Analysis Using Bi-Spectral Related Techniques.

Ralph Hippenstiel
EC/Hi
Naval Postgraduate School
Monterey, Ca 93943
e-mail hippenst@ece.nps.navy.mil

Abstract: This paper investigates the use of Bi-spectral related techniques to obtain detection clues from time-frequency representations. Earlier results have indicated that 1.5-D spectral techniques (a degenerate version of the Bi-spectrum) has potential Signal to Noise Ratio (SNR) gain over conventional techniques. This is partially due to the rejection of Gaussian like perturbations by the cumulant based techniques. The third order moment for Gaussian zero mean random processes is essentially zero. A comparison using synthetic data via conventional spectrogram, instantaneous power spectrum, and the cumulant based technique is provided.

1. Introduction

This paper examines the instantaneous power spectrum (IPS) and its cumulant based version (CM) relative to conventional spectral processing (i.e. spectrogram, Lofargram, periodogram). Section 2 provides the mathematical definition for IPS, CM, and LOFAR. Sections 3 and 4 allow a theoretical and experimental comparison, respectively. The comparison is performed using a deflection criterion (i.e. maximum signal magnitude versus standard deviation of the detector output under noise only conditions). Section 5 contains the conclusion.

2. IPS, CM and LOFAR

2.1 IPS:

IPS is a member of the Cohen's class [1], and is also known as the real part of the Rihaczec distribution. Our particular implementation differs in that filtering is employed as the data is processed (equation 1). Earlier results have shown that in contrast to the Wigner-Ville Distribution (WVD) no spectral cross modulation terms are generated. However, IPS has wider spectral peaks and superimposed spectral auto modulation terms [2]. These auto terms do not interfere with the interpretation of T-F surfaces, but can help in the detection of a weak stable spectral line component.

The digital implementation of IPS is given by

$$IPS(n,k) = \frac{1}{2} \sum_{m=-N/2}^{N/2-1} [x(n)x^*(n-m) + x^*(n)x(n+m)] w(m) \exp(-j2\pi km/N) \quad (1)$$

where k = frequency index, n = time index, N = data length, m = shift or delay of the implied instantaneous correlation function, and $w(\cdot)$ = lag window. Some earlier results are published in [2,3,4].

2.2 CM:

In general the Bi-spectrum is defined as

$$S_X(\omega_1, \omega_2) = \sum_{l_1} \sum_{l_2} C_X(l_1, l_2) e^{-j(\omega_1 l_1 + \omega_2 l_2)} \quad (2)$$

where the cumulant is given by [5]

$$C_X(l_1, l_2) = E(X^*(n)X(n+l_1)X(n+l_2)). \quad (3)$$

A degenerate version of the cumulant is given by

$$C_X(0, l_2) = E(X^*(n)X(n)X(n+l_2)) = E(|X(n)|^2 X(n+l_2)). \quad (4)$$

When replacing the expected value with the instantaneous value the degenerate Bi-spectrum, also called the 1½ D spectrum, becomes

$$S_X(\omega) = \sum_l |x(n)|^2 x(n+l) e^{-j\omega l}. \quad (5)$$

The cumulant version of IPS is given by

$$CM(n,k) = \frac{1}{2} \sum_{m=-N/2}^{N/2-1} [|x(n)|^2 x^*(n-m) + |x(n)|^2 x(n+m)] w(m) \exp(-j2\pi km/N) \quad (6)$$

where n = time index, k = frequency index, m = lag variable, and N = data length of segment under consideration.

Cross terms at locations where the true spectrum has no support are not created. Also contrary to the conventional Bi-spectrum none of the spectrally related components are suppressed.

2.3 LOFAR:

The LOFAR algorithm uses a Hamming windowed Fast Fourier Transform (FFT) and displays the magnitude of the transform. To keep T-F dimensions relative small, no zero padding is used.

$$P_x(n,k) = \left| \sum_{m=n}^{n+N-1} x(m) e^{-j2\pi km/N} \right| \quad (7)$$

where n = time index and k = frequency index and N is the transform length.

The IPS, CM, and LOFAR surfaces are averaged (in magnitude) over some segments of the time axis leaving only a display of spectral magnitude versus frequency. For each segment one spectral line is created resulting again in a time frequency display. The averaging technique when using magnitude squared values is known as power or incoherent averaging.

3. Theoretical results for averaged surfaces.

The analysis in this section is done by disregarding the window effects and assuming that under the noise only hypothesis (H_0) Gaussian white noise and under the signal only hypothesis (H_1) a deterministic signal of the form $x(n) = A \cos(2\pi k_0 n/N)$ is present. The value of k_0 , n , and N are assumed to be integers which ensures that the spectrum peaks at a bin center of an N -point Fourier transform. In all detection schemes (averaged IPS,

averaged CM, and averaged LOFAR) magnitudes are averaged since they are numerically more conveniently computed than the magnitude squared values.

The comparison is done for normalized versions of

$I(k) = \sum_n |IPS(n,k)|$,
 $C(k) = \sum_n |CM(n,k)|$, and
 $P(k) = \sum_n |\text{windowed Fourier transform } \{x(n_i,k)\}|$
 where n_i is the time counted from reference point n . In what follows the time (n) and frequency (k) dependency is usually suppressed.

3.1 Averaged IPS.

where $x(n) \sim N(0, \sigma^2)$ i.i.d. and M is the appropriate number of degrees of freedom.

Under noise only condition :

Under noise only condition (H_0) the random variable Y denoted by Y_0 has the following moments [6]:

$$E\{Y_0\} = \sigma^2, \\ E\{Y_0^2\} = [(N+5)/2] \sigma^4, \\ \text{var}(Y_0) = [(N+3)/2] \sigma^4.$$

Making the assumption that for large enough number of data points the random variable Y is approximately Gaussian distributed we can derive the variance for W as

$$\text{var}(W_0) \approx 1/M \text{ var}(Z_0) \quad (\text{for large } N) \\ \approx (1/M) 0.48 N \sigma^4.$$

A typical surface has about $N/3$ degrees of freedom [6], providing a standard deviation (under the noise only condition) of $\sigma(W_0) \approx 1.2 \sigma^2$.

Under signal only condition (at spectral location $\pm k_0$):

$$Z_1 = |Y_1| = (NA^2) \cos^2(2\pi k_0 n/N) \\ W_1 = (NA^2)/M \sum_n \cos^2(2\pi k_0 n/N) \\ \approx (NA^2)/2.$$

The deflection D_{PSI} for IPS is defined as

$$D_{IPS} = W_1/\sigma(W_0) = (NA^2/2)/(1.2 \sigma^2) = (N/2.4)(A^2/\sigma^2).$$

3.2 Averaged CM.

Under noise only condition:

$\text{var}(W_0) = 1/M \{(3/2-2/\pi)(6N+54)/4\} \sigma^6$
 $\sigma(W_0) = \text{SQRT}\{1/M (3/2-2/\pi)(6N+54)\}/2 \sigma^3$.
 A typical surface has about $N/3$ degrees of freedom [6], so the standard deviation becomes
 $\sigma(W_0) \approx \text{SQRT}\{3/N (3/2-2/\pi)(6N+54)\}/2 \sigma^3$.

Under signal only condition (at spectral location $\pm k_0$):

$$x(n) = A \cos(2\pi k_0 n/N)$$

The summation is lower bounded by 0.424

$$\frac{1}{M} \sum_{n=0}^{M-1} |(\cos^3(2\pi 1n/M))| > 0.424.$$

Hence $Z_1 > 0.424 NA^3/2$.

The deflection D_{CM} becomes

$$D_{CM} = (0.424 NA^3/2) \text{SQRT}(M)/\{\text{SQRT}\{(3/2-2/\pi)(6N+54)/4\} \sigma^3\}$$

$$\approx 0.1075 N (A/\sigma)^3.$$

3.3 LOFAR:

Under noise only condition:

$X(k) = X_r(k) + j X_i(k)$
 $= \sum_{m=n}^{n+N-1} x(m) \exp(-j(2\pi km/N))$; where n is the reference time

$\text{var}(W_0) = 1/M \text{var}(Z_0) = (1/M) (2-\pi/2)\sigma^2(N/2)$
 $\sigma(W_0) = \text{SQRT}\{(1/M) (2-\pi/2) (N/2)\} \sigma$

Under signal only condition (at spectral location $\pm k_0$):

$$X_i(k) = A N/2 @ k = \pm k_0$$

$$E(W_1) = E(Z_1) = A N/2$$

The deflection D_{LOFAR} becomes

$$D_{LOFAR} = (AN/2) / \{(1/M)(2-\pi/2)\sigma^2(N/2)\}^{1/2}$$

$$= N^{1/2} [3 A^2 / ((4-\pi)\sigma^2)]^{1/2}.$$

3.4 Deflection ratio comparison:

We can now plot the deflection ratio, that is compare and determine which detector is more advantageous under which conditions.

Comparing D_{IPS} and D_{LOFAR} :

$$D_{IPS} > D_{LOFAR}$$

simplifies to

$$N > 20.13 (\sigma^2/A^2)$$

Comparing D_{CM} and D_{LOFAR} :

$$D_{CM} > D_{LOFAR}$$

simplifies to

$$N > 302.4 (\sigma^4/A^4).$$

We notice that the processing length variable N , for IPS relative to LOFAR has a $(\text{SNR})^{-1}$ and for CM relative to LOFAR has a $(\text{SNR})^{-2}$ dependency.

4. Simulation results.

Ten test sets, consisting of sinusoidal signals embedded in white Gaussian noise are processed by the algorithms. Each test set contains seven sinusoids whose SNR ranges from -3 dB to -21 dB in 3 dB steps. The weakest sinusoid (bin 223-224 of 512 point transform) clearly shows up in the IPS and CM representation (figure 1.a & 1.b) but is not detectable in the LOFAR representation (figure 1.c). Figure 2 shows theoretical bounds. The straight solid lines demarcate the theoretical transitions points at which CM works better than LOFAR (upper straight line) and where IPS works better than LOFAR (lower straight line). Superimposed are the results from the simulation (zigzagging lines).

5. Conclusions.

It was demonstrated that processing gain can be obtained for IPS and CM relative to the LOFAR processing technique. This is a function of signal and processing parameters. It may allow glimpses of target signatures which otherwise maybe hidden. However, the processing gain when achievable is bought at an increase in processing cost. IPS and CM are actually tools designed more for transient type signals but as shown can be used to obtain gain on stable narrow band signals. Simplified processing was not attempted but deserves a closer look. Phase information (i.e. every T-F surface can be accessed in its complex data form, hence permits phase calculation) was not utilized.

7. References

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FIGURE 1. AVERAGED SPECTRA (y-axis=power,x-axis =frequency)

FIG 2. BOUNDARY FOR THEORETICAL & EXPERIMENTAL PERFORMANCE