The diversity-multiplexing tradeoff of the symmetric MIMO half-duplex relay channel

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Abstract—The diversity-multiplexing tradeoff (DMT) is obtained for the symmetric MIMO half-duplex (HD) relay channel where the source and the destination have n antennas each and the relay node has m antennas (hereafter, such a channel is referred to as an (n, m) -relay channel). The characterization of the DMT requires the joint eigenvalue distribution of three specially correlated central Wishart random matrices, which is derived using a related result in [1], [2]. The explicit characterization of the DMT, besides providing the theoretical benchmark for evaluating performance of practical cooperative protocols on this channel, reveals several interesting facts such as: a) the HD operation of the relay fundamentally limits relay channel performance in the sense that the DMT of the full-duplex (FD) relay channel can be strictly greater than that of the HD relay channel; b) an extra antenna at the relay node on a HD relay channel does not always improve the achievable diversity order, unlike that on an FD relay channel. While the fundamental DMTs of the two channels (FD and HD) coincide for a range of low multiplexing gains (MG), this range becomes smaller and the gap between the two (HD and FD) at higher MGs become larger with increasing number of antennas at the relay node, thus potentially justifying the extra cost of an FD relay. It is also proved that the DMTs of the HD and FD $(n, 1)$ -relay channels are the same. Finally, it is shown that the DMT of the $(1, m)$ relay channel can be achieved over different ranges of multiplexing gains by the dynamic decode-and-forward (DDF) and the quantize-and-forward (QF) protocols.

I. INTRODUCTION

Cooperation or relaying is an efficient technique to achieve higher transmission rates and/or better quality-of-service or reliability in wireless communication systems [3], [4], [5]. In the simplest model of a relay network, there is a single relay node that assists the source node to convey its message to the destination. The relay node has two modes of operation: 1) the listen mode, in which it receives the signal transmitted by the source; and 2) the transmit mode, in which it transmits some version of the received signal to the destination. If the relay can operate simultaneously in both these modes it is called a full-duplex (FD) relay and the corresponding channel, a FD relay channel. If the relay can operate in only one mode at any given time it is called a half-duplex (HD) relay and the corresponding channel is referred to as a HD relay channel.

FD relaying is hard to realize in practice due to the large difference between the power level of the transmitted and received signals. An HD relay on the other hand is practically feasible. It is the fundamental limit on the performance

achievable over the HD relay channel that is of interest in this paper.

Cooperative protocols for the HD relay channel can be categorized into two classes. If a protocol uses the channel state information at the relay to decide the time(s) at which to switch between the listen and transmit mode it is called a *dynamic* protocol (e.g., see dynamic-decode and forward (DDF) protocol in [6] or dynamic compress-and-forward (DCF) protocol [5]). If the switching time of the relay in a protocol is not a function of the channel states, it is called a *static* protocol (e.g., the static compress-and-forward (SCF) protocol of [2]). The highest achievable DMT over all static protocols is referred to the DMT of the static HD relay channel. The highest achievable DMT over all protocols (including dynamic protocols) is the fundamental DMT of the HD relay channel and it is this DMT that we obtain in this paper.

Note that transmit-receive durations of the relay node can be modulated to convey additional information to the receiver [7]. A cooperative protocol that utilizes this feature is called a *random* protocol. Otherwise, it is called a *fixed* protocol. However, since a random protocol can increase the capacity by at most one bit [5], there is no improvement in the DMT sense to be had by the use of random protocols over fixed protocols [8].

Thus, the highest achievable DMT with dynamic protocols is the fundamental DMT of the HD relay channel. Since static protocols are a subset of dynamic protocols, the DMT of the static HD relay channel found in [2] can be seen to be in general a lower bound on the DMT of the HD relay channel. On the other hand, the DMT of the dynamic half-duplex relay is described in [2] as being “more complex” and is left unsolved. It is precisely this problem that we solve in this paper.

The work in [5] proves that the DCF protocol is DMT optimal on a dynamic HD relay channel but it leaves open the problem of explicit characterizing that DMT. Finding the explicit DMT is important for example in comparing the performance of different cooperative protocols with the theoretical limit. For instance, as we show later in this paper, an explicit characterization of the fundamental DMT of a $(1, m)$ relay channel allows us to show that the DDF protocol and the QF [9] protocol together can achieve it without global CSIT at the relay node (whereas the DCF protocol requires this information). As noted earlier in [2], an explicit characterization of the DMT of the DCF protocol is difficult. This is due to the fact that unlike the static CF protocol, the switching time of the relay in the DCF protocol is dependent on the global CSIT available at the relay and should be chosen

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to maximize the mutual information from the source to the destination. After solving this optimization problem we find that evaluating the DMT requires the joint distribution of three mutually correlated Wishart matrices, which can be derived using a recent result by the authors in [1]. Here we consider the symmetric MIMO HD relay channel wherein the number of antennas at the source and the destination are equal. The more general asymmetric HD relay channel in which the numbers of antennas at the transmitter, receiver and relay are arbitrary will be reported in [10].

The rest of the paper is organized as follows. In Section II the system model is described and the asymptotic joint eigenvalue distribution of two specially correlated matrices is stated. In Section III, we derive the outage probability at the destination using an exact expression for the instantaneous mutual information at the destination. The negative SNR exponent of this probability yields the optimal diversity order. In Section IV, we provide a closed-form expression for the DMT of the (n, m) HD relay channel and discuss the performance of some existing protocols relative to the fundamental DMT of the channel. Section VI contains conclusions.

Notations 1: The symbols $(x)^+$, $x \wedge y$, $|x|$, $\det(X)$ and $(X)^\dagger$ represent the non-negative part of x , the minimum of x and y , the size of the set x , the determinant and the conjugate transpose of the matrix X , respectively. Moreover, $\mathcal{CN}(0, 1)$ represents the distribution of a complex Gaussian random variable with zero mean and unit variance. By $\text{diag}(\cdot)$, we denote a square diagonal matrix with the elements in its argument on the diagonal. I_n represents an $n \times n$ identity matrix, $\mathbb{C}$ the complex number field and $\mathbb{C}^{n \times m}$ the set of all $n \times m$ matrices with complex entries. $\Pr(\mathcal{E})$ represents the probability of the event $\mathcal{E}$. All the logarithms in this paper are to the base 2. Finally, any two functions $f(\rho)$ and $g(\rho)$ of ρ , where ρ is the signal to noise ratio (SNR) defined later, are said to be exponentially equal and denoted as $f(\rho) \doteq g(\rho)$ if,

$$\lim_{\rho \rightarrow \infty} \frac{\log(f(\rho))}{\log(\rho)} = \lim_{\rho \rightarrow \infty} \frac{\log(g(\rho))}{\log(\rho)}, \quad (1)$$

$\leq$ and $\geq$ signs are defined similarly.

II. SYSTEM MODEL AND PRELIMINARIES

A relay channel with a single relay as shown in Figure 1 is considered. The source and destination nodes have n antennas each whereas the relay node has m antennas. From now on we shall refer to such a relay channel as an (n, m) relay channel. $H_{SR} \in \mathbb{C}^{m \times n}$, $H_{SD} \in \mathbb{C}^{n \times n}$ and $H_{RD} \in \mathbb{C}^{n \times m}$ represent the channels between source and relay, source and destination and relay and destination, respectively. These matrices are mutually independent and the elements of these matrices are independent and identically distributed (iid) as $\mathcal{CN}(0, 1)$. The relay is assumed to operate in the half-duplex mode and uses the dynamic compress-and-forward (DCF) protocol. In this protocol the relay chooses a suitable fraction of time ($0 \leq t \leq 1$) for which it listens to the source transmission (this fraction of time will be referred to as phase one) and then compresses the received signal and forwards it to the destination during the remaining fraction of the time (i.e., in

phase two). In the DCF protocol, the relay node is assumed to know all the channel states of the network and uses this information for both selecting a suitable t and to optimally compress the received signal. However, the relay does not use this information to allocate power across different blocks of transmission. Thus, only a short term power allocation is allowed. Denoting the source transmitted signals in phases one and two as X_{S1} and X_{S2} , respectively, and the relay transmitted signal as X_R , the received signals per channel use at the relay and destination node are given as

$$Y_{D1} = H_{SD}X_{S1} + Z_{D1}, \quad Y_R = H_{SR}X_{S1} + Z_R, \quad \text{Phase one,} \\ Y_{D2} = H_{SD}X_{S2} + H_{RD}X_R + Z_{D2}, \quad \text{Phase two.}$$

Here $Z_{D1}, Z_{D2} \in \mathbb{C}^{n \times 1}$ and $Z_R \in \mathbb{C}^{m \times 1}$ represent the additive noise at the destination and relay nodes respectively. All the entries of these vectors are iid $\mathcal{CN}(0, 1)$.

Fig. 1. System model of the relay channel.

We end this section by recalling a distribution result from [1].

Theorem 1 (Theorem 1 of [1]): Let $H_1 \in \mathbb{C}^{n_1 \times k_1}$ and $H_2 \in \mathbb{C}^{n_1 \times m_1}$ be two mutually independent random matrices with i.i.d. $\mathcal{CN}(0, 1)$ entries and let $x_1 \geq x_2 \geq \dots \geq x_{(n_1 \wedge k_1)} > 0$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{(n_1 \wedge m_1)} > 0$ be the non-zero with probability 1 (w.p.1) eigenvalues of $V_1 \triangleq H_1^\dagger (I_{n_1} + \rho H_2 H_2^\dagger)^{-1} H_1$ and $V_2 \triangleq H_2 H_2^\dagger$, respectively, then the conditional asymptotic probability density function of the eigenvalues $\bar{x} \triangleq [x_1, x_2, \dots, x_{(n_1 \wedge k_1)}]$ given $\bar{\lambda} \triangleq [(\lambda_1, \lambda_2, \dots, \lambda_{(m_1 \wedge n_1)})]$ is

$$\mathbf{f}(\bar{x}|\bar{\lambda}) \doteq \prod_{j=1}^q (x_j^{2(q-j)+|n_1-k_1|} e^{-x_j}) \prod_{\substack{(u=1, v=1) \\ ((u+v) \geq (n_1+1))}}^{(p,q)} (e^{-\rho x_u \lambda_v}) \\ \prod_{i=1}^p (1 + \rho \lambda_i)^{k_1} \prod_{j=1}^q \prod_{i=1}^{(n_1-j) \wedge m_1} \left(\frac{1 - e^{-\rho x_j \lambda_i}}{\rho x_j \lambda_i} \right),$$

where $p = (m_1 \wedge n_1)$ and $q = (k_1 \wedge n_1)$.

III. OUTAGE PROBABILITY

In this section, we first derive an exact expression for the instantaneous mutual information at the destination, which the relay node tries to maximize using the channel state information it has. Defining the outage event as the event that this mutual information is less than the target data rate we see that the outage probability depends on the joint eigenvalue

distribution stated in the previous section. Using this distribution result we then establish the optimal DMT of the DCF protocol as the solution of an optimization problem. Closed form solutions for this optimization problem and hence the explicit expression for the DMT of the HD relay is provided in the following section.

The general approach to finding the fundamental DMT is to find an upper bound and then show that some particular protocol can achieve it. In [5], it was proved that the DCF protocol is DMT optimal on a dynamic HD relay channel and we restate the result here for easy reference:

Theorem 2 ([5]): For the random, dynamic state, half duplex relay channel with n antenna source, m antenna relay and k antenna destination, the DCF protocol is DMT optimal.

With this result it would be sufficient to establish the DMT of the DCF protocol. We follow the notations of [5]. Let $X_S \in \mathbb{C}^{n \times 1}$, $X_R \in \mathbb{C}^{m \times 1}$, $Y_R \in \mathbb{C}^{m \times 1}$ and $Y_D \in \mathbb{C}^{n \times 1}$ represent the signals transmitted by the source and relay and received by the relay and destination, respectively, in each channel use. Note that, we use the same symbol X_S , to denote the source signal per channel use for both phase one and phase two. The same is true for the received signal at the destination. The mutual information per channel use at the relay during phase one is denoted as $I(X_S; Y_R|p_1)$, where p_1 indicates phase one. Using similar notations it was shown in [5] that, for a given t , the instantaneous mutual information at the destination is given as

$$R_{CF} = tI(X_S; \hat{Y}_R, Y_D|p_1) + (1-t)I(X_S; Y_D|X_R, p_2) \quad (2)$$

subject to

$$tI(\hat{Y}_R; Y_R|Y_D, p_1) \leq (1-t)I(X_R; Y_D|p_2) \quad (3)$$

where $\hat{Y}_R$ is the auxiliary random vector which denotes the compressed signal at the relay and depends on Y_R and X_R . Note, constraint (3) is required to ensure that the compressed version, $\hat{Y}_R$, of the received signal, Y_R , during phase one at the relay can reach the destination error-free during phase two. We consider X_S and X_R to be mutually independent random vectors having entries i.i.d. as $\mathcal{CN}(0, P_S/n)$ and $\mathcal{CN}(0, P_R/m)$ respectively. Now assuming that the relay compresses Y_R , taking Y_D (received during phase one) as the side information, it can be easily shown that $\hat{Y}_R = Y_R + \hat{Z}_R$ (see Sec. 3 of [11]), where $\hat{Z}_R$ is a length- m random vector with i.i.d. $\mathcal{CN}(0, \hat{N}_R)$ entries and is independent from all other random variables. Note that since $I(\hat{Y}_R; Y_R|Y_D, p_1)$ is a function of $\hat{N}_R$, it also needs to satisfy equation (3). Now that we have specified the distribution of all the signals involved, the following can be easily proved

$$I(\hat{Y}_R; Y_R|Y_D, p_1) = \log \left(\frac{L_{S, RD}}{L_{S, D} \hat{N}_R^m} \right), \quad (4)$$

$$I(X_R; Y_D|p_2) = \log \left(\frac{L_{SR, D}}{L_{S, D}} \right) \quad \text{and} \quad (5)$$

$$R_{CF}(t, \hat{N}_R) = t \log \left(\frac{L_{S, RD}}{(\hat{N}_R + 1)^m} \right) + (1-t) \log(L_{S, D}), \quad (6)$$

where $R_{CF}(t, \hat{N}_R)$ represents the instantaneous mutual information at the destination (Equation (42) in [5]) and

$$L_{S, D} \triangleq \det \left(H_{SD} H_{SD}^\dagger \frac{P_S}{n} + I_n \right);$$

$$L_{SR, D} \triangleq \det \left(H_{SD} H_{SD}^\dagger \frac{P_S}{n} + H_{RD} H_{RD}^\dagger \frac{P_R}{m} + I_n \right);$$

$$L_{S, RD} \triangleq \det \left(H_{S, RD} H_{S, RD}^\dagger \frac{P_S}{n} + \begin{bmatrix} (\hat{N}_R + 1)I_m & 0_{m \times n} \\ 0_{n \times m} & I_n \end{bmatrix} \right);$$

$$L'_{S, RD} \triangleq \det \left(H_{S, RD} H_{S, RD}^\dagger \frac{P_S}{n} + I_{n+m} \right)$$

$$\text{and } H_{S, RD} \triangleq \begin{bmatrix} H_{SR} \\ H_{SD} \end{bmatrix} \text{ and } H_{SR, D} \triangleq [H_{SD} H_{RD}].$$

Remark 1: It should be emphasized at this point that by assuming that the source and relay node use independent Gaussian codebooks there is no loss of optimality since in [5], the optimality of the CF protocol was proved using exactly the same assumptions.

We note from equation (6) that the total mutual information at the destination depends both on t and $\hat{N}_R$. In the DCF protocol both these parameters should be chosen optimally, using the channel state information available at the relay. Thus the optimal outage probability at the destination node can be defined as follows

$$Pr(\mathcal{O}) = Pr \left(\sup_{(t, \hat{N}_R) \in \mathcal{R}} R_{CF}(t, \hat{N}_R) < r \log(\rho) \right), \text{ where} \\ \mathcal{R} = \left\{ (t, \hat{N}_R) : t \log \left(\frac{L_{S, RD}}{L_{S, D} \hat{N}_R^m} \right) \leq (1-t) \log \left(\frac{L_{SR, D}}{L_{S, D}} \right) \right\} \quad (7)$$

In order to find the outage probability we first derive an expression for the mutual information at the destination, which is given by the following

Lemma 1: The instantaneous mutual information at the destination is given as

$$\sup_{(t, \hat{N}_R) \in \mathcal{R}} R_{CF}(t, \hat{N}_R) \doteq \log(\rho) \left[\sum_{i=1}^n (1 - \alpha_i)^+ + \frac{\sum_{k=1}^p (1 - \delta_k)^+ + \sum_{j=1}^p (1 - \beta_j)^+}{\sum_{k=1}^p (1 - \delta_k)^+ + \sum_{j=1}^p (1 - \beta_j)^+} \right], \quad (8)$$

where α_i , $1 \leq i \leq n$, β_j , $1 \leq j \leq p$ and δ_k , $1 \leq k \leq p$ represent the negative SNR exponents of the ordered eigenvalues of $W_1 \triangleq H_{SD} H_{SD}^\dagger$, $W_2 \triangleq H_{SR} (I_n + \rho H_{SD}^\dagger H_{SD})^{-1} H_{SR}^\dagger$ and $W_3 \triangleq H_{RD}^\dagger (I_n + \rho H_{SD} H_{SD}^\dagger)^{-1} H_{RD}$ respectively and $p = (m \wedge n)$.

Let Λ_i be the diagonal matrix containing the eigenvalues of W_i for $1 \leq i \leq 3$. Now, since by definition H_{SR} and H_{RD} are unitarily invariant [12], the eigenvalues of W_2 and W_3 depends on W_1 through only its eigenvalues. Thus given the eigenvalues of W_1 , the conditional joint-distribution of the eigenvalues of W_2 and W_3 are independent (since H_{RD} and H_{SR} are mutually independent too). However, the conditional joint probability distribution functions (pdf), $f(\Lambda_2|\Lambda_1)$ and $f(\Lambda_3|\Lambda_1)$ of the eigenvalues of W_2 and W_3 are given by

Theorem 1. This give us the joint pdf of the eigenvalues of W_i for $1 \leq i \leq 3$ as follows

$$f(\Lambda_1, \Lambda_2, \Lambda_3) = f(\Lambda_3|\Lambda_1)f(\Lambda_2|\Lambda_1)f(\Lambda_1).$$

Finding the joint distribution of the negative SNR exponents of these eigenvalues is an easy algebraic step (a detailed proof can be found in [10]). Using the joint distribution of $\{\bar{\alpha}, \bar{\beta}, \bar{\delta}\}$ and defining $d_o = \lim_{\rho \rightarrow \infty} \frac{-\log(Pr(\mathcal{O}))}{\log(\rho)}$, we get the following

Theorem 3: At any multiplexing gain r , the optimal diversity order achievable by the DCF protocol is the solution of the following optimization problem:

$$d_o(r) = \min_{(\bar{\alpha}, \bar{\beta}, \bar{\delta})} \sum_{i=1}^n (2n + 2m - 2i + 1) \alpha_i + \sum_{j=1}^p (n + m - 2j + 1) (\beta_j + \delta_j) - 2mn + \sum_{\substack{i,j=1 \\ j+i \leq n}}^{n,p} \left[(1 - \alpha_i - \beta_j)^+ + (1 - \alpha_i - \delta_j)^+ \right],$$

subject to the following constraints

$$\begin{aligned} \sum_{i=1}^n (1 - \alpha_i) + \frac{\sum_{j=1}^p (1 - \beta_j) \sum_{k=1}^p (1 - \delta_k)}{\sum_{j=1}^p (1 - \beta_j) + \sum_{k=1}^p (1 - \delta_k)} &\leq r; \\ \forall 1 \leq j \leq p, \alpha_{n-j+1} + \beta_j &\geq 1; \\ \forall 1 \leq k \leq p, \alpha_{n-k+1} + \delta_k &\geq 1; \\ 0 \leq \alpha_1 \leq \dots \leq \alpha_n &\leq 1; \\ 0 \leq \beta_1 \leq \dots \leq \beta_p &\leq 1; \\ 0 \leq \delta_1 \leq \dots \leq \delta_p &\leq 1, \end{aligned}$$

where $\bar{\alpha} = [\alpha_1, \dots, \alpha_n]$, $\bar{\beta} = [\beta_1, \dots, \beta_p]$ and $\bar{\delta} = [\delta_1, \dots, \delta_p]$.

IV. FUNDAMENTAL DMT OF THE (n, m) RELAY CHANNEL

In this section, we first provide closed form solutions for the DMT of the MIMO HD relay channel for the (n, m) relay channel. We also compare the performances of some existing protocols with the optimal DMT of the HD relay channel.

Theorem 4: The highest achievable diversity order $d_o(r)$, at a multiplexing gain r , of a MIMO HD (n, m) relay channel is upper bounded by $d_u(r)$ where, $d_u(r)$ is given by the following

$$d_o(r) \leq d_u(r) = \begin{cases} \min_{\{1 \leq i \leq (3+p)\}} d_{U_i}(r), & m \leq n; \\ \min_{\{1 \leq i \leq (3+2p)\}} d_{U_i}(r), & m \geq n; \end{cases}$$

where, with $d_{(a,b)}(\hat{r})$ denoting the DMT of a MIMO point to point (PTP) channel [8] with a transmit and b receive antennas,

respectively

$$d_{U_1}(r) = d_{(n,n+m)}(r), \quad 0 \leq r \leq n;$$

$$d_{U_2}(r) = d_{(2n,2n)}(2r), \quad n - \frac{p}{2} \leq r \leq n;$$

$$d_{U_3}(r) = \sum_{l=1}^p (n + m - 2l + 1) \left(1 - \left(\frac{pr}{p-r} - l + 1 \right)^+ \right)^+ + n^2, \quad 0 \leq r \leq \frac{p}{2};$$

$$d_{U_{(3+N)}}(r) = N^2 + d_{(n-N),(n+2m-N)} \left(r - \frac{N}{2} \right), \quad \frac{N}{2} \leq r \leq \min \left\{ n - \frac{N}{2}, n - \frac{N^2}{(2p-N)} \right\};$$

$$d_{(3+p+N)}(r) = \sum_{i=1}^{(n-N)} (2n + m - N - 2i + 1) \left(1 - (a_N - i + 1)^+ \right)^+ + N^2, \quad \frac{Nn}{(N+n)} \leq r \leq n - \frac{N}{2};$$

and $a_N = \frac{n+r-\sqrt{(n-r)^2+4N(n-r)}}{2}$ and $1 \leq N \leq p$.

Remark 2: For a large number number of values of n ($1 \leq n \leq 10$) and m ($1 \leq m \leq 15$) the above upper bound was found (through numerical computation of the DMT) to be tight. We conjecture that, in Lemma 4, $d_u(r) = d_o(r)$ for all $r \in [0, n]$.

Remark 3: Note that, for $r \geq \frac{n}{2}$ the DMT of the (n, m) channel, where $m \geq \lceil \frac{3n}{2} \rceil$, does not depend on the number of antennas at the relay node. This is an interesting difference of the DMT of the HD relay channel to that of the FD relay channel for which every additional antenna at the relay improves the diversity order for all values of MG.

Lemma 2: The DMT of a half-duplex $(1, m)$ relay channel is given by the following

$$d_o(r) = \begin{cases} (m+1)(1-r), & 0 \leq r \leq \frac{1}{(m+1)}; \\ 1 + m \left(\frac{1-2r}{1-r} \right), & \frac{1}{(m+1)} \leq r \leq \frac{1}{2}; \\ 2(1-r), & \frac{1}{2} \leq r \leq 1. \end{cases} \quad (9)$$

Remark 4: Comparing this with the DMT of the DDF protocol on a $(1, m)$ relay channel [1], which is given as

$$d_{(1,m)}^{\text{DDF}}(r) \triangleq \begin{cases} (m+1)(1-r), & 0 \leq r \leq \frac{1}{(m+1)}; \\ 1 + m \left(\frac{1-2r}{1-r} \right), & \frac{1}{(m+1)} \leq r \leq \frac{1}{2}; \\ \left(\frac{1-r}{r} \right), & \frac{1}{2} \leq r \leq 1, \end{cases} \quad (10)$$

and that of a Static-CF protocol derived in [2] (see figure 2) we see that the optimal diversity orders of the DDF protocol for multiplexing gains larger than $\frac{1}{2}$ is strictly sub-optimal. The same is true for the SCF protocol for multiplexing gains less than $\frac{1}{2}$.

Lemma 3: The optimal diversity orders at any multiplexing gain r , of a half-duplex $(n, 1)$ relay channel is given by a piece-wise linear curve whose corner points at integer values of r are given as

$$d_o(r) = d_{(n+1,n)}(r) = (n-r)(n+1-r), \quad 0 \leq r \leq n. \quad (11)$$

Fig. 2. Comparison of DDF and SCF protocol with the optimal DMT on a (1,4) relay channel.

Remark 5: This curve also represents the cut-set upper bound of the corresponding full-duplex relay channel, [5]. Thus on an $(n, 1)$ relay channel, the full-duplex mode of operation of the relay node can not improve the performance any further. It was observed in [2] that, this upper bound is also attainable by the half-duplex SCF protocol. However, the result in [2] is based on several explicit examples, whereas in this paper we prove it analytically.

V. NO-CSIT AT THE RELAY NODE

The implementation of a cooperative protocol in a practical application depends on a lot of issues other than its DMT performance. Channel state information (CSI) required at different nodes is one of them. Compress-and-forward protocol requires global CSI (all the instantaneous channel matrices of the relay channel) at the relay node, which in some application might be hard or even impossible to achieve. Therefore, it is important to investigate whether there exist cooperative protocols which do not require CSIT and can achieve the fundamental DMT of the MIMO HD relay channel. In this section, we address this problem and answer the above questions partially.

Theorem 5: The fundamental DMT of the $(1, m)$ relay channel can be achieved by the DDF protocol, for multiplexing gains in the range $0 \leq r \leq \frac{1}{2}$ and the quantize-and-forward protocol, for multiplexing gains in the range $\frac{1}{2} \leq r \leq 1$.

Remark 6: It should be noted that neither of the above mentioned cooperative protocols require CSIT at the relay node or at any other node.

Proof of Theorem 5: Lemma 2 provides the optimal DMT on a $(1, m)$ relay channel. Comparing this with the achievable DMT of the DDF protocol on this channel (given in equation (10), derived in [1]), it is evident that the fundamental DMT of the $(1, m)$ relay channel can be achieved by the DDF protocol for $0 \leq r \leq \frac{1}{2}$.

In [9], it was proved that the QF protocol can achieve a diversity order of $2(1 - r)$ on the $(1, 1)$ relay channel. The $(1, m)$ relay channel can be converted into a $(1, 1)$ relay channel by allowing only (any) one antenna to receive and transmit at the relay node. Thus if for multiplexing gains in the

range $\frac{1}{2} \leq r \leq 1$, the relay node chooses an arbitrary antenna (keeping all other $(k - 1)$ antennas inactive) and operates via the quantize-and-forward protocol, a diversity order of $2(1 - r)$ can be achieved (Theorem 3.1 in [9]), which is the fundamental diversity order of the $(1, m)$ relay channel for $r \geq \frac{1}{2}$. ■

VI. CONCLUSION

We have characterized the DMT of a three node half-duplex MIMO relay channel, where the source and destination nodes have n antennas each and the relay has m antennas. It is found that the HD mode of operation of the relay node on a relay channel fundamentally limits its performance in the sense that the fundamental DMT of the HD relay channel is not the same as that of the fundamental DMT of the full-duplex (FD) relay channel except for the $(n, 1)$ case. Another distinct feature of the fundamental DMT of the HD relay channel over that of its FD counter part is that, an extra antenna at the relay node does not always increase the achievable diversity order on the channel. Finally, we have shown that the requirement of global CSIT at the relay node is not necessary on a $(1, m)$ relay channel, by showing that the fundamental DMT of this channel can be achieved by DDF and QF protocols for different multiplexing gains, neither of which utilizes any CSIT at the relay node.

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