Use of the Kurtosis Statistic in the Frequency Domain as an Aid in Detecting Random Signals

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Abstract—**Power spectral density estimation is often employed as a method for signal detection. For signals which occur randomly, a frequency domain kurtosis estimate supplements the power spectral density estimate and, in some cases, can be employed to detect their presence. This has been verified from experiments with real data of randomly occurring signals. In order to better understand the detection of randomly occurring signals, sinusoidal and narrow-band Gaussian signals are considered, which when modeled to represent a fading or multipath environment, are received as non-Gaussian in terms of a frequency domain kurtosis estimate. Several fading and multipath propagation probability density distributions of practical interest are considered, including Rayleigh and log-normal. The model is generalized to handle transient and frequency modulated signals by taking into account the probability of the signal being in a specific frequency range over the total data interval. It is shown that this model produces kurtosis values consistent with real data measurements.**

The ability of the power spectral density estimate and the frequency domain kurtosis estimate to detect randomly occurring signals, generated from the model, is compared using the deflection criterion. It is shown, for the cases considered, that over a large range of conditions, the power spectral density estimate is a better statistic based on the deflection criterion. However, there is a small range of conditions over which it appears that the frequency domain kurtosis estimate has an advantage. The real data that initiated this analytical investigation are also presented.

I. INTRODUCTION

IN MANY IMPORTANT signal processing applications, including underwater acoustics, an estimate of the power spectral density (PSD) of the received data is often employed for signal detection. The data are first transformed into the frequency domain by utilizing the discrete Fourier transform (DFT), which can be efficiently executed by an algorithm called the fast Fourier transform (FFT). At this point, the data are considered to be in the frequency domain and an estimate of the PSD can be easily obtained. Often, this estimate consists of averaging together a sufficient number of individual FFT spectrums or periodograms to ensure consistent results [1].

The PSD is essentially a sum of the estimates of the second-order moments for both the real and imaginary parts of each frequency component in the frequency domain. If the frequency domain signals are randomly occurring and not Gaussian distributed, then higher order moments of the complex frequency components may contain additional information that could be utilized in signal processing. The objective of this paper is to compare the PSD technique for signal processing with a new method which computes the frequency domain kurtosis (FDK) [2] for the real and imaginary parts of the complex frequency components. Kurtosis is defined as a ratio of a fourth-order central moment to the square of a second-order central moment.

Using the Neyman-Pearson theory in the time domain, Ferguson [3], has shown that kurtosis is a locally optimum detection statistic under certain conditions. The reader is referred to Ferguson's work for the details; however, it can be simply said that it is concerned with detecting outliers from an otherwise Gaussian sample. The outliers are equivalent to the randomly occurring signal that is to be detected. By extending this idea to the frequency domain and based on analyses of real underwater acoustics data, we have found conditions under which the FDK indicates the presence of randomly occurring signals [2], [4]. Both time and frequency domain analyses of the real data have been performed. By setting the frequency parameter equal to zero in a DFT, it can be shown that the time domain is a special case of the frequency domain. Analogous results should also hold in the spatial domain; however, we will only consider the frequency domain here. In addition, the results are applicable to both active and passive sonar, although we will concentrate on the latter application rather than the former. The objective of this paper is to analytically determine the potential for exploiting kurtosis estimation in the frequency domain to indicate the presence of randomly occurring signals. To accomplish this, we introduce a model for the received data which contains the effects of amplitude and phase fluctuation of the signal. In addition, to be more realistic, transient and frequency modulation effects of the signal are also incorporated into the model. References which support this model will be cited in the text. To justify the results presented here, the PSD and FDK estimates will be compared in the last section using the real underwater acoustic data that initiated this work. However, subsequent data have also supported the analytical work presented here.

These results should also apply in other fields where the detection of a randomly occurring signal is important. For example, the detection of variable stars in astronomy may benefit from this approach.

II. FREQUENCY DOMAIN KURTOSIS

Let $x(i, q) = x[(i + (q - 1)M)h]$, $i = 0, 1, \cdots, M - 1$, $q = 1, 2, \cdots, n$ represent the real discrete data where h is the

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interval between successive observations of the process. We will use the same definition for the DFT, as given in [5]. The DFT is defined as

$$X(q, F_p) = \sqrt{h/M} \sum_{i=0}^{M-1} W_i x(i, q) \exp(-jF_p i) \quad (1)$$

where the symbols are identified as

$j = \sqrt{-1}$.

$F_p = 2\pi f_p h$ is the pth radian frequency component, $p = 0, 1, \cdots, M-1$, and $f_p = p/Mh$ Hz. For simplicity, we shall resume the window weights equal one, i.e., $W_i = 1$, for all i, and $h = 1$.

The power spectrum estimate is defined as

$$P(F_p) = (1/n) \sum_{q=1}^{n} X(q, F_p) X^*(q, F_p) \quad (2)$$

where the asterisk represents complex conjugate. The variance of the periodogram does not go to zero as $M \to \infty$, and therefore the periodogram is not a consistent estimate of the PSD [5]. In the PSD estimate considered here, n nonoverlapped DFT segments are averaged to ensure that each frequency component represents a consistent PSD estimate [5]. Thus (2) is an asymptotically unbiased estimate of the power spectral density [5].

The FDK estimate was discussed in terms of a detection problem in [6]. The main concern here is to determine the sensitivity of the PSD and FDK estimates to randomly occurring signals, assuming a sufficient number of DFT segments is available. The FDK represents a measure for the probability distribution over a time interval consisting of many DFT segments. Many of the arctic segments, which will be discussed in the last section, showed highly dynamic frequency components. This was due to the highly dynamic nature of ice sounds. These dynamic components were easily descerned using a spectrogram. The advantages of the DFK for estimating the statistical behavior of ice sounds over the spectrogram are that an operator is not needed and that it produces a quantitative measure of the probability distribution. This measure can also be used to distinguish stationary sinusoids and Gaussian signals from ice sounds.

It should also be pointed out that overlapped segments have also been studied to reduce the variance in the PSD estimate for another application [7], but this technique will not be treated here.

The FDK is defined separately for the real and imaginary parts of (1). Only the results for the real part will be discussed here. The imaginary part derivation is identical. In practice, both parts are computed, since each can contribute information, and they can be combined into one algorithm, if desired.

References [2] and [4] discuss an analysis using the FDK method for arctic under-ice ambient noise data. For a Gaussian distribution, kurtosis will have a value around 3 within some confidence bound determined by the number of samples used in the estimate [8]. For randomly occurring signals that produce non-Gaussian distributions, the kurtosis estimate can be less than 3 or it can have a value much greater. Several cases are examined in the paper to demonstrate the range of kurtosis values for various situations.

Techniques for optimally processing signals contaminated by under-ice ambient noise as well as other noise environments are presented in [9] and [10].

The FDK is defined by taking the expected value of the fourth-order central moment and the square of the expected value of the second-order central moment separately, and then forming the ratio. The result of this operation is

$$K(F_P) = E\{[X(q, F_p)]^4\}/\{E[(X(q, F_p))^2]\}^2. \quad (3)$$

Before proceeding further, we need to define a model for the received data. Our goal is to compare the FDK and PSD estimates under some conditions which are known to occur in underwater acoustic detection problems, but have not been explicitly evaluated in this way before. The model we employ assumes that the transmitted or radiated signal is acted upon multiplicatively by the medium which causes amplitude modulation and frequency spreading to occur. This is obviously not the most general model possible, but it is adequate to answer some important sonar design questions.

The input, $x(i, q)$, will be a zero-mean process which is composed of an additive mixture of signal and noise of the form

$$x(i, q) = N(i, q) + m(i, q)s(i, q) \quad (4)$$

where $m(i, q)$ modulates the signal and will either represent the effects of the propagation medium or reflect a physical characteristic of the transmitted or radiated signal $s(i, q)$. The components $N(i, q)$ and $s(i, q)$ are zero-mean stationary processes and $N(i, q)$, $m(i, q)$ and $s(i, q)$, will be assumed to be mutually independent from each other. For the particular choices of $m(i, q)$ and $s(i, q)$ given in the text, $x(i, q)$ will be stationary in the wide sense.

Our model for the fading received signal $m(i, q)s(i, q)$ assumes that the total effects of the amplitude fluctuations due to multipath interference or to nonstationarities of the source, receiver, or of the medium can be simply included in the multiplicative function $m(i, q)$. On the other hand, the phase fluctuations of the signal will be contained in the function $s(i, q)$ itself. This approach applies to sound propagating in the ocean [11] and to electromagnetic communication systems [12]. Later, we will generalize this model to include transient and frequency modulated signals.

The temporal statistical effects of the multiplicative term and the frequency modulation term, to be introduced later, were considered in a different setting than here, in [13], and found to cause the resultant data measured over a time interval to be non-Gaussian. We have also found this to be the case based on the FDK.

Substituting (4) into (1) and (2) and taking the expected value, we obtain the result for the PSD as follows:

$$E\{P(F_p)\} = (1/M) \sum_{i_1=0}^{M-1} \sum_{i_2=0}^{M-1} R_N(i_1 - i_2) \cdot \exp(-jF_p(i_1 - i_2)) + (1/M) \cdot \sum_{i_1=0}^{M-1} \sum_{i_2=0}^{M-1} E\{m(i_1, q)m(i_2, q)\} R_s(i_1 - i_2) \cdot \exp(-jF_p(i_1 - i_2)) \tag{5}$$

where we have defined the components

$$R_N(i_1 - i_2) = E(N(i_1, q)N(i_2, q))$$
$$R_s(i_1 - i_2) = E(s(i_1, q)s(i_2, q)).$$

When it is obvious, in succeeding equations, we shall not indicate the index for the sum symbols.

The FDK is obtained by substituting (4) into (1) and (3), and after some manipulations, the terms of this expression are given by

$$E\{(X(q, F_p))^4\} = (1/M^2)OP_4\{E[N(i_1, q)N(i_2, q)N(i_3, q)N(i_4, q)]\} + 6(1/M)OP_2\{R_N(i_1 - i_2)\}(1/M)OP_2\{E[m(i_1, q) \cdot m(i_2, q)]R_s(i_1 - i_2)\} + (1/M^2)OP_4\{E[m(i_1, q) \cdot m(i_2, q)m(i_3, q)m(i_4, q)]E[s(i_1, q)s(i_2, q) \cdot s(i_3, q)s(i_4, q)]\} \tag{6a}$$

and

$$E\{(X(q, F_p))^2\} = (1/M)OP_2\{R_N(i_1 - i_2)\} + (1/M)OP_2 \cdot \{E[m(i_1, q)m(i_2, q)]R_s(i_1 - i_2)\} \tag{6b}$$

where we have defined the operators

$$OP_4\{\ \} = \sum\sum\sum\sum\{\ \} \cos(F_p i_1)\cos(F_p i_2)\cos(F_p i_3) \cos(F_p i_4)$$
$$OP_2\{\ \} = \sum\sum\{\ \} \cos(F_p i_1)\cos(F_p i_2).$$

The FDK estimate is defined as

$$K(F_p) = (1/n) \sum_{q=1}^{n} (X(q, F_p))^4 \Big/ \left\{ (1/n) \sum_{q=1}^{n} (X(q, F_p))^2 \right\}^2. \tag{7}$$

It was proven in [8] that a kurtosis estimate as given in (7) is an asymptotically (as $n \to \infty$) unbiased estimate of the kurtosis, if the distribution is Gaussian. In practice, the true value of the FDK estimate can only be approached asymptotically. We will only be concerned with examining the analytical properties of the FDK as defined in (3).

It was proven in [6] that the variance of the FDK estimate does not depend on the discrete Fourier transform size M. But, the variance does decrease with increasing n [6]. Therefore, the real data comparison of the PSD and FDK estimates, to be presented in Section V, will be based on (2) and (7), respectively, with n sufficiently large to ensure consistent results. In addition, the variance of both (2) and (7), calculated assuming independent and identically distributed (i.i.d.) Gaussian statistics, have been shown to approach zero proportionally with $1/n$ as $n \to \infty$; for example, see [5] and [8], respectively. For the theoretical results of the paper the FDK and PSD are obtained from (3) and (5), respectively.

III. NOISE CONSIDERATIONS

Let $s(i, q) = 0$ and assume that the noise samples are identically distributed and statistically independent. Then, as M approaches ∞, $K(F_p)$ approaches 3 for each p and for any probability distribution function of $N(i, q)$. The proof of this result can be obtained by defining $R_N(i_1 - i_2) = R_N \delta_{i_1 i_2}$, where $\delta_{i_1 i_2}$ is the Kronecker delta function, expanding the first term of (6a) and letting M become large. This result suggests that as $M \to \infty$ for independent samples at the input, the output probability distribution function of the DFT will approach a Gaussian probability distribution independent of the input probability distribution as measured by the FDK.

Let $s(i, 1) = 0$, and assume that the noise samples are temporally dependent but Gaussian distributed. Then, based on (3), the FDK equals 3. The proof of this result is straightforward. However, the result suggests that the modulation parameter of the signal is important in obtaining non-Gaussian output statistics, i.e., $K(F_p) \neq 3$.

Hereafter, we shall assume that the input noise is independent and identically distributed and Gaussian, with PSD R_N. The real part of the noise PSD is, then, $R_N/2$.

IV. SLOWLY FADING SIGNAL

Signal level fluctuations have been measured in underwater acoustic propagation studies [14]-[18]. In some cases, the received signal level has been measured to fluctuate over a period of seconds as much as 50 dB [18]. For different experimental trials, the actual amplitude probability distribution function estimate for a purely sinusoidal transmitted signal, however, was not exactly repeated, although it was close to Rayleigh in some trails and Gaussian in others [15], [16], [18]. Reference [19] presents data that shows that the amplitude fluctuations can follow a Rayleigh or log-normal probability distribution depending upon the propagation path. In addition, there is a much slower fluctuation, on the order of 10 min or more, superimposed on those "faster" fluctuations [14], [17]. A summary of detection problems arising from signals propagating in the ocean can be found in [20]. Also, the effects of fluctuation on the sonar equation was discussed in [21]. Many of the references cited above used kurtosis and other statistical measures to describe fluctuation.

For the rest of the paper we will only consider cases in which the fluctuations are slow with respect to the data segment Mh, but may be considered changing according to a probability law over the total detection interval nMh. These cases are supported by the measured fluctuations reported in the references cited above, and from practical limitations of implementing large DFT's.

To summarize, if $m(i, q)$ is slowly changing with respect to Mh, or equivalent to the length of the DFT, but changing according to a probability law over the detection interval, then we may approximate the fluctuation to occur after each segment as $m(i, q) = m(q)$ and, therefore, we can rewrite (3) as follows:

$$K(F_p) = (3 + 6A_1 + FA_2)/(1 + A_1)^2 \tag{8}$$

where we have defined the parameters

$$A_1 = E(m^2)(1/M)OP_2\{R_2(i_1 - i_2)\}/R_N/2$$

$$A_2 = (E(m^2))^2(1/M^2)OP_4\{E[s(i_1, q)s(i_2, q) \cdot s(i_3, q)s(i_4, q)]\}/(R_N/2)^2$$

$$F = E(m^4)/(E(m^2))^2.$$

Also, $m(q)$ will be assumed independent, i.e.,

$$E[m(q_1)m(q_2)] = \begin{cases} E[m^2(q_1)], & q_1 = q_2 \\ E[m(q_1)]E[m(q_2)], & q_1 \neq q_2. \end{cases}$$

A. Sinusoidal Signal

Let the signal be given by $s(i, q) = \cos(F_k i + \phi)$, where we have neglected the index q for convenience, and ϕ represents a slowly changing random phase distributed uniformly between $0 - 2\pi$.

After substituting the sinusoidal signal into (8), we find, after some simplification, that

$$K(F_p) = \{3 + 3A_1 + 3/8FA_1{}^2\}/\{1 + 1/2A_1\}^2 \tag{9}$$

where

$$1/2 \cos(F_k(i_1 - i_2))$$

replaces

$$R_s(i_1 - i_2)$$

in (8). If M is sufficiently large, then

$$(1/M)OP_2\{\cos(F_k(i_1 - i_2))\}$$

approaches $M/2$, if F_p equals F_k, and is zero otherwise.

From (5), it can be shown that

$$E[P(F_p)] = R_N(1 + A_1).$$

The Appendix provides a proof showing that the parameter A_1 for the PSD and the FDK are equal. The parameter A_1 has units of a signal-to-noise ratio (SNR) as defined in (8).

If the signal-to-noise ratio A_1 approaches ∞, then $K(F_p)$ approaches $1.5F$. This result could possibly be used to advantage to determine the propagation conditions. But, additional data are required to verify this result. For example, suppose the PSD shows a large frequency component but the FDK is 1.5. This would indicate that F equals 1 and no fading or fluctuation exists in the propagation path of the signal. In sonar, this result could be indicative of a short-range signal propagating without multipath [22]. On the other hand, if $K(F_p)$ is, say, greater than 3, and at the same time the PSD indicates a large frequency component, then this would suggest that F is greater than 1 and fading or multipath propagation existed. For a Rayleigh fading environment, F equals 2, and for a log-normal environment, F can be very large depending on the variance of the distribution. However, the measured data currently available [19] suggest that F is between 2 and 3 depending upon frequency. We included a larger spread in F, which is theoretically justified, to show its effect on the FDK. Nevertheless, there may be cases of practical interest where F is larger than the current data suggest. For example, the effect on F of receiver and source motion and of high sea states needs to be experimentally measured.

Table I lists various values of F for some specific distribution functions which have been associated with fading and fluctuating signals in underwater acoustics, communication, and radar applications.

B. Narrow-Band Gaussian Signal

In many applications, the radiated or transmitted signal is not a pure sinusoid but rather, a periodic random signal. For example, radiated ship noise is composed of random narrow-band noise [22], [23]. We shall now examine the FDK for a narrow-band Gaussian signal. The results of the following discussion are limited to the multiplicative model given in (4).

Let the signal be zero-mean Gaussian so that the following relationship can be utilized in (3)

$$\begin{aligned} E[s(i_1, q)s(i_2, q)s(i_3, q)s(i_4, q)] &= E[s(i_1, q)s(i_2, q)]E[s(i_3, q)s(i_4, q)] \\ &+ E[s(i_1, q)s(i_3, q)]E[s(i_2, q)s(i_4, q)] \\ &+ E[s(i_1, q)s(i_4, q)]E[s(i_2, q)s(i_3, q)]. \end{aligned} \tag{10}$$

After substituting (10) into (3) and employing the slow fading condition as well as assuming that the noise i.i.d. and Gaussian, as we will throughout, we obtain for the FDK the expression

$$K(F_p) = 3(1 + 2A_1 + FA_1{}^2)/(1 + A_1)^2 \tag{11}$$

where A_1, as defined in (8), represents a signal-to-noise ratio (SNR).

For F equal to 1, $K(F_p)$ equals 3. However, for all other values of F greater than 1, the FDK is greater than 3.

Fig. 1 represents a plot of the FDK as a function of F and SNR. The estimated signal spectrum at the frequency F_p will be denoted by $\hat{S}_s(F_p)$ in Figs. 1 and 2. From Fig. 1, the FDK may have significantly high values even if the SNR which

TABLE I
AMPLITUDE PROBABILITY DENSITIES AND THEIR CORRESPONDING F VALUES

DENSITY NAME	F	DENSITY FUNCTION
HALF-NORMAL	1.5	$f(x)=(2/\pi)^{1/2}\sigma^{-1}\exp(-(1/2)\sigma^{-2}x^2)$ $x \geq 0$
UNIFORM	1.8	$f(x)=1/a$ $0 \leq x \leq a$
RAYLEIGH	2.0	$f(x)=(x/\sigma^2)\exp(-(1/2)\sigma^{-2}x^2)$ $x \geq 0$
GAUSSIAN	3.0	$f(x)=(2\pi)^{-1/2}\sigma^{-1}\exp(-(1/2)\sigma^{-2}x^2)$ $-\infty < x < \infty$
GAMMA	$\Gamma(\eta)\frac{(\eta+3)!}{((\eta+1)!)^2}$ $\eta \to 1$ $F \to 6$ $\eta \to \infty$ $F \to 1$	$f(x)=\frac{\lambda^\eta}{\Gamma(\eta)}x^{\eta-1}\exp(-\lambda x)$ $x \geq 0$, $\lambda > 0$
LOG-NORMAL	$\exp(4\sigma^2)$ $\sigma^2=.25$, $F=2.72$ $\sigma^2=.5$, $F=7.4$ $\sigma^2=.75$, $F=20$ $\sigma^2=1.0$, $F=55$	$f(x)=(\sigma x\sqrt{2\pi})^{-1}\exp(-(1/2)\sigma^{-2}(\ln x-u)^2)$ $x \geq 0$

incorporates the fluctuation is small. This means that for fluctuating signals, the PSD estimate, since it is averaged over many DFT's, may not be the best method for detection. The FDK, on the other hand, is sensitive to the probability distribution of the signal and, consequently, may lead to a solution of this problem.

Therefore, for long integration times, this theory suggests the possibility of detecting randomly occurring signals at lower SNR using the FDK instead of the PSD method. The data presently available to support this theory are the arctic under-ice ambient-noise data [2], [4] which show that in some cases the FDK can detect randomly occurring signals whereas the PSD method cannot. As we mentioned above, this result is not surprising for the PSD, but it is unexpected for the FDK. The randomly occurring signals in the under-ice data were due to ice movement which suggests a different statistical model for $m(i,q)$. Namely, ice dynamics produce frequency modulated signals. Therefore, in order to include the under-ice data in the FDK formula, we will assume that the modulating function can be written as $m(q) = am_1(q)$, where a is a random variable, for a particular frequency, which takes on two values, $a = 0$, and $a = 1$, with the probabilities $P(a = 0) = 1 - L$ and $P(a = 1) = L$. Thus the parameter a modulates the signal in the frequency domain by turning it on and off. This produces a mixture probability density function for the signal in the frequency domain for a particular frequency. The other component $m_1(i, q)$ will represent the propagation conditions as discussed above. Therefore, taking the random variable a into account, we obtain the relationship for the FDK as

$$K(F_p) = 3(1 + L(2A_1 + FA_1^2))/(1 + LA_1)^2 \quad (12)$$

where A_1 is defined above. If L equals 1, then (12) reduces to the results given in Fig. 1. The parameters L and F are actually functions of frequency. However, we have not explicitly shown this dependence for notational simplicity.

Fig. 2 represents a plot of specific values for $K(F_p)$ under different combinations of parameters L, A_1 and F. In order to emphasize the influence of L on the total SNR we have also defined the effective SNR, as measured by the PSD, over the total data length and is equal to LA_1. For moderate effective SNR (LA_1), when L is quite small the momentary SNR A_1 may be quite high. This high momentary SNR has a strong influence on high-order statistical moments (e.g., kurtosis) and consequently on the usefulness of the detection method presented. These results suggests that for signals that occur randomly, as would a transient or a frequency modulated signal over long integration times, the FDK estimate may be a better detection statistic than the PSD estimate method. If the medium is also a fading environment, which occurs often in underwater acoustics, then the FDK is significantly enhanced as shown in the figure. These analytical results for the FDK agree with the experimentally measured values of the real data which are presented later.

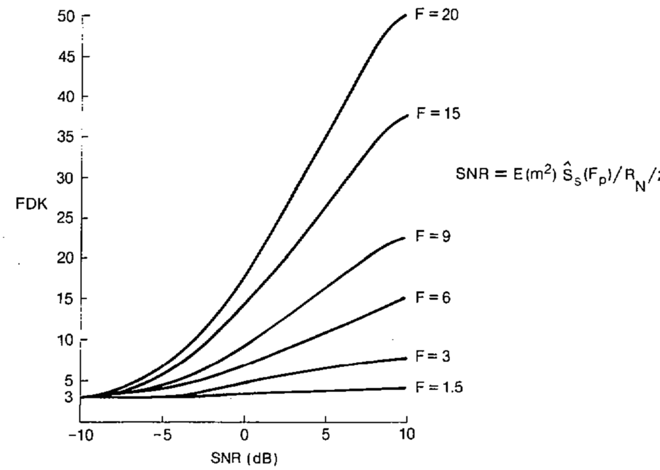

Fig. 1. FDK versus SNR for a narrow-band Gaussian signal in a fluctuating environment indicated by the values of F.

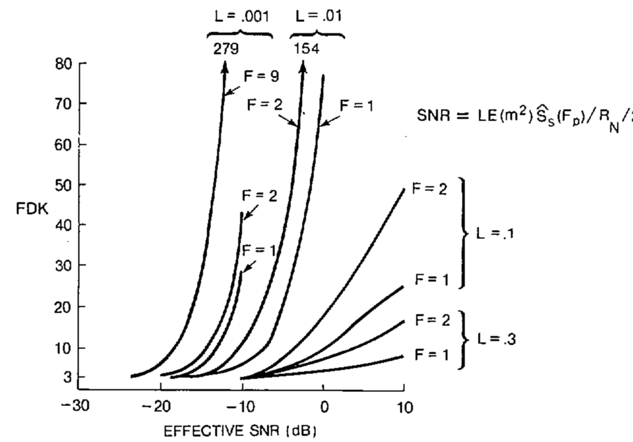

Fig. 2. FDK versus effective SNR for a randomly occurring narrow-band Gaussian signal in a fluctuating environment indicated by the values of L and F, respectively.

The detection performance of the FDK and PSD was considered in [6] for long integration times. The technique employed was based on Cramer's proof [24] which shows that both statistics converge to a Gaussian probability distribution for sufficiently large n. To emphasize this, we have called the result in [6] the asymptotic probability of detection (APD). The results show that the APD for the PSD is generally larger than the corresponding APD for the FDK evaluated over the parameters L and F. However, there is a region for small L which expands with increasing F that gives a larger APD for the FDK.

We can summarize the results in [6] by saying that for a randomly occurring signal characterized by the parameter L, the asymptotic probability of detection value based on the FDK estimate was larger than the corresponding asymptotic probability of detection value based on the PSD estimate over a region where L was small. Therefore, only for small L was the FDK a better detection statistic based on the APD.

Another way of comparing the relative merits of the FDK and PSD is based on the deflection criterion.

According to Lawson and Uhlenbeck [25], a detectability criterion can be defined in terms of a function of the deflection. The detectability criterion is defined as a ratio of the difference between the averaged value of a function when signal and noise are present and the averaged value of the function when noise only is present, to the standard deviation of the function under noise only conditions. The function is either the FDK or the PSD. Therefore, the detectability criteria for the FDK and PSD estimates can be stated as

$$D(K) = 3L(F - L)A_1{}^2\sqrt{n}/[\sqrt{24}(1 + LA_1)^2]$$

and

$$D(PSD) = LA_1\sqrt{n}/\sqrt{2},$$

respectively. Where consistent with this subsection, the parameter K is greater than or equal to 3 and n, the number of DFT's used in each estimate, is assumed sufficiently large. The mean and variance of the FDK were obtained from [8]. In both criteria, the underlying noise is composed of i.i.d. Gaussian samples.

As L approaches zero, the detectability criteria $D(K)$ and $D(PSD)$ also approach zero. However, if L equals 1 and F equals 1 then $D(K)$ equals 0, since both noise and signal are Gaussian.

The ratio of the two criteria defines a relative efficiency for the two methods. Therefore, we define the relative efficiency ($\mathcal{E}$) of the FDK and PSD estimates for $L \neq 0$ as

$$\mathcal{E}[K, PSD] = D(K)/D(PSD) = 3(F - L)A_1/[\sqrt{12}(1 + LA_1)^2].$$

In Fig. 3, the relative efficiency is plotted as a function of L for a few values of A_1 and F. This figure shows, that only for small values of L, will the relative efficiency be greater than one. Therefore, based on the deflection criterion, the FDK will be a better statistic than the PSD for detecting randomly occurring signals only over a limited range of L. These results agree with the results based on the APD obtained in [6]. But even in those cases where the PSD is a better detection statistic the FDK can still be useful. For example, the physical evidence to be discussed in the next section, suggests the existence of measurable randomly occurring signals, and therefore, the FDK estimate may provide additional information for detecting and classifying those signals.

V. REAL DATA EXAMPLES

It has been observed that Arctic under-ice noise is at times composed of highly dynamic narrow-band components [26]. The data were collected during the 1980 Arctic Ocean experiment [27].

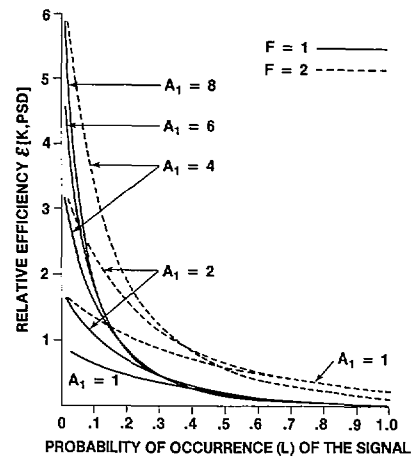

Fig. 3. Relative efficiency of FDK relative to PSD versus probability of occurrence (L) of the signal.

The specific data that were analyzed were recorded on April 23-24, 1980 from a pack ice camp in the Arctic Ocean, located at 86°N latitude, 25°W longitude. At this location, the bottom depth was approximately 4000 m. The measurement system consisted of a broad-band, omnidirectional, hydrophone suspended to a depth of 91 m from a sonobuoy located in a lead. Under the influence of arctic currents, the pack ice was slowly moving. This movement caused rifting and cracking of ice, which occurred at times throughout the experiments and represented a structured acoustic noise source. Both impulsive and burst noise were identified in the data and were probably created by tensile cracks and rubbing ice masses.

In order to verify the results of this paper, we have included real data examples, shown in Figs. 4 and 5, which represent signals generated from ice movement. A complete decription of these data is not needed here since it can be found in [2], [4]. Essentially, ice movement produces transient and frequency modulated signals. In the analysis of the real data the mean, variance, skew, and kurtosis were estimated for the real and imaginary parts of each frequency component. The kurtosis estimate appeared to be more significant and a theoretical analysis was, therefore, undertaken. The following figures pertain only to the kurtosis estimates.

The data were processed utilizing a 1024-point fast Fourier transform (FFT). In Figs. 4 and 5, the top curve represents the PSD estimate in decibels versus frequency. The data in Fig. 4 were first filtered through a low-pass filter with a bandwidth of 2500 Hz and then sampled at a 10-kHz rate, giving a resolution of 10 Hz. The PSD estimate was obtained by appropriately averaging the output of 1000 consecutive FFT's which gave an overall time interval of 1.7 min.

The bottom curve of Figs. 4 and 5 is the corresponding FDK estimate for the real part of the FFT output. This method of displaying the FDK estimate was introduced in [2]. Like the PSD estimate, the FDK estimate for Fig. 4 was obtained by appropriately averaging over the 1000 consecutive FFT outputs. For many of the frequency locations, over a wide bandwidth in Fig. 4, the FDK estimates deviate significantly from the Gaussian assumption and thereby indicate the location of non-Gaussian signals. This conclusion cannot be obtained from the PSD estimate alone. In

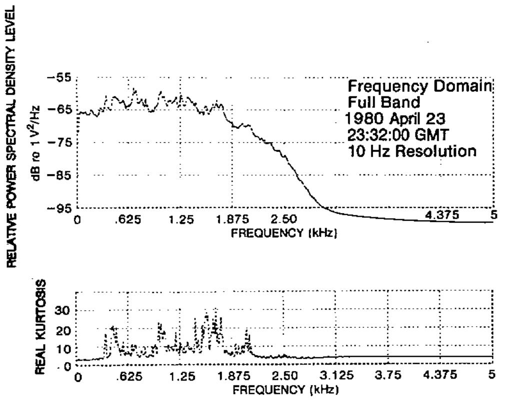

Fig. 4. Real arctic data at 10-Hz resolution. Top curve: Power spectrum density estimate. Bottom curve: Corresponding frequency domain kurtosis estimate.

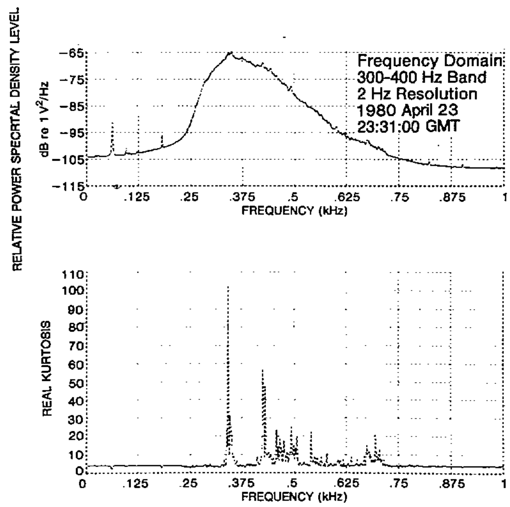

Fig. 5. Real arctic data at 2-Hz resolution. Top curve: Power spectrum density estimate. Bottom curve: Corresponding frequency domain kurtosis estimate.

fact, it would be difficult to detect the ice induced signal given the PSD estimate in Fig. 4, since one would also have to know the PSD noise level. Therefore, the advantage of the FDK estimate is that its actual value is significant, since we are looking for deviations from the Gaussian value of 3, whereas the PSD estimate represents a relative value. It will also be obvious from the figures that the FDK estimate is independent of the slope of the spectrum.

Fig. 5 represents a different data set than Fig. 4. The data were first passed through a 100-Hz bandpass filter centered at 350 Hz and then sampled at a 2-kHz rate which gave a resolution of 2 Hz. The estimates were obtained by averaging 750 consecutive FFT outputs giving an overall time interval of 6.25 min. In the top curve, on the left in Fig. 5, is a 60-Hz tonal due to an electrical ground loop. The corresponding FDK estimate, indicated in the bottom curve by a "dip", has a value of 1.8. The next "dips" at higher frequencies are due to harmonics of the 60-Hz ground loop. These results were predicted by (9) in the text since, as the SNR approaches infinity, $K(F_p)$ approaches $1.5F$. For a ground loop, F equal 1 because the signal is not being propagated through the medium. However, due to system noise, the FDK estimate is prevented from reaching 1.5. The non-Gaussian nature of the output in this case, is due to the phase fluctuation of the sinusoid after each FFT. This was probably caused by the generator not producing exactly a 60-Hz voltage source. In addition, the generator frequency had a tendency to drift over time.

Within the passband of the filter, which is clearly seen in the top curve in Fig 5, the statistics of the signal at many frequency locations deviate significantly from Gaussian, indicating the presence of non-Gaussian signals. We also notice some small indication of a signal present in the corresponding PSD estimate at some of the frequency locations. However, the PSD estimate does not contain information on the non-Gaussian nature of the signals and, therefore, in order to detect these signals, the noise-only PSD estimate must be known. These results again show the advantage of the FDK estimate over the PSD in detecting non-Gaussian signals in some cases of practical importance.

These results have also been verified by simulations. Reference [6] compared the results of a limited simulation with theoretical predictions. But probably more significant than simulations, were the corroboration with other real data. It has been found that the rotors of helicopters generate frequency modulated components, due to changing Doppler, that produced high FDK values [28]. Often these frequencies were also measured by the PSD. So, the frequencies producing the high FDK values could be identified by the PSD estimate. These and other real data measurements tend to support the theoretical results of this paper.

In closing, we shall make one additional observation concerning the bandwidth of the signal. If $s(i, q)$ is a wide-band fluctuating signal, (12) would still be valid. The FDK estimate, however, would be measured over the bandwidth of the signal. Therefore, non-Gaussian broad-band signals would still register kurtosis estimates, but across their entire band. This may be an advantage over the PSD method for detecting non-Gaussian signals since the actual FDK estimate is significant and not its relative value, as is true for the PSD method when the noise level is not known.

SUMMARY

We have considered using the frequency domain kurtosis estimate to obtain additional information about the frequency components of a received randomly occurring signal. The FDK measure was defined in the frequency domain for the real and imaginary parts of each frequency component as the ratio of the averaged value of the fourth-order central moment, to the square of the second-order central moment of a DFT output. Both sinusoidal and narrow-band Gaussian signals were investigated where each were modeled as propagating through a fading or multipath environment. In addition, transient and frequency modulated signals were modeled by introducing a mixing parameter L into the FDK. A theoretical derivation shows that, under some practical conditions which are known to exist in underwater acoustics as well as other environments, the FDK estimate gives an additional measure for randomly occurring signals. This measure can, under certain conditions outlined above, be more significant than the PSD estimate. In other cases, the FDK estimate reflects the propagation condition and, therefore, enhances the information gained by the PSD estimate.

APPENDIX

The parameter A_1 of the PSD and the FDK estimates can be shown to be equal by defining the relationship

$$R_s(i_1 - i_2) = \int_{-1/2}^{1/2} S_s(f) \exp(j\omega(i_1 - i_2))\, df \qquad (13)$$

where $S_s(f)$ is the two-sided spectrum of the real symmetric correlation function $R_s(i_1 - i_2)$ and ω is the radian frequency. Substituting (13) into the expression

$$(1/M)OP_2\{R_s(i_1 - i_2)\}$$

which represents the estimate of $S_s(F_p)$ for the real part of the PSD, we obtain for $F_p \neq 0$

$$S_s(F_p) = 1/2 \int_{-1/2}^{1/2} S_s(f)(W_M(\omega - F_p))^2\, df \qquad (14)$$

where we have neglected some terms which converge to zero as $M \to \infty$ and

$$(W_M(x))^2 = (1/M)\{(\sin(Mx/2)/\sin(x/2))\}^2$$

is called the Bartlett spectrum window [29], [30]. The PSD estimate of $S_s(F_p)$ is equal to twice the value obtained in (14). However, since the real part of the noise PSD estimate is also reduced by 2, the parameter A_1 is the same for both the PSD and the FDK.

If the frequency component F_p equals zero, then the parameter A_1 of the FDK and PSD are also equal. This can be easily verified by comparing (5) and (6b).

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