Field Decomposition for Analysis of Hyperbolic Media

M. Havrilla

Air Force Institute of Technology, Department of ECE, 2950 Hobson Way, 45433, WPAFB, USA
michael.havrilla@afit.edu

Abstract – A field decomposition of Maxwell’s equations for uniaxial hyperbolic media based on the 2D Helmholtz theorem is provided. This decomposition is particularly useful since it produces TE and TM field sets that are orthogonal and allows for identification of two independent scalar potential formulations. It is shown that these formulations provide substantial mathematical simplification, enhanced physical insight, ease of boundary condition enforcement and a trivial field recovery process. An example involving a parallel-plate waveguide filled with hyperbolic media is given to demonstrate the functionality and practicality of the decomposition.

I. INTRODUCTION

Recent advancements in technology have made the fabrication of complex media, such as uniaxial hyperbolic media [1], possible. In order to understand how electromagnetic waves interact with such materials, Maxwell’s equations are employed. Depending on the form of the material tensor properties, Maxwell’s equations may be decomposed [2]-[3] into reduced form to simplify the solution process. As pointed out in [2], many decompositions are possible. The goal here is to present a decomposition based on the 2D Helmholtz theorem that is valid for uniaxial (e.g., hyperbolic) media. It will be shown in Section II that this decomposition produces TE and TM orthogonal field sets. Scalar potentials and associated boundary conditions are subsequently identified that lead to further mathematical simplification, enhanced physical insight, ease of boundary condition enforcement and a trivial field recovery process. Green’s functions for a parallel-plate waveguide filled with hyperbolic media are derived in Section III to demonstrate the simplicity and usefulness of the decomposition. Section IV provides concluding remarks. Note, an $\exp(j\omega t)$ time dependence is assumed and suppressed.

II. FIELD DECOMPOSITION BASED ON THE 2D HELMHOLTZ THEOREM

Maxwell’s curl equations may be written in transverse and longitudinal parts [4] for uniaxial media as follows

$$\begin{aligned} \nabla_t \times \hat{z} E_z + \hat{z} \times \frac{\partial \vec{E}_t}{\partial z} &= -\vec{J}_{ht} - j\omega\mu_t \vec{H}_t & \nabla_t \times \vec{E}_t &= -\hat{z} J_{hz} - \hat{z} j\omega\mu_z H_z \\ \nabla_t \times \hat{z} H_z + \hat{z} \times \frac{\partial \vec{H}_t}{\partial z} &= \vec{J}_{et} + j\omega\epsilon_t \vec{E}_t & \nabla_t \times \vec{H}_t &= \hat{z} J_{ez} + \hat{z} j\omega\epsilon_z E_z \end{aligned} \quad (1)$$

Upon decomposing the transverse fields and currents into orthogonal lamellar (curl free) and rotational parts via the 2D Helmholtz theorem, the respective linearly independent TE^z and TM^z field sets are identified, namely

$$\begin{aligned} \nabla_t \times \hat{z} H_z + \hat{z} \times \frac{\partial \vec{H}_{t\ell}}{\partial z} &= \vec{J}_{etr} + j\omega\epsilon_t \vec{E}_{tr} & \nabla_t \times \hat{z} E_z + \hat{z} \times \frac{\partial \vec{E}_{t\ell}}{\partial z} &= -\vec{J}_{ht\ell} - j\omega\mu_t \vec{H}_{tr} \\ \hat{z} \times \frac{\partial \vec{E}_{tr}}{\partial z} &= -\vec{J}_{ht\ell} - j\omega\mu_t \vec{H}_{t\ell} & \dots TE^z &, & \hat{z} \times \frac{\partial \vec{H}_{tr}}{\partial z} &= \vec{J}_{et\ell} + j\omega\epsilon_t \vec{E}_{t\ell} & \dots TM^z, \\ \nabla_t \times \vec{E}_{tr} &= -\hat{z} J_{hz} - \hat{z} j\omega\mu_z H_z & & & \nabla_t \times \vec{H}_{tr} &= \hat{z} J_{ez} + \hat{z} j\omega\epsilon_z E_z \end{aligned} \quad (2)$$

where the subscripts ℓ and r represent lamellar and rotational parts, respectively. This simplifying decomposition is very insightful since it shows, in the TM^z field set, that $J_{ez}, \vec{J}_{et\ell}, \vec{J}_{ht\ell}$ maintains a transverse lamellar and z-directed electric field and a transverse rotational magnetic field, as one would physically expect from Maxwell’s equations. Complementary insight holds for the TE^z field set. A scalar potential formulation,

which leads to further simplification, is readily identified for each orthogonal field set by explicitly identifying the lamellar and rotational fields and currents in terms of potential functions $\Phi, \theta, \pi, \psi, u_e, u_h, v_e, v_h$, namely

$$2D \text{ Helmholtz Theorem} \rightarrow \begin{aligned} \vec{E}_t &= \vec{E}_{t\ell} + \vec{E}_{tr} = \nabla_t \Phi + \nabla_t \times \hat{z} \theta & \vec{J}_{et} &= \vec{J}_{et\ell} + \vec{J}_{etr} = \nabla_t u_e + \nabla_t \times \hat{z} v_e \\ \vec{H}_t &= \vec{H}_{t\ell} + \vec{H}_{tr} = \nabla_t \pi + \nabla_t \times \hat{z} \psi & \vec{J}_{ht} &= \vec{J}_{ht\ell} + \vec{J}_{htr} = \nabla_t u_h + \nabla_t \times \hat{z} v_h \end{aligned} \quad (3)$$

Substitution of (3) into (2) gives the respective TE^z and TM^z scalar potential formulations (with $k_t^2 = \omega^2 \epsilon_t \mu_t$)

$$\begin{aligned} -\frac{\partial^2 \theta}{\partial z^2} - \frac{\mu_t}{\mu_z} \nabla_t^2 \theta - k_t^2 \theta &= -\frac{\mu_t}{\mu_z} J_{hz} + \frac{\partial u_h}{\partial z} - j\omega \mu_t v_e & -\frac{\partial^2 \psi}{\partial z^2} - \frac{\epsilon_t}{\epsilon_z} \nabla_t^2 \psi - k_t^2 \psi &= \frac{\epsilon_t}{\epsilon_z} J_{ez} - \frac{\partial u_e}{\partial z} - j\omega \epsilon_t v_h \\ \pi &= -\frac{1}{j\omega \mu_t} \left(\frac{\partial \theta}{\partial z} + u_h \right) & \Phi &= \frac{1}{j\omega \epsilon_t} \left(\frac{\partial \psi}{\partial z} - u_e \right) \\ H_z &= \frac{1}{j\omega \mu_z} (\nabla_t^2 \theta - J_{hz}) & E_z &= -\frac{1}{j\omega \epsilon_z} (\nabla_t^2 \psi + J_{ez}) \end{aligned} \quad TE^z, \quad TM^z. \quad (4)$$

Boundary conditions at a perfect conductor ($\vec{E}_{tang} = 0$) and material interface ($\vec{E}_{tang}, \vec{H}_{tang}$ continuous), for interfaces perpendicular to z , are readily identified by taking key note that the lamellar and rotational fields are orthogonal, leading to the simple TE^z and TM^z boundary condition relations, respectively

$$\theta = 0; \theta_1 = \theta_2, \pi_1 = \pi_2 \dots TE^z \quad \Phi = 0; \Phi_1 = \Phi_2, \psi_1 = \psi_2 \dots TM^z. \quad (5)$$

The overall procedure is to solve the wave equations for θ and ψ , compute π and Φ , enforce boundary conditions, then perform field recovery via (3) and (4). To demonstrate the simplicity and usefulness of this analysis, the Green's functions for a parallel-plate waveguide filled with hyperbolic media are computed next.

III. GREEN'S FUNCTIONS OF A PARALLEL-PLATE WAVEGUIDE FILLED WITH A HYPERBOLIC MEDIUM

Consider a parallel-plate waveguide with PEC interfaces located at $z=0, d$ and filled with a uniaxial hyperbolic medium having permittivity $\vec{\epsilon} = \hat{x}\hat{x}\epsilon_t + \hat{y}\hat{y}\epsilon_t + \hat{z}\hat{z}\epsilon_z$ (with $\epsilon_t > 0$ and $\epsilon_z < 0$) and permeability μ_0 . It will be assumed, for the sake of brevity, only a z -directed electric current source exists in the structure, thus only TM^z fields need to be considered. Other current sources are accommodated via (4). The Green's functions for this structure can be found via superposition of principal and scattered waves satisfying the equations

$$-\frac{\partial^2 \psi^p}{\partial z^2} - \frac{\epsilon_t}{\epsilon_z} \nabla_t^2 \psi^p - k_t^2 \psi^p = \frac{\epsilon_t}{\epsilon_z} J_{ez}, \quad -\frac{\partial^2 \psi^s}{\partial z^2} - \frac{\epsilon_t}{\epsilon_z} \nabla_t^2 \psi^s - k_t^2 \psi^s = 0 \quad (6)$$

with $k_t^2 = \omega^2 \epsilon_t \mu_0$. Fourier transformation on $\vec{\rho}$ and z leads to the principal solution

$$[k_z^2 - (k_t^2 - \frac{\epsilon_t}{\epsilon_z} k_\rho^2)] \tilde{\psi}^p = (k_z^2 - k_{zp}^2) \tilde{\psi}^p = \frac{\epsilon_t}{\epsilon_z} \tilde{J}_{ez} \Rightarrow \tilde{\psi}^p(\vec{k}_\rho, k_z) = \frac{\epsilon_t}{\epsilon_z} \frac{\tilde{J}_{ez}(\vec{k}_\rho, k_z)}{(k_z - k_{zp})(k_z + k_{zp})}. \quad (7)$$

Fourier inversion to the $\vec{k}_\rho$ domain is accomplished using complex-plane analysis, resulting in

$$\tilde{\psi}^p(\vec{k}_\rho, z) = \int_0^d \frac{\epsilon_t e^{-jk_{zp}|z-z'|}}{j2\epsilon_z k_{zp}} \tilde{J}_{ez}(\vec{k}_\rho, z') dz'. \quad (8)$$

The scattered solution is easily found to be simple harmonic functions, thus the total solutions are

$$\begin{aligned} \tilde{\psi}(\vec{k}_\rho, z) &= \int_0^d \frac{\epsilon_t e^{-jk_{zp}|z-z'|}}{j2\epsilon_z k_{zp}} \tilde{J}_{ez}(\vec{k}_\rho, z') dz' + W^+ e^{-jk_{zp}z} + W^- e^{jk_{zp}z} \\ \tilde{\Phi}(\vec{k}_\rho, z) &= \frac{1}{j\omega \epsilon_t} \frac{\partial \psi}{\partial z} = \int_0^d \frac{j \operatorname{sgn}(z-z') e^{-jk_{zp}|z-z'|}}{\omega 2\epsilon_z} \tilde{J}_{ez}(\vec{k}_\rho, z') dz' - \frac{k_{zp}}{\omega \epsilon_t} W^+ e^{-jk_{zp}z} + \frac{k_{zp}}{\omega \epsilon_t} W^- e^{jk_{zp}z}, \end{aligned} \quad (9)$$

where W^+ and W^- are the up and down-going wave amplitudes along the z -direction and $\operatorname{sgn}(z-z')$ is +1 for $z > z'$ and -1 for $z < z'$. Enforcing boundary conditions ($\tilde{\Phi} = 0$) at the bottom and top PEC plates, with the aid of (5), leads to the result

$$\begin{aligned}\tilde{\psi}(\vec{k}_\rho, z) &= \int_0^d -\frac{\epsilon_t}{2\epsilon_z k_{zp}} \frac{\cos k_{zp}(d - |z - z'|) + \cos k_{zp}(d - z - z')}{\sin k_{zp}d} \tilde{J}_{ez}(\vec{k}_\rho, z') dz' \\ \tilde{\Phi}(\vec{k}_\rho, z) &= \int_0^d \frac{j}{2\omega\epsilon_z} \frac{\text{sgn}(z - z') \sin k_{zp}(d - |z - z'|) + \sin k_{zp}(d - z - z')}{\sin k_{zp}d} \tilde{J}_{ez}(\vec{k}_\rho, z') dz'\end{aligned}\quad (10)$$

Field recovery follows immediately via the relations in (3),(4) and using $\tilde{J}_{ez} = \hat{z} \cdot \tilde{\vec{J}}_e$, resulting in

$$\vec{E}_{t\ell} = \nabla_t \Phi \Rightarrow \vec{E}_{t\ell} = j\vec{k}_\rho \tilde{\Phi} = \int_0^d \left[\frac{j\vec{k}_\rho \hat{z} j}{2\omega\epsilon_z} \frac{\text{sgn}(z - z') \sin k_{zp}(d - |z - z'|) + \sin k_{zp}(d - z - z')}{\sin k_{zp}d} \right] \cdot \tilde{\vec{J}}_e(\vec{k}_\rho, z') dz' \quad (11)$$

$$\vec{H}_{tr} = \nabla_t \times \hat{z} \psi \Rightarrow \vec{H}_{tr} = j\vec{k}_\rho \times \hat{z} \tilde{\psi} = \int_0^d \left[-\frac{j\vec{k}_\rho \times \hat{z} \epsilon_t}{2\epsilon_z k_{zp}} \frac{\cos k_{zp}(d - |z - z'|) + \cos k_{zp}(d - z - z')}{\sin k_{zp}d} \right] \cdot \tilde{\vec{J}}_e(\vec{k}_\rho, z') dz' \quad (12)$$

$$\vec{E}_z = \hat{z} E_z \Rightarrow \vec{E}_z = \int_0^d \left[-\frac{\hat{z} \hat{z} k_\rho^2}{j\omega\epsilon_z} \frac{\epsilon_t}{2\epsilon_z k_{zp}} \frac{\cos k_{zp}(d - |z - z'|) + \cos k_{zp}(d - z - z')}{\sin k_{zp}d} - \frac{\hat{z} \hat{z} \delta(z - z')}{j\omega\epsilon_z} \right] \cdot \tilde{\vec{J}}_e(\vec{k}_\rho, z') dz' \quad (13)$$

The desired spectral domain dyadic Green's functions are therefore identified as the bracketed quantities in the integrands of (11)-(13). Again, the physical insight gained by this simplifying analysis reveals the anticipated result that a z-directed electric current will maintain a transverse lamellar ($j\vec{k}_\rho$) and z-directed electric field and a transverse rotational ($j\vec{k}_\rho \times \hat{z}$) magnetic field. The cos and sin terms reveal the anticipated standing waves along the z-direction and the poles resulting from $\sin k_{zp}d = 0$ are the expected discrete parallel-plate waveguide modes. The $\text{sgn}(z - z')$ function highlights the fact that the transverse lamellar electric field is discontinuous across the source region, which is again expected due to the dipole nature of the z-directed current. The spatial domain Green's functions are found via Fourier inversion, the details are omitted for brevity. These Green's functions can be utilized for nondestructive evaluation of hyperbolic media.

IV. CONCLUSION

A decomposition based on the 2D Helmholtz theorem for both transverse fields and currents in a uniaxial medium was presented. It was shown that this decomposition led to orthogonal TE^z and TM^z field sets. Scalar potentials were subsequently identified along with simplifying boundary conditions, which allowed for efficient and physically-insightful results. Field recovery was shown to be straightforward. An example of a parallel-plate waveguide filled with a uniaxial hyperbolic media was presented to demonstrate the simplicity and usefulness of this technique. Future work will include the analysis of a hyperbolic media half space and the generalization of these results to gyrotropic bianisotropic media.

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