Dynamic Leader-follower Algorithms in Mobile Multi-agent Networks

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***Abstract*—In this paper, we consider a Linear Time-Varying (LTV) model to describe the dynamics of a leader-follower algorithms with mobile agents. We first develop regularity conditions on the LTV system matrices, according to a random motion of the agents and the underlying communication protocol. We then study the convergence of all agents to the state of the leader, and show that this requires the underlying LTV system to be asymptotically stable. We introduce the notion of slices as non-overlapping partitions of the LTV system matrices, and relate the convergence of the multi-agent network to the length of these slices. Finally, we demonstrate the convergence and steady-state of a dynamic leader-follower network through simulations.**

***Index Terms*—Leader-follower, LTV systems, Asymptotic stability, Dynamic networks.**

I. Introduction

In leader-follower networks, the leader nodes play the role of an input, and their influence is propagated throughout the network. The follower nodes, on the other hand, update their states according to the information they receive from their neighbors through an iterative, distributed algorithm, [1]. The goal is to converge to the state of one leader (or a function of multiple leaders). A common approach is through weighted averaging of the states of the neighboring nodes. In this approach, each follower state is guaranteed to converge to the leader(s) state(s), if the underlying network graph is connected, details can be found in [1], [2].

Leader-follower networks have been widely used in many practical applications such as vehicle formation control, and robotic systems [3], [4]. The study of leader-follower networks can also be cast in the context of sensor localization problem, where only a few sensors in the network, referred to as anchors, are aware of their exact locations, and steer the entire network towards finding their locations [5]. A large amount of research has focused on how to control leader-follower networks, [6]–[12].

In this paper, we consider a leader-follower network where the agents move randomly in a bounded region of interest and exchange information with nearby agents in their communication radius. We assume that only one agent at each iteration, updates its state as a linear combination of the neighboring states. We consider different update scenarios based on whether or not there is a leader within the communication region of the updating sensor, and note that an agent may not always be able to find a neighbor.

We provide a Linear Time-Varying (LTV) model, which describes the leader-follower dynamics and show that the steady-state of the network is independent of the followers' initial states if the corresponding LTV system is asymptotically stable. In order to derive this result, we partition the entire chain of the LTV system matrices into non-overlapping *slices*. We define a slice as the product of consecutive system matrices and link the convergence to the length of the slices.

We now describe the rest of the paper. In Section II, we formulate the problem and describe the dynamics of a leader-follower network. In Section III, we provide the sufficient conditions for the convergence of the network to the state(s) of the leader(s). We provide simulation results in Section IV, and finally Section V concludes the paper.

II. Problem Formulation

Consider a network of N mobile agents, consisting of a set of s leaders denoted by κ, and a set of n followers denoted by Ω. We assume that all of the agents are moving arbitrarily in a bounded region of interest and have a limited communication radius, r. The agents can exchange information if they find another agent within their communication region. For the ith agent, we denote the set of neighbors (not including agent i) at time k by $\mathcal{N}_i(k)$, and define $\mathcal{D}_i(k) = \mathcal{N}_i(k) \cup \{i\}$.

A. Leader-follower dynamics

Let $\mathbf{u}(k) \in \mathbb{R}^s$ and $\mathbf{x}(k) \in \mathbb{R}^n$ be the state of the leaders and followers at time k, respectively, concatenated in vectors of appropriate lengths. We consider the following linear time-varying system to describe the leader-follower dynamics:

$$\mathbf{x}(k+1) = P_k \mathbf{x}(k) + B_k \mathbf{u}(k), \qquad k \geq 0, \tag{1}$$

where $P_k \in \mathbb{R}^{n \times n}$ is the time-varying system matrix, and $B_k \in \mathbb{R}^{n \times s}$ is the time-varying input matrix. The leaders inject information to the system, while their states are unaffected by other agents. Thus, we have

$$u_m(k+1) = u_m(k) = u_m(0), \tag{2}$$

where $u_m(k)$ is the (scalar) state of the mth leader at time k. On the other hand, the state of a follower at each time is influenced by its neighboring agents. Note that since all agents are mobile, the neighboring interactions change over time, and it is not guaranteed for an agent to find a leader among the neighboring agents at time k. It is also possible that an agents does not find any other agent within its communication radius. We summarize these state-update scenarios for agent $i \in \Omega$ in the following:

(i) When there is no neighbor, i.e. $\mathcal{N}_i(k) = \emptyset$, the agent maintains its current state, and we have

$$x_i(k+1) = x_i(k), \tag{3}$$

in which $x_i(k)$ is the (scalar) state of the ith follower at time k.

(ii) When all of the neighboring agents are followers, i.e. $\mathcal{N}_i(k) \cap \kappa = \emptyset$, the agent updates its state as a linear combination of the neighboring states, and we have

$$x_i(k+1) = \sum_{j \in \mathcal{D}_i(k)} (P_k)_{i,j} x_j(k), \tag{4}$$

where $(P_k)_{i,j}$'s are the updating coefficients assigned to the neighboring agents at time k.

(iii) When there is at least one leader among the neighbors, i.e. $\mathcal{N}_i(k) \cap \kappa \neq \emptyset$, the agent update is a linear combination of the neighboring states, and we have

$$\begin{aligned} x_i(k+1) &= \sum_{j \in \mathcal{D}_i(k) \cap \Omega} (P_k)_{i,j} x_j(k) \\ &+ \sum_{m \in \mathcal{D}_i(k) \cap \kappa} (B_k)_{i,m} u_m(0), \end{aligned} \tag{5}$$

in which $(B_k)_{i,m}$'s are the updating coefficients assigned to the neighboring leaders at time k.

B. Assumptions

We now enlist the assumptions made on the above updates:
A0: When there is no leader among the neighbors, an agent updates its state as a *linear-convex* combination of the neighboring nodes, and we have

$$\sum_{j \in \mathcal{D}_i(k) \cap \Omega} (P_k)_{i,j} = 1. \tag{6}$$

A1: The weights assigned to the neighboring nodes can not be arbitrarily close to zero:

$$0 < \beta_1 \leq (P_k)_{i,j} < 1, \qquad \forall j \in \mathcal{D}_i(k), \beta_1 \in \mathbb{R}. \tag{7}$$

Also it can be inferred from Eq. (7) that when an update occurs, a non-zero self-weight is always assigned to the agent's current state, which does not let the agent to completely forget the past information.
A2: When there is a leader among the neighbors, a certain amount of information is always contributed by the leader, which on the other hand restricts the amount of (unreliable) information received from other neighboring nodes, i.e.

$$\sum_{j \in \mathcal{D}_i(k) \cap \Omega} (P_k)_{i,j} \leq \beta_2 < 1. \tag{8}$$

The above assumptions are not uncommon in the related literature. While stochasticity and non-negativity are standard in multi-agent fusion and relevant applications, Eqs. (7) and (8) ensure that no agent may be entrusted with the role of a leader. Note that under these assumptions, the LTV system matrices, $\{P_k\}$'s, are always non-negative, stochastic, or sub-stochastic. Also, without loss of generality, we assume that only one agent can update its state at any time. Hence, a system matrix at time k is the identity matrix, except for at most one row, which can be either stochastic or sub-stochastic.

In the remaining of this paper, we consider a network containing only one leader, i.e. we assume $|\kappa| = 1$. The goal for the entire network is to converge to the state of the leader, i.e. we would like

$$\lim_{k \to \infty} \mathbf{x}(k) = \mathbf{1}_n u, \tag{9}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ collects the states of all of the followers, u is the scalar state of the (single) leader that is known and does not change over time, and $\mathbf{1}_n$ is the $n \times 1$ column vector of n 1's. It can be verified that when there are multiple leaders, the agents converge to a linear-convex combination of the leader states. In the next section, we propose a framework to study the conditions under which the leader-follower network converges to the state of a leader.

III. CONVERGENCE

Before we provide our main results, let us start this section with a brief discussion on the convergence of an infinite product of stochastic and sub-stochastic matrices to zero, which we use later to study the convergence of the leader-follower network.

A. Convergence of an infinite product of stochastic and sub-stochastic matrices

In the following, we investigate the convergence of

$$\lim_{k \to \infty} P_k P_{k-1} \dots P_0, \tag{10}$$

in which P_k is the LTV system matrix at time k, as described in Section II. Convergence of an infinite product of (sub-)stochastic matrices is an ongoing research, and often involves computation of the *Joint Spectral Radius (JSR)*, [13], of all matrices in the product, which is an NP-hard problem even in the case of only two matrices, [14].

In [15], we provide an alternative approach, which links the norm properties of subsets of system matrices to the convergence of the infinite product in Eq. (10). In particular, we divide system matrices into non-overlapping *slices*. We define a slice as the smallest product of consecutive systems matrices such that each slice has a subunit infinity norm, i.e. all row sums are strictly less than 1, and all slices cover the entire sequence of system matrices.

Slice representation is depicted in Fig. 1, where the jth slice is denoted by M_j. Each slice is initiated by a sub-stochastic matrix [15], and the length of the jth slice is defined as

$$|M_j| = m_j - m_{j-1}, \qquad m_{-1} = 0. \tag{11}$$

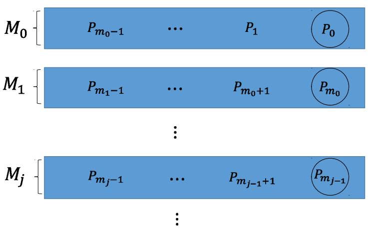

Fig. 1. Slice representation

Using the slice notation, we can study the convergence of

$$\lim_{t\to\infty} M_t M_{t-1} \dots M_0, \tag{12}$$

instead of Eq. 10, note that $k > t$. In [15], we provide the largest upper bound on the infinity norm of a slice, which is strictly less than one and approaches one as the length of a slice increases. Thus, the convergence of the product of all slices (to zero) depends on the size of the slices. The following theorem summarizes the results in [15]:

Theorem 1. *With assumption* **A0-A2**, *Eq.* (12) *converges to zero in either of the following cases:*

(i) The length of each slice is bounded, i.e.

$$|M_j| \leq N < \infty, \qquad \forall j,\ N \in \mathbb{N}; \tag{13}$$

(ii) The length of an infinite subset of slices, denoted by J_1, is bounded, i.e.

$$|M_j| \leq N_1 < \infty, \qquad \forall M_j \in J_1, \tag{14}$$
$$|M_j| < \infty, \qquad \forall M_j \notin J_1; \tag{15}$$

(iii) There exists a subset of slices, denoted by J_2, such that

$$\exists M_j \in J_2: \quad |M_j| \leq \frac{1}{\ln(\beta_1)} \ln\left(\frac{1-e^{(-\gamma_2 i^{-\gamma_1})}}{1-\beta_2}\right) + 1, \tag{16}$$

for every $i \in \mathbb{N}$, and $|M_j| < \infty, j \notin J_2$.

The complete proof is beyond the scope of this paper and can be found in [15]. We now explain the intuition behind Theorem 1:

Case (i): The upper bound on the infinity norm of a slice is a function of the slice length [15]. In this case, a bound on the length of each slice leads to an upper bound on the corresponding infinity norm, which is a positive number strictly less than one. Thus, the infinity norm of the infinite product of slices converges to zero.

Case (ii): We can partition the entire chain of slices into two sets: one set includes an infinite number of slices with bounded length and thus with a subunit infinity norm, whereas the other set includes the remaining slices with finite but unbounded length. Similar to the previous case, the product of slices in the first set converges to zero, which in turn leads to the convergence of the whole product to zero, i.e. the absolute asymptotic stability of the system.

Case (iii): When there exist an infinite subset of slices whose lengths are not bounded, but do not grow faster than the exponential growth in Eq. (16), we can still guarantee the asymptotic stability of the system. If the slices in this subset are such that there exist a slice with length following Eq. (16) for every $i \in \mathbb{N}$, the infinite product of slices converges to a zero, and thus the system is absolutely asymptotically stable. Note that the order does not necessarily matter in Eq. (16).

In what follows we use the above results to study the convergence of the *dynamic* leader-follower network, described by Eq. (3) with the LTV matrix form in Eq. (1), to the state of a leader, under the assumptions described in Section II.

B. Leader-follower convergence

In this section, we characterize the asymptotic behavior of the dynamic leader-follower algorithm. When there are multiple leaders in the network, the convergence of the followers to a linear-convex combination of the leaders may be considered, see e.g. [5], [16]. However, in what follows we consider one leader in the network, i.e. $s = 1$. We provide our main result in the following theorem:

Theorem 2. *Consider a network of n followers and one leader with the following update:*

$$\mathbf{x}(k+1) = P_k\mathbf{x}(k) + B_k u, \qquad k \geq 0, \tag{17}$$

in which u is the state of the leader. With assumption **A0-A2**, *in addition to the following*

$$\sum_j (P_k)_{i,j} + (B_k)_{i,j} = 1, \qquad \forall k, \tag{18}$$

all agents asymptotically converge to the state of the leader.

We refer the reader to [15] for the detailed proof. Instead, in what follows we briefly give the intuition behind the proof of Theorem 2. Using the slice notation, we first represent the updates in Eq. (17) as

$$\mathbf{y}(t+1) = M_t\mathbf{y}(t) + N_t u, \qquad k \geq 0, \tag{19}$$

where $\mathbf{y}(0) = \mathbf{x}(0)$. We then use the fact that the spectral radius of each slice is subunit, to show that Eq. (19) converges to a limit, $\mathbf{y}^*$. We further show that this limit is unique, and for the limiting states of the followers, $\mathbf{x}^*$, we have

$$\mathbf{x}^* = \mathbf{y}^* = M_t\mathbf{y}^* + N_t u = (I_n - M_t)^{-1} N_t u. \tag{20}$$

Finally, we show the convergence of the network to the state of the leader by proving that $(I_n - M_t)^{-1} N_t = \mathbf{1}_n$.

IV. Simulations

In this section, we present the key concepts of the theoretical results described in this paper with simple illustrations. In Fig. 2, we show the dynamic leader-follower algorithm, with $n = 4$ mobile agents, and 1 mobile leader. Each node can only explore a restricted region, and the communication radius of all agents is set to $r = 1.5$. Fig. 2 (top left) shows the random trajectories of each node for the first 30 iterations. In Fig. 2 (top right), we show an instance where none of the agents can find the leader or other agents within their communication radius, hence no update occurs. Fig. 2 (bottom left) shows the case when only one follower communicates with the leader; resulting in a sub-stochastic system matrix, whereas Fig. 2 (bottom right) depicts the information exchange between two follower nodes. This setup can be extended to any scenario with arbitrary motion models, network sizes and configurations, where the communication radii and random motion models ensure that the information reaches from the leader to each follower node. Also note that the setup can be extended to the case where many followers are never in the communication radius of the leader and the information must be propagated within the followers.

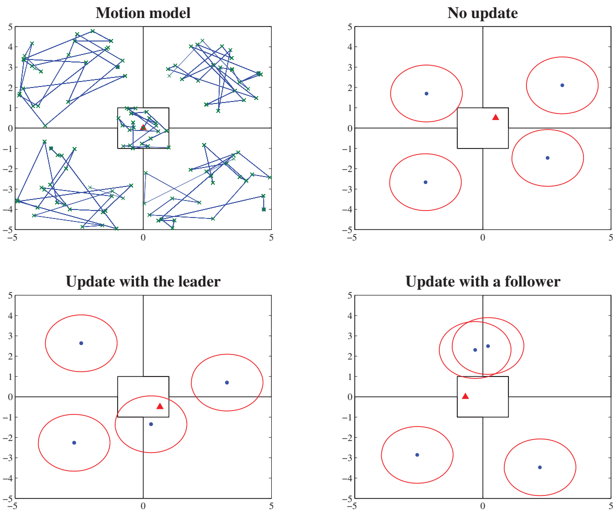

Fig. 2. Updates and motion in a leader-follower network of size 5; red triangle indicates the leader; blue circles represent the followers;

Finally, Fig. 3 shows the state trajectories at the followers converging to the state of the leader, chosen at $u = 3$. The convergence is guaranteed as long as (i) each sensor assigns a non-zero weight to its past state, and (ii) no agent is assigns an arbitrarily large weight on any neighboring node ensured by Assumptions **A0-A2**.

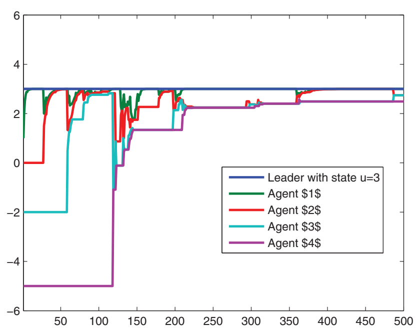

Fig. 3. Convergence of $n = 4$ agents in a dynamic leader-follower network; blue line indicates the state of the leader. 500 iterations are shown for better visibility of the transients.

V. Conclusion

In this paper, we study the leader-follower problem in mobile multi-agent networks. In particular, we study the convergence of an arbitrary number of mobile agents to the state of one leader. We model the corresponding leader-follower algorithm as an LTV system whose system matrices are random and can be either identity, stochastic, or sub-stochastic. We show that the network converges to the state of the leader, if the infinite product of system matrices converges to zero, which does not necessarily require bounded slice lengths.

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