THRESHOLD EFFECTS IN TIME-DELAY ESTIMATION VIA CROSS-CORRELATION

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ABSTRACT

The estimate of the difference in time of arrival of a common signal received at two sensors, each of which also receives uncorrelated noise, is shown to exhibit a thresholding effect similar to that in pulse position modulation. Thus, as the post detection signal-to-noise ratio decreases, the probability of an anomalous estimate (an estimate that is outside $\pm T_c/2$, where T_c is the correlation time of the process) increases suddenly and precipitously. Theoretical results for the probability of an anomalous estimate are presented and verified experimentally. Threshold is shown to depend most strongly on the two quantities M and A . $M = 2T_o/T_c$, where $\pm T_o$ is the extent of the cross-correlation function searched for a peak; M is thus the effective number of independent correlation bins. $A = \{B_s T\}^{1/2} \{ (S/N) / [(S/N)^2 + (1 + S/N)^2]^{1/2} \}$ where B_s is the statistical bandwidth of the process, T is the total observation time, and S/N is the signal-to-noise ratio at the sensors. Depending on M , threshold is shown to occur in the region $3 < A < 5$.

1. INTRODUCTION

The estimate of the difference in the time of arrival of a random signal received at two or more sensors is an important practical problem in underwater acoustics. One common method for estimating the time delay is to use a cross-correlator. This instrumentation is illustrated in Fig. 1. I show two sensors, one of which receives a random signal $s(t)$ plus uncorrelated noise $n_1(t)$, and the other of which receives a time delayed version of the signal, $s(t - D_o)$, plus uncorrelated noise, $n_2(t)$. The signal variance is S , the uncorrelated noises have common variance N and the pre-integration signal-to-noise ratio is S/N . The time delay estimate is formed, conceptually, by inserting a delay D in one channel, averaging for time T and generating a correlation value denoted by $z(D)$. D is then varied throughout the range $\pm T_o$ and the value of D corresponding to the peak value of $z(D)$ is used as the estimate of the true value D_o .

The variance of the estimate $\hat{D}$ is of interest. For small estimation errors the variance of D can be determined using a linearized analysis. I want to investigate what happens as the estimation errors

get large, that is, when does the small error, linear analysis break down? I will show that a more interesting effect than a simple increase in

Figure 1. Cross-correlation time delay estimator

variance occurs as the errors get large. In fact, I will show that at some point, as the post-integration signal-to-noise ratio decreases, the probability of large error (what are called anomalous estimates) increases suddenly. This is a thresholding effect similar to that seen in non-linear communication systems such as pulse position modulation and frequency modulation [1, p. 627, p. 661].

I will show first the results of an approximate theory for the probability of an anomaly, and will then present experimental (that is, computer simulation) results verifying the theory. I will then compare the observed results for the variance of the time delay estimate with the linear theory to determine the range of applicability of that theory.

In Fig. 2 I show the correlator output $z(D)$ for both high and low signal to noise ratios. First, however, note that there is a time scale T_c , the correlation time, over which the process $s(t)$ is well correlated, and beyond which the correlation falls to zero.

Assume that the true time delay D_o is zero. Let $\pm T_o$ be the region of the time delay searched, and assume that T_o is much smaller than the total averaging time T . The correlator output for high signal-to-noise ratio is shown in the top illustration. Here the errors are small and the peak of the correlator output occurs close to the true value, say within $\pm T_c/2$. This is the situation that must occur for the linear or small error analysis to apply. As the signal-to-noise ratio

continues to decrease, the correlator output is observed to behave as shown in the bottom illustration. Now large values of $z(\hat{D})$ begin to occur far from the true value D_0 . These are anomalous estimates.

Figure 2. Correlator output for high and low SNR

2. APPROXIMATE THEORY FOR PROBABILITY OF ANOMALY

In Fig. 3 I outline the development of the approximate theory for the probability of an anomaly. First, since the signal decorrelates over time T_c , and since we search a region $\pm T_0$, we expect there to be roughly $2T_0/T_c$ independent values of $z(\hat{D})$. I'll call this number M . Let us denote these M independent values by z_0 through z_m .

1. ASSUME THERE ARE

$$M = 2T_0/T_c$$

INDEPENDENT VALUES OF $Z(\hat{D})$. CALL THESE $Z_0, \dots, Z_M$. Z_0 IS OUTPUT AT TRUE DELAY.

2. APPROXIMATE AN ANOMALY BY THE EVENT.

$$\{Z_M > Z_0 \text{ FOR AT LEAST ONE VALUE OF } Z_M\}$$

3. THEN

$$PR(\text{ANOMALY}) = 1 - \int_{-\infty}^{\infty} P(Z_0) \left[\int_{-\infty}^{Z_0} P(Z_M) dZ_M \right]^{M-1} dZ_0$$

4. $PR(\text{ANOMALY})$ DEPENDS ON M AND ON POST INTEGRATION SNR - A

$$A = (BT)^{\frac{1}{2}} \frac{S/\%}{[(S/N)^2 + (1+S/N)^2]^{\frac{1}{2}}}$$

Figure 3. Approximate theory for probability of anomaly

An anomaly occurs when the peak of the correlator output closest to zero is not the largest value in $\pm T_0$. We approximate this by the event such that at least one of the z_m exceeds z_0 . The probability of an anomaly can then be expressed, as shown, in

terms of the probability density functions of z_0 and z_m . This probability can be calculated numerically after assuming some form for the probability density functions. I will show results using both a Gaussian approximation to the probability density functions and then using a more refined estimate incorporating the first few terms of the Gram-Charlier approximation to the actual density function [2].

The probability of anomaly will be shown to depend on M the number of independent values of $z(\hat{D})$, and on the postintegration signal-to-noise ratio A , which, as shown, depends on the square root of the bandwidth time product and on the preintegration signal-to-noise ratio. My results apply only for large bandwidth time product.

3. PROBABILITY OF ANOMALY: THEORY AND EXPERIMENT

In Fig. 4 I compare the theoretical and experimental results for probability of anomaly versus postintegration signal-to-noise ratio for $M = 16$ and a bandwidth time product of 110. The two theoretical curves labelled Gaussian Approximation and Gram-Charlier Approximation are quite close, and only differ noticeably for the larger values of A . The curve labelled Upper Bound comes from a simple analytic approximation to the theoretical results and is seen to be reasonably accurate for large values of A .

Figure 4. Probability of anomaly vs. postintegration SNR, $M=16$, $B_s T=110$: theory and experiment

The observed values are indicated by the x's; the bars indicate 95% confidence intervals. The observed values were generated by a large number of computer simulation experiments. For these experi-

ments both the signal and noise sequences had low pass Gaussian spectra. This spectrum was chosen to avoid complicating effects due to sidelobes in the correlation functions. Details of the experiment are given in [2]. The theoretical and experimental results agree very well over the entire range of signal-to-noise ratio shown. The theory does appear to overestimate the probability of anomaly slightly for the largest value of A tested. The threshold value of A, where the probability of anomaly increases rapidly, is approximately $A = 4$.

Figure 5 shows theoretical and experimental results for probability of anomaly versus signal-to-noise ratio for M equal to 4 and 64, along with the previous theoretical curve for M=16. Only M was varied, the bandwidth time product was held constant. Again the data and theory agree well.

Figure 5. Probability of anomaly versus postintegration SNR, $M=4, 64$, $B_s T=110$: theory and experiment

As M increases for a constant value of post integration signal-to-noise ratio the probability of anomaly increases. This occurs because as M increases there are simply more chances for a noise bin to exceed the peak at the true value of delay, since the number of noise bins increases linearly with M.

4. VARIANCE OF TIME DELAY ESTIMATE

Figure 6 shows theoretical and experimental results for the variance of the time delay estimate as a function of preintegration signal-to-noise ratio for the previous case of $M=16$ and bandwidth time product equal 110. The curve labelled Linear Theory is the theoretical prediction based on the commonly used linear analysis. The curve labelled

Nonlinear Correction shows the predicted variance using a correction to the linear theory which allows for somewhat larger estimation errors [2]. I will discuss the third solid curve below.

Figure 6. Time delay variance versus preintegration SNR, $M=16$, $B_s T=110$: Gate Mode

The experimental variances were computed for two different modes of operation which I call Gated and No Gate. In the Gated mode the correlation peak closest to the known true delay that also fell within a specified delay gate width, was used to calculate the variance. This restricts the variance calculation to apply to local errors only. The linear theory also applies only for local errors. In a practical case this would be a valid mode of operation if there were some a priori information about the true value of time delay.

If there is no a priori information, other than that the time delay is expected to fall within $\pm T_0$, then the full range of the correlator must be searched for the peak. This is what I mean by the No Gate mode.

We can now examine the range of validity of the variance predictions from the linear theory. I show experimental results for the Gated modes for two gate widths, $\pm T_c/2$ and $\pm T_c$. The experimental variances for both gates are quite close for signal-to-noise ratios above 0.6 but the larger gate has a greater variance for the lower signal-to-noise ratios. The experimentally observed variances agree exactly with the linear theory for high signal-to-noise ratio and remain within 10% of the variance predicted by the linear theory for signal-to-noise ratios greater than 0.8 to 1. This range of signal-to-noise ratio corresponds to a postintegration signal-to-noise ratio of 4, which was identified earlier as a rough threshold

value. In this range of signal-to-noise ratio the experimental results still agree well with the predictions based on the nonlinear correction. However, at sufficiently low signal-to-noise ratios even the nonlinear correction is no longer adequate. At very low signal-to-noise ratios the experimental results also depend on the gate width; the observed variances are still quite repeatable, however, so in principle they can be predicted by a sufficiently complex theory. From these results we see that the range of applicability of the linear theory can be predicted quite accurately when the correlator is used in the Gated mode, at least for the Gaussian spectral shape used here.

The theoretical variance for the No Gate mode is shown by the curve to the extreme right. This variance rises rapidly with decreasing signal-to-noise ratio. This effect is shown more clearly in Fig. 7.

Figure 7. Time delay variance versus preintegration SNR, $M=16$, $B_s T=110$: No Gate mode

Figure 7 shows the theoretical and experimental variance for the ungated mode along with the theoretical prediction for the linear theory. At a signal-to-noise ratio of zero, the variance for the No Gate mode is equal to $T_0^2/3$, since this is the variance of a variable that is uniformly distributed in $\pm T_0$. At other signal-to-noise ratios the theoretical variance for No Gate is equal to $T_0^2/3$ times the probability of an anomaly, added to the variance from the linear theory. This is what I have plotted as the line labelled Theoretical Variance for No Gate. The observations again agree quite well with the theory.

From Fig. 7 it is clear that the variance due to the anomalies dominates the contribution due to the local errors. The predictions from the linear theory in fact are not applicable for the No Gate mode until a signal-to-noise ratio greater than 2.5 to 3. This is roughly a factor of three larger than that found using the Gated mode only. These conclusions only apply for the $M=16$ case presented here. Generally, as M increases the anomalies become more important. It should be remembered, however, that at the high signal-to-noise ratios the anomalous estimates, which cause the increased variance, are very rare; performance should be interpreted with this in mind.

5. SUMMARY

Three main conclusions have been reached by this study. First, the approximate theory for the probability of an anomaly agrees well with experimental results. Second, in the Gated mode the observed variance of the time delay estimates is related to the linear theory in a predictable manner. Thus the range of validity of the theory can be established for the particular spectral shape used. Finally, in the No Gate mode, the variance due to the anomalies can dominate, and must then be considered in performance predictions.

6. REFERENCES

1. J. M. Wozencraft and I. M. Jacobs, Principles of Communication Engineering, Wiley, New York, 1965.
2. J. P. Ianniello, Time delay estimation via cross-correlation in the presence of large estimation errors, in preparation.