$r(t) = -\dot{s}(t)/\omega. \quad (\text{D-Method Solution}) \quad (11)$$

The value to be used for ω is optional, and as before, the components $p(t)$ and $q(t)$ are obtained by substituting (11) into (2). However, if one keeps ω variable and substitutes (11) into (2) and (5), then $\mathcal{V}(p,q)$ becomes a function of ω which is minimized at $\omega = \omega_d$, where

$$\omega_d^4 = \frac{\int_{-\infty}^{\infty} |\dot{s}(t)|^2 dt}{\int_{-\infty}^{\infty} |s(t)|^2 dt} = \frac{\int_{-\infty}^{\infty} \nu^4 |\hat{s}(\nu)|^2 d\nu}{\int_{-\infty}^{\infty} |\hat{s}(\nu)|^2 d\nu} \quad (12)$$

Hence the D-method suggests yet another value of ω that might be used for the "carrier", viz., $\omega = \omega_d$.

Finally, we note that in applications to discretely sampled data $\{s_n\} = \{s(n\tau)\}$, better results are obtained by using a two sided rather than a one sided difference operator to approximate the derivative $\dot{s}$ occurring in (11); i.e., the n 'th value of $r(t)$ is set equal to

$$r_n = -\frac{s[(n+1)\tau] - s[(n-1)\tau]}{2\omega\tau}. \quad (13)$$

5. *Comparisons.* Space does not permit a detailed discussion, and so we shall only hit some of the highlights of our studies with both computer simulations and empirical data.

(i) *The General Narrow Band Case:* A comparison of the auxiliary functions for the variational and Gabor solutions, specifically the multiplicative factors of $\hat{s}(\nu)$ occurring in (6) and (9), show that these factors differ by less than 1 per cent when the signal power spectrum is confined to a band whose length is 30% of a central frequency [5]. Hence for narrow band signals $s(t)$, the Variational and Gabor solutions are practically identical.

(ii) *Sinusoid with a Sinc Function Modulation.* Consider the pulse-like waveform

$$s(t) = 2A \frac{\sin(bt)}{t} \cos(\omega_0 t). \quad (14)$$

Letting r_{gab} denote the Gabor solution for the auxiliary function, it turns out that

$$r_{gab}(t) = 2A \frac{\sin(bt)}{t} \sin(\omega_0 t).$$

It also turns out that $\bar{\omega} = \omega_0$, and setting $\omega = \bar{\omega} = \omega_0$ and $r = r_{gab}$ in (2), we get $p = p_{gab}$ and $q = q_{gab}$, where

$$\left. \begin{aligned} p_{gab}(t) &= s_0(t) = 2A \frac{\sin(bt)}{t} \\ q_{gab}(t) &= 0 \end{aligned} \right\} \quad (15)$$

We now compare these results with the variational solution. Table 1 shows the ratio ω_V/ω_0 as a function of the half-bandwidth b . The values of ω_V were calculated using Newton's Algorithm, as described above. It can also be shown that the variational components for any given waveform $s(t)$ satisfy the equation [5]

$$\dot{p}(t) \sin \omega t + \dot{q}(t) \cos \omega t = 0. \quad (16)$$

Hence the variational components cannot satisfy either $p(t) \equiv 0$ or $q(t) \equiv 0$ (as is the case for the Gabor solution shown in (15); however in the present example the variational quadrature component $q(t)$ was always found to be small, in the sense that the ratio of the q -energy to the total energy of the envelope function was never

found to exceed 3% for all values of b in the range $[0,1]$. We also note that the spectrum $\hat{p}_{var}(\nu)$ is not absolutely flat, as is the case for $\hat{p}_{gab}(\nu)$ [5].

Table 1

The Ratio ω_V/ω_0 as a Function of the Half-Bandwidth b

b	ω_V/ω_0	b	ω_V/ω_0
.00	1.0000000000	.10	1.0000075015
.01	1.0000000008	.20	1.0001200458
.02	1.0000000120	.30	1.0006070233
.03	1.0000000608	.40	1.0019092917
.04	1.0000001920	.50	1.0046068877
.05	1.0000004688	.60	1.0093371399
.06	1.0000009721	.70	1.0166484494
.07	1.0000018009	.80	1.0268227834
.08	1.0000030724	.90	1.0397671365
.09	1.0000049216	1.00	1.0550760263

Another parameter of interest is the instantaneous frequency, which is defined to be the time-derivative of the argument of the analytic signal $z(t) = s(t) + ir(t)$ [2]; i.e.,

$$\dot{\phi}(t) = \frac{s(t) \dot{r}(t) - r(t) \dot{s}(t)}{s^2(t) + r^2(t)} \quad (17)$$

Different choices of $r(t)$ lead to different definitions of $\dot{\phi}$. In general $\dot{\phi}(t)$ is a time-varying quantity, but for the waveform (14) it turns out that $\dot{\phi}_{gab}(t) \equiv \omega_0$, whereas

$$\dot{\phi}_{var}(0) = \omega_0 \left\{ 2 - \frac{\omega_0}{b} \left(\text{atan}[1+b/\omega_0] - \text{atan}[1-b/\omega_0] \right) \right\}.$$

(iii) Doppler Distorted Radar Waveforms. A computer simulation was undertaken to compare D-Method results with those that would be obtained with a standard heterodyne receiver. The simulation was carried out for both point and extended targets, and the transmitted radar waveform was taken to be of the form (14). The simulation modelled the Doppler induced: (1) shift in carrier, (2) change of scale in the modulation function argument, and (3) the $\sin x/x$ range fading that occurs as the target motion carries the target outside the range gate. For wideband systems, this latter effect can be very significant. The power spectra of the envelope functions $p+iq$ produced by the D-transform and heterodyne receiver were compared, and in all cases were found to be graphically identical (although there were small numerical differences of no practical importance.) The value for ω used in the calculations was taken to be equal to ω_0 (i.e., the radar LO frequency) since no practical advantage was found in the use of some other value, such as ω_V .

(iv) Empirical Data. The data in this example were taken with a bench top radar illuminating a pendulum consisting of a swinging steel sphere. The system is described in [7]. The data sample rate was 51.2 GHz, and each radar pulse waveform had approximately 20 zero crossings. The pulses grow and fade as the sphere swings in and out of the radar range gate, as can be seen in Figure 1, which shows a trace of 40,000 instantaneous samples. There are 128 samples in each pulse, and the data is "reconstituted", so that Doppler induced frequency shifts are scaled by a multiplicative factor equal to 1.2×10^6 , as described in [7]. The power spectrum for a typical pulse is shown in Figure 2. The spectra are sharply peaked, and the spectral spread σ was generally on the order of 2 GHz, where

$$\sigma^2 := \frac{\int_0^\infty (\omega - \bar{\omega})^2 |\hat{s}(\omega)|^2 d\omega}{\int_0^\infty |\hat{s}(\omega)|^2 d\omega}.$$

For pulses with large S/N ratios the calculated values of $\bar{\omega}$ and ω_V were almost always found to differ by less than 3%. The agreement between $\bar{\omega}$ and ω_d (or ω_V and ω_d) was not quite as good, and there was generally a large difference between $\bar{\omega}$ and ω_m . The mode ω_m is measured in increments of $\Delta f = 1/T$ where T is the observation time, and some of the difference between ω_m and $\bar{\omega}$ is due to this quantization effect. In fact, for 64 point DFT's $\Delta f = 0.8$ GHz, and the values for ω_m were found to toggle between 6.4 and 7.2 GHz. Figure 3 shows a plot of $\bar{\omega}$ and ω_V over the entire data set of 40,000 samples. There is one value of each for each pulse whose signal strength exceeded a certain threshold, which for the case shown was 175 pulses.

Although $\bar{\omega}$ and ω_V were found to be in close agreement, the time histories of the variational and Gabor values for instantaneous frequency were found to behave quite differently.

Figure 4 shows $s(t)$ for a typical pulse, and the p and q components provided by the D-Transform, Gabor, and variational solutions. The curves labeled "env" are the envelopes $\pm \sqrt{p^2(t) + q^2(t)}$, and all three envelopes can be seen to have the qualitative features that one would expect to see in an envelope function. In this particular example the in-phase and quadrature components that are produced by the three techniques are in close agreement, and it remains to be determined which if any of these solutions is best suited for the Doppler processing of radar data. Indeed, for ultra wideband systems it might be appropriate to dispense with the calculation of the p and q components, and to infer target velocity information directly from calculated values of ω_V , $\bar{\omega}$, the instantaneous frequencies, or some such other quantity.

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Figure 1

FIGURE 3
Time History of the Mean
and Variational Frequencies
(One sample per pulse.)

FIGURE 4
In-Phase and Quadrature Components as
Constructed by the D-Transform, Gabor,
and Variational Techniques