A DYNAMIC SPECTRAL TRANSFORM AND ITS STATISTICAL CHARACTERISTICS

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ABSTRACT

A generalized spectral transform is defined by extending the kernel of the conventional sectionalized Fourier transform (SFT). The generalized transform accumulates signal energy along narrow dynamic spectral channels which may be made to conform to the instantaneous frequency dynamics of a given signal. This property may be used to achieve optimum detection of a deterministically known signal, or to estimate the spectral dynamics of an unknown signal over the temporal limits of the transform. As an initial step toward achieving the general spectral transform, the canted spectral transform (CST) is defined by using a quadratic phase kernel. The statistical properties of the CST are derived and compared with those of the conventional SFT. Statistical distributions of the peak cant variable for simulated signals in Gaussian noise provide a basis for determining the performance of the CST in practical applications.

INTRODUCTION

The sectionalized Fourier transform (SFT) accumulates signal energy along narrow spectrally invariant channels (frequency bins) over the temporal limits of the transform integration interval. This restriction on signal spectral dynamics is imposed by the kernel $\exp \{ -i 2 \pi f t \}$. The restriction is arbitrary - one may accumulate signal energy along any dynamic spectral path by choosing a suitable kernel function to use in the transform. To extend the phase kernel from linear to quadratic is a step toward the generalized concept. This form of the spectral transform is called the canted spectral transform (CST). It accumulates signal energy along spectral paths which are canted (or linearly sloping) with respect to time.

FORMULATION OF THE CST

The discrete form of the canted spectral transform (CST) to be studied in this paper is

$$F(k, v) =$$

$$\frac{1}{W} \left| \sum_{n=0}^{N-1} w_n s_{n+n_0} e^{-i 2 \pi \left[k \frac{n}{N} + \frac{v}{2} \left(\frac{n}{N} \right)^2 \right]} \right| \quad (1)$$

where

- s_j are the signal samples over the interval $n_0 \leq j < n_0 + N - 1$
- $k = 0, 1, \dots, N-1$ is the frequency index
- $v = 0, \pm 1, \pm 2, \pm 3, \dots, \pm v_{\max}$ is the cant index
- n is the time index (or sample number) relative to n_0
- n_0 is the initial time index
- $w_n = w_{N-n}$ is the window function over $0 \leq n \leq N-1$

and

$$W = \sum_{n=0}^{N-1} w_n \quad \text{is the windowed signal averaging factor}$$

The signal $s(t)$ is assumed to be uniformly sampled over time increments Δt . The transform integration time T is therefore $N \Delta t$. The bin width of the spectral channels is $1/T$, centered at $f = k/T$. The frequency slide $f = \Delta f/T = \Delta k/T^2$ of the spectral channels is v/T^2 . Thus, the cant v is the integer number k of frequency bins shifted over the transform integration time T . The factor W is equal to N for the conventional rectangular window (i.e. no window), and equals $N/2$ for the Hanning window. The spectral channel width $1/T$ is generally chosen to encompass the short-term bandwidth of the narrowband signal components of interest, and the range of the cant variable is chosen to encompass the anticipated frequency shift of the signal components over the transform integration time. As a consequence, the class of narrowband signals most suitable for analysis by the CST, is that whose member's instantaneous frequency deviates from linear over time T no greater than approximately $1/T$.

Since the CST is structured to accumulate signal energy along canted or sloping channels in the frequency-time plane, let $s(t)$ be a signal whose instantaneous frequency varies linearly with time. That is, let

$$s_{n_0+n} = e^{i \left[2\pi k_0 \frac{n}{N} + \pi v_0 \left(\frac{n}{N} \right)^2 + \Theta_0 \right]} \quad (2)$$

where

$k_0 = f_0 T$
 f_0 is the instantaneous frequency of $s(t)$ at $t_0 = n_0 \Delta t$
 $v_0 = \dot{f}_0 T^2$
 $\dot{f}_0$ is the time derivative of the instantaneous frequency at t_0
 and
 Θ_0 is a constant phase term.

In this event, the CST becomes, from (1) and (2)

$$F(k, v) = \frac{1}{W} \left| \sum_{n=0}^{N-1} w_n e^{-i\pi \left[2(k-k_0) \frac{n}{N} + (v-v_0) \left(\frac{n}{N} \right)^2 \right]} \right|$$

Letting $x = k - k_0$ and $y = v - v_0$, defines the characteristic function of the CST as

$$Z(x, y) = \frac{1}{W} \left| \sum_{n=0}^{N-1} w_n e^{-i\pi \left[2x \frac{n}{N} + y \left(\frac{n}{N} \right)^2 \right]} \right|$$

Graphs of the CST characteristic function for the rectangular window (i.e. no window) function are shown in Fig. 1. The topology is mapped over integer increments of x and y about initial offsets of either 0 or $1/2$. As a visual aid, the discrete values along the x -axis are connected by straight lines. The offsets of $1/2$ in either or both the x and y axis are intended to depict the effect of noninteger values for the signal parameters k_0 and v_0 on the resulting topology. In the upper left-hand diagram, the parameters k_0 and v_0 are both integers. Consequently, a peak value of 1 is achieved for $x = y = 0$. In the upper right-hand diagram, the parameter k_0 is depicted to be an odd multiple of $1/2$. That is, the initial frequency is depicted to start at the edge of a CST bin, while the cant v_0 remains an integer. In the lower left-hand diagram, the parameter k_0 is an integer while the cant parameter v_0 is depicted to be an odd multiple of $1/2$. In this situation, the cant can never be fully compensated by the integer variable v . The lower right-hand diagram depicts the situa-

tion where both k_0 and v_0 are odd multiples of $1/2$. In reality, the two scale offsets can fall anywhere between $\pm 1/2$. In Fig. 1, only the extreme offsets are depicted.

Fig.1 - Topology of the CST Characteristic Function $Z(x,y)$ for a rectangular window. The upper left-hand diagram illustrates the case when k_0 and v_0 are integers. In the upper right-hand diagram, v_0 is an integer but k_0 is an odd multiple of $1/2$; that is, k_0 occurs at a bin edge. The two peaks occur for $y = v - v_0 = \pm 1$. In the lower left-hand diagram, k_0 is an integer but v_0 is an odd multiple of $1/2$; that is, v_0 is midway between two cant values of v . The two central peaks are equal and occur for $y = v - v_0 = \pm 1/2$. In the lower right-hand diagram, both k_0 and v_0 are odd multiples of $1/2$.

To study the effect of a shading window function on the CST, the Hanning window is used. In this case, the window function w_n is $\sin^2(\pi n/N)$ and $N/W = 2$. The graphical plots of the characteristic function with Hanning window are shown in Fig. 2.

Fig.2 - Topology of the CST Characteristic Function $Z(x,y)$ for a Hanning window. The rationale and peak characteristics of the four diagrams are the same as described in Fig. 1. The topology for the shaded window is, however, more smooth. And, the peak of the topology decays less rapidly (along the line of minimum phase excursion $x = -y/2$) as the absolute value of y increases.

CANT SELECTIVITY

A study of the characteristic function topology for the two window functions (Figs.

1 and 2) reveals marked differences in the cant selectivity. In the case of the rectangular (no window case, since $w_n = 1$ over the integration interval), the peak of the contour is more sharply defined. The variation of the contour over the x, y plane is, however, more erratic. At the larger values of y , the spread of the peak along the x -axis is relatively broad. On the other hand, the topology of the characteristic function for the Hanning window is relatively smooth, and it displays a more gradual fall-off along the line $y = -2x$. For the purpose of estimating the cant of a linear-frequency-slide signal, the rectangular window would prove superior; particularly, in a noisy environment.

To obtain a quantitative measure of the cant selectivity, the peak value of the characteristic function over x was computed as a function of y for the two window functions. These values involve offsets within $\pm 1/2$ about integer values of x and y . The results are plotted in Fig. 3 as a function of integer values of the cant variable. The diagrams display the expected value (small circle) and the range of variation of $Z(x, y)$ due to random offsets. The value of $Z(x, y)$ at each y in Fig. 3 are uniquely dependent on the specific offsets. Thus, the indicated variations along the abscissa are not independent but are highly correlated. The results demonstrate the superiority of the rectangular window in resolving the cant of a narrowband signal in a noisy background environment.

Fig. 3 - Cant Selectivity of the CST Characteristic Function $Z(x, y)$ for the rectangular and Hanning windows. The vertical lines depict the range of variation in CST output for k_o and v_o uniformly distributed over ± 0.5 about given integers. The specific values at points along the abscissa are not independent but perfectly correlated. That is, given the values for k_o and v_o , the CST outputs are uniquely determined at the points along the abscissa and will fall within the range indicated in the figure.

CST STATISTICS FOR A SIGNAL IN NOISE

Consider the canted spectral transform of a signal comprised of a linear frequency-modulated sinusoid and random Gaussian distributed noise. Let

$$s_n = a \sin \left[\pi \frac{n}{N} \left(2k_o + v_o \frac{n}{N} \right) \right] + \eta_n \quad (3)$$

where η_n is a zero-mean Gaussian variable with variance σ^2 . Let, $X_j = F_i(k, v_j)$ be a sample CST statistic for $v = v_j$, ($j = 1, 2, \dots, n$), where k is chosen to include k_o within its bin width of $\pm 1/2$. Define $\tilde{v}$ as the specific value of v whereby $F_i(k, \tilde{v}) \leq F_i(k, v)$ for a given value of k . Define $\tilde{X} = F_i(k, \tilde{v})$; such that, $X_j \leq \tilde{X}$ for all j , and $\tilde{v}$ is a member of the set $\{v_j\}$. Of interest will be the distributions of $\tilde{v}$ as a function of the inband signal-to-noise ratio r and fractional offsets in the parameters k_o and v_o .

The desired distributions were measured experimentally using (3) in equation (1), for a rectangular window. The CST parameters were; $N=1024$, $k=200$, and $v = -20, \dots, 0, \dots, 20$. The signal parameters included $k_o = 200$ and 200.5 , $v_o = -0.5, 0, 9.5$, and 10 ; and in-band signal-to-noise ratio $r = 20$ dB, 10 dB, 6 dB, 3 dB, 0 dB, -3 dB, and -6 dB. Using Gaussian noise statistics, more than 20,000 samples of the CST statistic were obtained for each set of parameters in measuring the v distributions.

DISTRIBUTIONS OF THE PEAK CANT

As expected, the distributions of $\tilde{v}$ are independent of the parameter v_o as long as v_o is an interior cant within the range of the cants v_j , and is not in close proximity to an end value. The distributions are independent on the offsets of both k_o and v_o from their nearest integer values. Figure 4 shows the distributions of $\tilde{v}$ as a function of the inband signal-to-noise ratio r , for a rectangular window and for fractional offsets of the parameter k_o and v_o . The upper left-hand diagram shows the distributions when both k_o and v_o are integers. The upper right-hand diagram illustrates the effect of k_o being offset by $1/2$; that is, to the edge of the k bin. The lower left-hand diagram illustrates the case where v_o is midway between two integer cants. The lower right-hand diagram illustrates the case where both k_o and v_o are offset by $1/2$. From the figure, one can readily see that a cant shift of one nearly compensates for the degradation induced by the signal frequency occurring at the edge of the spectral bin. Further, the degradation resulting from the signal cant being midway between two integer cant values is partially compensated when the signal frequency is offset from the bin center. In practice, the values of k_o and v_o can be expected to be uniformly distributed over ± 0.5 from an integer value.

Distributions of $\tilde{v}$ as a function of the inband signal-to-noise ratio r and for fractional offsets of the two signal parameters were also computed for a Hanning window. The distributions in the k bin did not differ greatly from those realized with the rectangular window. The peak values are approximately the same, although the distributions about the peak values were somewhat broader. This broadening effect about the peaks can be attributed to the broader bandwidth of the CST for a Hanning window. As evidenced

Fig. 4 - Distributions of the peak cant $\hat{\nu}$ for a ramp-frequency signal in random noise, a rectangular window, and selected values of inband signal-to-noise ratio r . The upper left-hand diagram illustrates the case when the signal parameters k_0 and ν_0 are integers. In the upper right-hand diagram, ν_0 is an integer but k_0 is an odd multiple of $1/2$; that is, k_0 is at the bin edge. In the lower left-hand diagram, k_0 is in the center of the k bin but ν_0 is an odd multiple of $1/2$; that is, ν_0 is midway between two cant values. In the lower right-hand diagram, both k_0 and ν_0 are odd multiples of $1/2$. The number of contiguous cant values, in all cases, is 41.

from Figs. 1 and 2, the distributions for values of k remote from k_0 can be significantly different for the two window functions.

SUMMARY AND CONCLUSIONS

The conventional sectionalized Fourier transform (SFT) accumulates signal energy in narrow spectral channels or frequency bins which are constant over the integration time of the transform. It has been shown that by extending the phase kernel of the SFT, signal energy over narrow spectral channels that vary dynamically over the integration period can be accumulated to achieve optimum detection sensitivity for a narrowband signal whose instantaneous-frequency dynamics are known. When the spectral dynamics of the signal are unknown, the parameters of the generalized kernel may be varied to obtain estimates of the input signal frequency dynamics. The canted spectral transform (SFT), which employs a quadratic phase kernel, is an initial step toward achieving a generalized spectral transform.

The statistical properties of the CST

reveal that shaded window functions are generally inferior to the rectangular window (commonly referred to as no window) in discriminating the cant of signals whose instantaneous frequency varies linearly with time. Although the primary utility of the CST is in detecting signals with linearly varying frequency characteristics, it also has merit as a conventional spectrum analyzer by appropriately exploiting the distribution characteristics of the peak cant variable. The statistical properties of the CST for both signal and noise provide a basis for determining the performance of the transform in practical applications.

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