

BER and SER Analyses for M -ary Modulation Schemes Under Symmetric Alpha-Stable Noise

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Abstract—Symmetric alpha-stable (S α S) distribution has been widely used to model undesirable impulsive noise disturbance in many scenarios. Due to lack of probability density function (pdf) of S α S distribution (except Cauchy and Gaussian cases), the general closed-form expression of the bit error probability or symbol error probability for M -ary modulation schemes has not been derived yet, preventing the derivation of the exact coding gain from being feasible. By employing geometric power involved in zero-order statistics, we create a mapping mechanism which is consistently continuous along the entire range of characteristic values. Then we derive the accurate bit error probability and symbol error probability of M -ary modulation schemes under S α S noise. Our obtained derivations agree well with our simulations, which provide a unified framework for uncoded systems using M -ary modulation under additive white Gaussian noise (AWGN) and additive white symmetric alpha-stable noise (AWS α SN). Also, it enables the design of capacity approaching codes especially for higher-order modulation scenarios.

Index Terms— M -ary modulation, symmetric alpha-stable noise (S α S), probability density function (pdf), smart grid, non-Gaussian, geometric signal-to-noise ratio (GSNR), zero-order statistics.

I. INTRODUCTION

ALPHA-STABLE noise processes with infinite-variance that satisfy generalized central limit theorem (GCLT) can appear in many practical scenarios. The symmetric alpha-stable (S α S) process with heavy tailed distributions is shown to give a good approximation to real-world impulsive noise, such as in power lines noise, underwater acoustic noise, and atmospheric noise [1]–[4].

Because communication systems originally designed for Gaussian noise may perform very poorly in impulsive noise, several channel coding schemes using binary phase shift-keying (BPSK) modulation were proposed for performance improvement [2], [5]–[8]. However, the closed-form expression of probability density function (pdf) of S α S distribution

generally does not exist, except for the scenarios of Gaussian and Cauchy distributions with characteristic exponent $\alpha = 2$ and $\alpha = 1$, respectively. The performance of an uncoded system especially for higher-order modulation under arbitrary S α S noise ($0 < \alpha \leq 2$) remain unexplored, presenting the exact coding gain from being derived.

The use of high-order modulation can achieve higher speed communication by allowing more bits per symbol transmitted in parallel [9]. The need for high capacity transmission with M -ary ($M \geq 4$) modulation such as M -ary quadrature amplitude modulation (MQAM) and M -ary phase-shift keying (MPSK), is highly demanded in impulsive noise environments. This trend can be seen more clearly in power line communication (PLC) applications. The HomePlug Green PHY (GP) supports quadrature phase shift-keying (QPSK) modulation in smart grid applications, while the HomePlug AV provides up to 1024-QAM in its physical layer. Moreover, the HomePlug AV2 takes much higher orders of modulation 4096-QAM into consideration.

As [10] stated that all complex baseband additive white S α S noise (AWS α SN) samples are independent of each other for any α . If the carrier and bandpass sampling frequencies satisfy a certain condition, the real and imaginary components of the baseband AWS α SN sample can be independent. This feature provides an elegant way to implement M -ary modulation schemes under AWS α SN channel. The authors of [5] proposed a p -norm metric for Viterbi decoding under AWS α SN channel, and obtained an approximate bit error probability through empirical experiments for $1 < \alpha \leq 2$. The approximation is efficient when α is close to 2 (the Gaussian case), while the gap between the approximation and theoretical value increases as the value of α gets smaller. Moreover, the authors of [5] only considered the empirical approximation for BPSK signaling, but didn't derive the theoretical results for higher-order constellations.

Conventionally, the bit error probability and symbol error probability of a communication system can be measured by the ratio of signal energy per bit E_b to the noise power spectral density N_0 or signal-to-noise ratio (SNR) using second-order statistics. However, the alpha-stable process has infinite variance, for which neither the classical second-order statistics nor

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the higher-order statistics are well defined. According to the zero-order statistics framework [11], geometric signal-to-noise power ratio (GSNR) defined by geometric power is a suitable indicator for evaluating channel qualities.

To overcome the aforementioned problems, in this paper we develop the performance modeling technique/framework under the S α S noise. Motivated by the Cauchy distribution, the only case that has closed-form pdf for no-Gaussian distribution with $0 < \alpha < 2$, we create a mapping mechanism from standardized normal distribution to general S α S distribution by utilizing the closed-form pdf of Cauchy distribution. Then the analytical performance expressions for binary phase-shift keying (BPSK) modulation can be readily derived. Because the geometric power is consistently continuous along the entire range of values of α , the analytical expressions of M -ary signaling under S α S noise can be derived from its Gaussian counterpart. Using our developed mapping mechanism, one can derive the coding gains under S α S noise for systems employing different types of modulations, including, but not limited to, BPSK, QPSK, MQAM, etc.

The rest of the paper is organized as follows. Section II describes system model and the GSNR involved in zero-order statistics. Section III derives the analytical expressions for M -ary modulations. Section IV evaluates our obtained analytical bit error probability and symbol error probability through simulations. The paper concludes with Section V.

II. THE SYSTEM MODEL

We consider the n -th transmit signal sample $s_i(n)$ ($i = 1, 2, \dots, M$) using M -ary modulation and Gray mapping, corrupted by a complex AWS α SN channel. Then the corresponding received signal sample $r(n)$ is given by

$$r(n) = s_i(n) + w(n), \quad (1)$$

where $w(n)$ is the complex symmetric α -stable (S α S) noise sample, which is assumed to be independent and identically distributed (i.i.d.) for its real and imaginary components.

Generally, a random alpha-stable variable U is called following $S(\alpha, \beta, \gamma_s, \mu)$ distribution [12] can be represented as $U \sim S(\alpha, \beta, \gamma_s, \mu)$, which has characteristic function

$$\phi_U(\omega) = \begin{cases} \exp \left\{ -\gamma_s^\alpha |\omega|^\alpha \left[1 - j\beta \text{sign}(\omega) \tan \left(\frac{\pi\alpha}{2} \right) + j\mu\omega \right] \right\}, & \text{for } \alpha \neq 1; \\ \exp \left\{ -\gamma_s |\omega| \left[1 + j\beta \text{sign}(\omega) \frac{2}{\pi} \ln |\omega| \right] + j\mu\omega \right\}, & \text{for } \alpha = 1, \end{cases} \quad (2)$$

ω is the frequency-domain variable, α is usually called the characteristic exponent and determines the heaviness of the tails for the distribution such that $\alpha \in (0, 2]$, β is the skew parameter with $\beta \in [-1, +1]$, γ_s is a scale parameter of distribution that controls the spread satisfying $\gamma_s \in (0, +\infty)$, and μ is the location parameter of the distribution such that $\mu \in (-\infty, +\infty)$.

If $U \sim S(\alpha, 0, \gamma_s, 0)$ with β and μ are equal to zero, it follows symmetric alpha-stable (S α S) distribution. Thus the

characteristic function of a univariate S α S distribution can be simplified to

$$\phi_U(\omega) = \exp(-\gamma_s^\alpha |\omega|^\alpha). \quad (3)$$

When $\alpha = 2$, the S α S distribution is converted to Gaussian distribution with finite variance $\sigma^2 = 2\gamma_s^2$. When $\alpha < 2$, the S α S distribution is algebraic-tailed with infinite variance, and only Cauchy distribution has closed-form pdf. Smaller value of α indicates the heavier tail of the density function, resulting in more impulsive noise. We employ GSNR to measure the noise impulsiveness, which is defined based on the zero-order statistics framework. Even there is no close form pdf for S α S distribution, we can represent it using integral method as [13]

$$f(u; \alpha) = \frac{1}{\pi} \int_0^\infty \exp(-\gamma_s^\alpha |\omega|^\alpha) \cos(u\omega) d\omega. \quad (4)$$

Let U be a logarithmic-order variable. According to [11], the geometric power of U is defined as:

$$S_0 = S_0(U) \triangleq e^{\mathbb{E}[\log |U|]} = \gamma_s C_g^{(\frac{1}{\alpha}-1)}, \quad (5)$$

where $\mathbb{E}(\cdot)$ denotes taking expectation operation, C_g is the exponential of Eulers constant and $C_g \approx 1.78$.

The GSNR Υ_G designed involving the Gaussian case ($\alpha = 2$) can be expressed as

$$\Upsilon_G = \frac{1}{2C_g} \left(\frac{A}{S_0} \right)^2 = \frac{A^2}{2C_g^{(\frac{2}{\alpha}-1)} \gamma_s^2}, \quad (6)$$

where A is the root mean square (RMS) amplitude of the transmitted signal.

For the real or the imaginary of complex baseband S α S noise, we have

$$\frac{E_s}{N_0} = \frac{1}{2} \Upsilon_G = \frac{A^2}{4C_g^{(\frac{2}{\alpha}-1)} \gamma_s^2}, \quad (7)$$

where E_s is the signal energy per symbol.

III. GENERAL PERFORMANCE ANALYSIS FOR M -ARY SCHEMES UNDER S α S NOISE

Notice that only Cauchy distribution has closed-form pdf in algebraic-tailed distributions when $0 < \alpha < 2$. We first consider the bit error probability of the Gaussian cases ($\alpha = 2$) and the Cauchy case ($\alpha = 1$), respectively, where both of the two distributions have close-form standard pdf in alpha-stable framework. Then we indicate that the lower limit of integration in derivation of the bit error probability is closely related to the geometric power, which is consistently continuous along the entire range of values for $\alpha \in (0, 2]$. Thus we can generate a mapping mechanism from standardized normal distribution to an arbitrary S α S distribution. Subsequently, the bit error probability of BPSK signaling is derived. Based on the framework of Gaussian counterpart, we can further extend to obtain the analytical expressions of M -ary modulation system under S α S noise.

A. Performance Analysis for BPSK Modulation

A random variable Y follows general normal (or Gaussian) distribution with mean μ and variance σ^2 can be represented as $Y \sim N(\mu, \sigma^2)$. The probability density function of Gaussian distribution is given by

$$g(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right). \quad (8)$$

We can set $\mu = 0$ and $\sigma^2 = 1$ in a general normal distribution, deriving a standard normal distribution. The standardized process for a general normal distributed variable Y is implemented by changing Y to $Z = (Y - \mu)/\sigma$, i.e., $Z \sim N(0, 1)$, yielding

$$g(y)dy = g_0(z)dz, \quad (9)$$

where $g_0(z)$ is the pdf of standard normal distribution given by

$$g_0(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right). \quad (10)$$

According to [14], the analytical bit error probability of BPSK modulation under AWGN channel can be expressed as the form of Q-function

$$\mathcal{P}_b^{\text{BPSK}} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right), \quad (11)$$

where $Q(\cdot)$ is the Q-function defined as:

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} \exp\left(-\frac{z^2}{2}\right) dz = \int_x^{+\infty} g_0(z)dz. \quad (12)$$

By substituting Eq. (11) into Eq. (12), we have

$$\mathcal{P}_b^{\text{BPSK}} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = \int_{\sqrt{\frac{2E_b}{N_0}}}^{+\infty} g_0(z)dz. \quad (13)$$

According to [12], if random variable $U \sim S(\alpha, 0, \gamma_s, 0)$, then $V = U/\gamma_s$ follows the so-called standard S α S distribution, i.e., $V \sim S(\alpha, 0, 1, 0)$. The standardized process for general S α S distributed variable yields

$$f(u; \alpha)du = f_0(v; \alpha)dv, \quad (14)$$

where $f_0(v; \alpha)$ is the pdf of standard S α S distribution by setting $\gamma_s = 1$ in Eq. (4), i.e.,

$$f_0(v; \alpha) = \frac{1}{\pi} \int_0^{+\infty} \exp(-|\omega|^\alpha) \cos(v\omega) d\omega, \quad (15)$$

where the integral formula for $f_0(v; \alpha)$ can be implemented by [15] effectively. Referring to Eq. (12), we can define the tail probability function for S α S distribution as

$$Q_s(x; \alpha) \triangleq \int_x^{+\infty} f_0(v; \alpha)dv. \quad (16)$$

Notice that we have $\sigma^2 = 2\gamma_s^2$ in Gaussian case ($\alpha = 2$). By making substitution for $v = \sqrt{2}z$, the Q-function of Eq. (12)

can be expressed as

$$\begin{aligned} Q(x) &= \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz \\ &= \int_{\sqrt{2}x}^{+\infty} \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{v^2}{4}\right) dv \\ &= \int_{\sqrt{2}x}^{+\infty} f_0(v; \alpha = 2)dv. \end{aligned} \quad (17)$$

In Gaussian case, the relation between the Q-function and the tail probability function is shown below

$$Q(x) = Q_s(\sqrt{2}x; \alpha = 2). \quad (18)$$

Now, we consider the bit error probability of BPSK modulation in the general Cauchy distribution ($\alpha = 1$) which is the only case that has closed-form pdf in algebraic-tailed distributions. The probability density function of Cauchy distribution can be represented as

$$f(u; \alpha = 1) = \frac{\gamma_s}{\pi} \frac{1}{\gamma_s^2 + u^2}. \quad (19)$$

We assume that the bipolar bits has equal transmission probability. Then the bit error probability of Cauchy case can be calculated as

$$\begin{aligned} \mathcal{P}_{b,\alpha}^{\text{BPSK}} &= \Pr\{U > A\} = \int_A^{+\infty} f(u; \alpha = 1)du \\ &= \int_A^{+\infty} \frac{\gamma_s}{\pi} \frac{1}{\gamma_s^2 + u^2} du = \int_{\frac{A}{\gamma_s}}^{+\infty} \frac{1}{\pi} \frac{1}{1 + v^2} dv. \end{aligned} \quad (20)$$

Then we have

$$\mathcal{P}_{b,\alpha}^{\text{BPSK}} = Q_s\left(\frac{A}{\gamma_s}; \alpha = 1\right) = \int_{\frac{A}{\gamma_s}}^{+\infty} f_0(v; \alpha = 1)dv, \quad (21)$$

where $f_0(v; \alpha = 1)$ is the pdf of standard Cauchy distribution given by

$$f_0(v; \alpha = 1) = \frac{1}{\pi} \frac{1}{1 + v^2}. \quad (22)$$

We recall that the bit error probability of BPSK modulation in Gaussian case can be written as

$$\mathcal{P}_b^{\text{BPSK}} = Q_s\left(\frac{A}{\gamma_s}; \alpha = 2\right) = \int_{\frac{A}{\gamma_s}}^{+\infty} f_0(v; \alpha = 2)dv. \quad (23)$$

According to Eq. (7), we obtain

$$\frac{A}{\gamma_s} = \sqrt{4C_g^{(\frac{2}{\alpha}-1)} \frac{E_s}{N_0}}. \quad (24)$$

We can see that the lower limit of integration $2\sqrt{E_s/N_0}$ in Eq. (23) is equal to A/γ_s in Eq. (21) when $\alpha = 2$ in Eq. (24). Because the geometric power is defined consistently continuous for $\alpha \in (0, 2]$, we can generate the following performance mapping mechanism of BPSK modulation system from the standard Gaussian to general S α S distribution as follows

$$\begin{aligned} \mathcal{P}_b^{\text{BPSK}} &= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = Q_s\left(2\sqrt{\frac{E_s}{N_0}}; \alpha = 2\right) \\ &\stackrel{CM}{=} Q_s\left(\frac{A}{\gamma_s}; \alpha\right) = Q_s\left(\sqrt{4C_g^{(\frac{2}{\alpha}-1)} \frac{E_s}{N_0}}; \alpha\right) = \mathcal{P}_{b,\alpha}^{\text{BPSK}}, \end{aligned} \quad (25)$$

where the operation $\stackrel{CM}{=}$ denotes consistently mapping (CM), which is continuous for the whole entire of α , i.e., $\alpha \in (0, 2]$.

B. Extension to MPSK Modulation

First we consider bit error rate (BER) and symbol error rate (SER) performance of the MPSK under AWGN channel. The relation between E_s/N_0 and E_b/N_0 is given by

$$\frac{E_s}{N_0} = k \frac{E_b}{N_0}, \quad (26)$$

where $k = R_c \log_2 M$ represents the information bits per symbol, R_c is the code rate of the system and $R_c = 1$ in our case for uncoded system.

As [14] indicated that the MPSK has comparable performance as MQAM when $M = 4$, however, the latter is superior to the former when $M \geq 4$. Thus, we only consider the representative QPSK with an initial phase $\pi/4$ instead of MPSK modulation, and show its SER performance as

$$\begin{aligned} \mathcal{P}_M^{\text{QPSK}} &= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right) \\ &= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right). \end{aligned} \quad (27)$$

The BER of QPSK modulation is equivalent to that of the BPSK case shown in Eq. (13).

Based on the mapping mechanism of BPSK signaling under general S α S noise, $\alpha \in (0, 2]$, in Eq. (25), we drive

$$Q\left(\sqrt{\frac{2E_s}{N_0}}\right) \stackrel{CM}{=} Q_s\left(\sqrt{4C_g^{(\frac{2}{\alpha}-1)} \frac{E_s}{N_0}}; \alpha\right). \quad (28)$$

According to Eq. (27) and Eq. (11), then the symbol error probability and the equivalent bit error probability of QPSK signaling under general S α S noise, $\alpha \in (0, 2]$, can be represented as

$$\begin{aligned} \mathcal{P}_{M,\alpha}^{\text{QPSK}} &= 2Q_s\left(\sqrt{2C_g^{(\frac{2}{\alpha}-1)} \frac{E_s}{N_0}}; \alpha\right) - Q_s^2\left(\sqrt{2C_g^{(\frac{2}{\alpha}-1)} \frac{E_s}{N_0}}; \alpha\right) \\ &= 2Q_s\left(\sqrt{4C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0}}; \alpha\right) - Q_s^2\left(\sqrt{4C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0}}; \alpha\right) \end{aligned} \quad (29)$$

and

$$\mathcal{P}_{b,\alpha}^{\text{QPSK}} = Q_s\left(\sqrt{4C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0}}; \alpha\right). \quad (30)$$

C. Extension to MQAM Modulation

According to [16], the bit error probability of MQAM signaling ($M \geq 4$) with square constellation can be well

approximated by using the first two dominant term, i.e.,

$$\begin{aligned} \mathcal{P}_b^{\text{MQAM}} &\approx \frac{\sqrt{M}-1}{\sqrt{M} \log_2 \sqrt{M}} \text{erfc}\left(\sqrt{\frac{3 \frac{E_s}{N_0}}{2(M-1)}}\right) \\ &\quad + \frac{\sqrt{M}-2}{\sqrt{M} \log_2 \sqrt{M}} \text{erfc}\left(3\sqrt{\frac{3 \frac{E_s}{N_0}}{2(M-1)}}\right) \\ &= \eta_1 Q\left(\sqrt{\frac{3 \frac{E_b}{N_0} k}{M-1}}\right) + \eta_2 Q\left(3\sqrt{\frac{3 \frac{E_b}{N_0} k}{M-1}}\right), \end{aligned} \quad (31)$$

where

$$\begin{cases} \eta_1 = \frac{4}{k} \left(1 - \frac{1}{\sqrt{M}}\right); \\ \eta_2 = \frac{4}{k} \left(1 - \frac{2}{\sqrt{M}}\right). \end{cases} \quad (32)$$

and $\text{erfc}(\cdot)$ is the complementary error-function defined as:

$$\text{erfc}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_x^{+\infty} \exp(-z^2) dz, \quad (33)$$

and the complementary error-function can be further expressed in terms of the Q-function as

$$\text{erfc}(x) = 2Q(\sqrt{2}x). \quad (34)$$

The symbol error probability of square MQAM modulation is shown below [17]

$$\begin{aligned} \mathcal{P}_M^{\text{MQAM}} &= \xi_1 Q\left(\sqrt{\frac{3 \frac{E_s}{N_0}}{M-1}}\right) - \xi_2 Q^2\left(\sqrt{\frac{3 \frac{E_s}{N_0}}{M-1}}\right) \\ &= \xi_1 Q\left(\sqrt{\frac{3 \frac{E_b}{N_0} k}{M-1}}\right) - \xi_2 Q^2\left(\sqrt{\frac{3 \frac{E_b}{N_0} k}{M-1}}\right) \end{aligned} \quad (35)$$

where,

$$\begin{cases} \xi_1 = 4 \left(1 - \frac{1}{\sqrt{M}}\right); \\ \xi_2 = 4 \left(1 - \frac{1}{\sqrt{M}}\right)^2. \end{cases} \quad (36)$$

Based on the consistently mapping in Eq. (28), we have

$$Q\left(\sqrt{\frac{E_s}{N_0}}\right) \stackrel{CM}{=} Q_s\left(\sqrt{2C_g^{(\frac{2}{\alpha}-1)} \frac{E_s}{N_0}}; \alpha\right). \quad (37)$$

Similarly, according to Eq. (35) and Eq. (31), after some algebraic manipulation, the symbol error probability and the bit error probability of MQAM signaling under general S α S noise, $\alpha \in (0, 2]$, are described as

$$\begin{aligned} \mathcal{P}_{M,\alpha}^{\text{QPSK}} &= \xi_1 Q_s\left(\sqrt{\frac{6C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0} k}{M-1}}; \alpha\right) \\ &\quad - \xi_2 Q_s^2\left(\sqrt{\frac{6C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0} k}{M-1}}; \alpha\right) \end{aligned} \quad (38)$$

and

$$\mathcal{P}_{b,\alpha}^{\text{MQAM}} \approx \eta_1 Q_s \left(\sqrt{\frac{6C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0} k}{M-1}}; \alpha \right) + \eta_2 Q_s \left(3 \sqrt{\frac{6C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0} k}{M-1}}; \alpha \right). \quad (39)$$

D. Asymptotic Performance of M -ary Modulation

In order to evaluate the asymptotic performance, the tail probability in a limited case is given by [13]

$$\lim_{v \rightarrow \infty} Q_s(v; \alpha) \rightarrow \frac{C_\alpha}{v^\alpha}, \quad (40)$$

where,

$$C_\alpha = \frac{1}{\pi} \Gamma(\alpha) \sin\left(\frac{\pi\alpha}{2}\right), \quad (41)$$

the operation \rightarrow represents asymptotic, and the gamma function is $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

1). According to Eq. (25) and and Eq. (30), as $(E_b/N_0) \rightarrow \infty$, the asymptotic bit error provability of both BPSK and QPSK signaling under S α S noise has the form

$$\mathcal{P}_{b,\alpha}^{\text{BPSK}} \rightarrow C_\alpha \left(\sqrt{4C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0}} \right)^{-\alpha}. \quad (42)$$

2). According to Eq. (29), as $(E_b/N_0) \rightarrow \infty$, the asymptotic symbol error probability of QPSK signaling under S α S noise has the form

$$\mathcal{P}_{M,\alpha}^{\text{QPSK}} \rightarrow 2C_\alpha \left(\sqrt{4C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0}} \right)^{-\alpha} - C_\alpha^2 \left(\sqrt{4C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0}} \right)^{-2\alpha}. \quad (43)$$

3). According to Eq. (38) and Eq. (39), as $(E_b/N_0) \rightarrow \infty$, the asymptotic performance of MQAM signaling ($M \geq 4$) under S α S noise has the form

$$\mathcal{P}_{M,\alpha}^{\text{MQAM}} \rightarrow \xi_1 C_\alpha \left(\sqrt{\frac{6C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0} k}{M-1}} \right)^{-\alpha} - \xi_2 C_\alpha^2 \left(\sqrt{\frac{6C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0} k}{M-1}} \right)^{-2\alpha}, \quad (44)$$

and

$$\mathcal{P}_{b,\alpha}^{\text{MQAM}} \rightarrow \eta_1 C_\alpha \left(\sqrt{\frac{6C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0} k}{M-1}} \right)^{-\alpha} + \eta_2 C_\alpha \left(3 \sqrt{\frac{6C_g^{(\frac{2}{\alpha}-1)} \frac{E_b}{N_0} k}{M-1}} \right)^{-\alpha}. \quad (45)$$

IV. NUMERICAL SIMULATION RESULTS

We derived the analytical expressions of BPSK, QPSK and MQAM modulation for uncoded systems under S α S noise. We employ Gray-coded constellation in each modulation scheme. The simulations are based on the S α S framework $\alpha \in (0, 2]$, taking 10^7 independent trials for each E_b/N_0 . For a given characteristic exponent α , a group of curves including the asymptotic expression, the simulation, and our analytical expression are used for comparison.

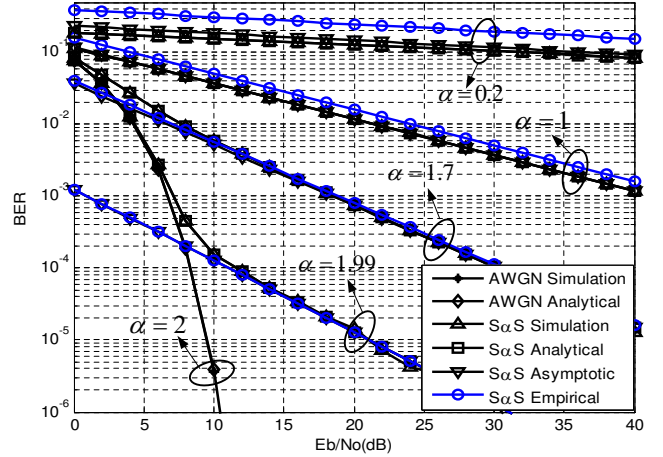


Fig. 1. The BER of BPSK signaling under S α S noise corresponding to α equals 2, 1.99, 1.7, and 0.2, respectively.

We show the BER of BPSK signaling in Fig. 1. In order to have a deep insight, we add the empirical approximation obtained in [5] for further comparison. The BER performance of BPSK signaling under Gaussian noise are used as benchmark. We can see from Fig. 1 that the simulation results are well matched with our analytical BER expressions one-by-one even for small values of α . For a given α , three curves (the asymptotic expression, the simulation, and our analytical expression) are almost overlapped in large SNR region. The analytical BER curve under S α S noise is close to the one under Gaussian case in low SNR region when α approaching to 2, while it deviated when α gets smaller.

On the other hand, empirical approximation using $Q_s(2\sqrt{E_s/N_0})$ in [5] is very close to the exact BER performance when the value of α approaching to 2, such as $\alpha = 1.99$ and $\alpha = 1.7$. However, this approximation deviates from the three curves (the asymptotic expression, the simulation, and our analytical expression) when α gets smaller. It can be seen more clearly from Eq. (30) that the value $2\sqrt{E_s/N_0}$ is approximate to $\sqrt{4C_g^{[(2/\alpha)-1]}(E_s/N_0)}$ when α is close to 2. We can deduce that the approximation used in [5] is not accurate for small values of α and higher-order modulation types.

The BER and SER results for M -ary signaling under S α S noise ($\alpha = 1.9$) are shown in Fig. 2 and Fig. 3, respectively. Also, we employ the Gaussian case as benchmark. The trend for the BER and SER curves under S α S noise corresponding to different order of modulation is similar to the Gaussian

case, i.e., the BER and SER degrade as the modulation order M increases.

It can be observed that the QPSK and QAM signaling have almost the same performance either in Gaussian or in impulsive noise channels. Consider impulsive noise channel, the SER of the QAM in Eq. (38) ($M = 4$) is equal to that of the QPSK in Eq. (29), and the BER of the QAM in Eq. (39) dominated by the first term is approximate to that of the QPSK in Eq. (30). Moreover, we can see that the asymptotic curves can only approximate to the exact performance for large SNR if higher-order modulation is implemented. For example, as shown in Fig. 3, the E_b/N_0 is needed above 10dB to get a better SER approximation for QAM signaling, however, it is required up to 30dB for 1024-QAM signaling. This can be seen more clearly in Eq. (44) that we can get better asymptotic performance for $\sqrt{6kC_g^{[(2/\alpha)-1]}(E_b/N_0)/(M-1)}$ by increasing E_b/N_0 in the numerator when a larger M is used in the denominator of the square root.

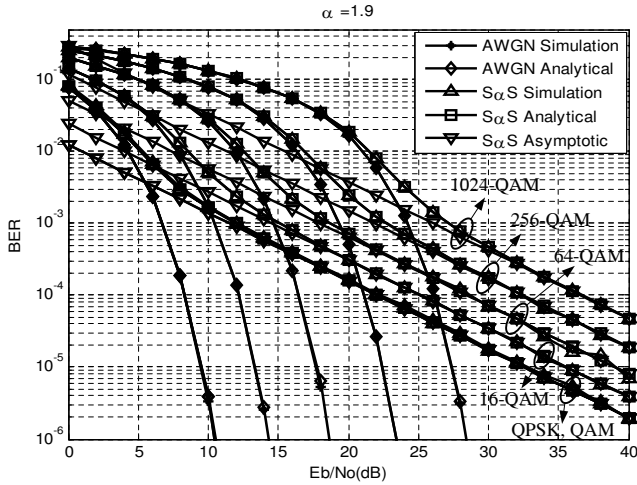


Fig. 2. The BER of M -ary signaling under $S\alpha S$ noise with $\alpha = 1.9$.

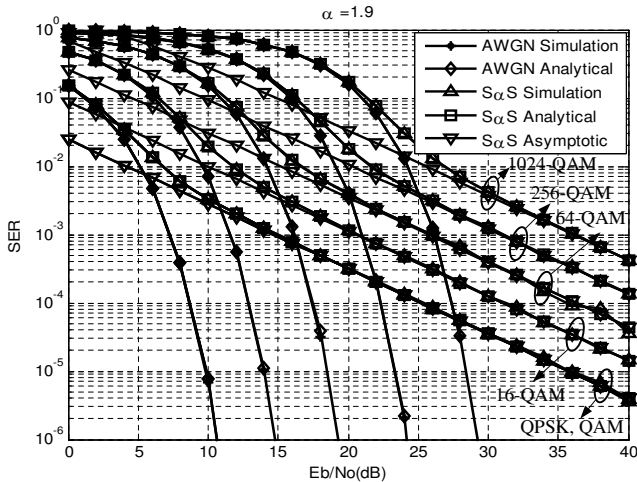


Fig. 3. The SER of M -ary signaling under $S\alpha S$ noise with $\alpha = 1.9$.

V. CONCLUSIONS

By employing zero-order statistics, we created a consistently continuous mapping mechanism acting as a linkage between the Gaussian and the general $S\alpha S$ framework. Using our developed mapping mechanism, we can simply derive the analytical BER and SER expressions of M -ary signaling including BPSK, QPSK, and MQAM under $S\alpha S$ noise. We deduced that the analytical expressions for other conventional modulated signals under $S\alpha S$ noise can be also derived by using this mapping mechanism based on its Gaussian counterpart. Our analytical expressions have been verified through simulations, and provide a valuable benchmark for deriving the exact coding gain under $AWS\alpha SN$ channels.

REFERENCES

- [1] J. Lin, M. Nassar, and B. Evans, "Non-Parametric Impulsive Noise Mitigation in OFDM Systems Using Sparse Bayesian Learning," in *IEEE Global Telecommunications Conference (GLOBECOM 2011)*, Dec. 2011, pp. 1–5.
- [2] M. Souryal, E. Larsson, B. Peric, and B. Vojcic, "Soft-Decision Metrics for Coded Orthogonal Signaling in Symmetric Alpha-Stable Noise," *IEEE Transactions on Signal Processing*, vol. 56, no. 1, pp. 266–273, Jan. 2008.
- [3] M. Chitre, S. Kuselan, and V. Pallayil, "Ambient noise imaging in warm shallow waters; robust statistical algorithms and range estimation," *The Journal of the Acoustical Society of America*, vol. 132, no. 2, pp. 838–847, 2012.
- [4] D. Zha and T. Qiu, "Underwater sources location in non-Gaussian impulsive noise environments," *Digital Signal Processing*, vol. 16, no. 2, pp. 149–163, 2006.
- [5] M. Chitre, J. Potter, and S. Ong, "Viterbi Decoding of Convolutional Codes in Symmetric Alpha-Stable Noise," *IEEE Transactions on Communications*, vol. 55, no. 12, pp. 2230–2233, Dec. 2007.
- [6] T. Saleh, I. Marsland, and M. El-Tanany, "Simplified LLR-based Viterbi decoder for convolutional codes in symmetric alpha-stable noise," in *IEEE Canadian Conference on Electrical Computer Engineering (CCECE)*, April 2012, pp. 1–4.
- [7] T. Shehata, I. Marsland, and M. El-Tanany, "Near Optimal Viterbi Decoders for Convolutional Codes in Symmetric Alpha-Stable Noise," in *IEEE 72nd Vehicular Technology Conference Fall (VTC 2010-Fall)*, Sept. 2010, pp. 1–5.
- [8] S. Mohammad, L. Heng-Siong, and C. Teong-Chee, "Decoding of Turbo Codes in Symmetric Alpha-Stable Noise," *ISRN Signal Processing*, vol. 2011, 2011.
- [9] J. Forney, G.D. and G. Ungerboeck, "Modulation and coding for linear Gaussian channels," *IEEE Transactions on Information Theory*, vol. 44, no. 6, pp. 2384–2415, Oct. 1998.
- [10] A. Mahmood, M. Chitre, and M. Armand, "PSK Communication with Passband Additive Symmetric alpha-Stable Noise," *IEEE Transactions on Communications*, vol. 60, no. 10, pp. 2990–3000, October 2012.
- [11] J. Gonzalez, J. Paredes, and G. Arce, "Zero-Order Statistics: A Mathematical Framework for the Processing and Characterization of Very Impulsive Signals," *IEEE Transactions on Signal Processing*, vol. 54, no. 10, pp. 3839–3851, Oct. 2006.
- [12] J. P. Nolan, *Stable distributions: models for heavy-tailed data*. Birkhauser, 2003.
- [13] G. Sureka and K. Kiasaleh, "Sub-Optimum Receiver Architecture for AWGN Channel with Symmetric Alpha-Stable Interference," *IEEE Transactions on Communications*, vol. 61, no. 5, pp. 1926–1935, May 2013.
- [14] J. G. Proakis, *Spread spectrum signals for digital communications*. Wiley Online Library, 2001.
- [15] J. P. Nolan, "Numerical calculation of stable densities and distribution functions," *Communications in statistics. Stochastic models*, vol. 13, no. 4, pp. 759–774, 1997.
- [16] K. Cho and D. Yoon, "On the general BER expression of one- and two-dimensional amplitude modulations," *IEEE Transactions on Communications*, vol. 50, no. 7, pp. 1074–1080, 2002.
- [17] M. K. Simon and M.-S. Alouini, "A unified approach to the performance analysis of digital communication over generalized fading channels," *Proceedings of the IEEE*, vol. 86, no. 9, pp. 1860–1877, 1998.