Approaching the Ergodic Capacity with Lattice Coding

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Abstract—It is known that lattice coding can achieve the capacity of the additive white Gaussian noise (AWGN) channel. This paper addresses the performance of lattice codes in the ergodic fading channel. Using nested lattice codes and ambiguity decoding, we show that the rates achieved by lattice coding and decoding are within a constant gap of the capacity of the ergodic channel at moderate and high signal-to-noise-ratio (SNR), and within a gap that decreases quadratically with the SNR for the low SNR regime.

Index Terms—Channel capacity, nested lattice codes, ergodic channel, minimum mean-square error (MMSE) estimation.

I. INTRODUCTION

Körner and Marton showed an example where linear codes outperform random codes [1], opening the door for structured codes to compete with random codes. Erez and Zamir showed that nested lattice codes achieve the capacity of the point-to-point AWGN channel [2]. The ingredients include common randomness via a dither variable and MMSE scaling at the receiver. Following this result, Erez, Listyn and Zamir [3] proved the existence of lattices with good properties that achieve the performance promised in [2]. El Gamal *et al.* [4] showed that nested lattice codes achieve the diversity-multiplexing tradeoff of multiple-input-multiple-output (MIMO) flat fading channels. It was shown in [5] that lattice codes approach the capacity of the Gaussian relay network for arbitrary topologies, showing that the random codebook performance in quantize-map-and-forward [6] can be achieved with lattices. Song and Devroye [7] showed that lattice codes can achieve the same performance of random codes for the Gaussian relay channel.

Nazer and Gastpar [8] introduced the *compute-and-forward* strategy where relays in a network decode integer combinations of signals from multiple transmitters. It was also shown in [9] that a modified version of the compute-and-forward scheme operates within a constant gap to the sum capacity of the K -user multiple-access channel (MAC).

Most of the literature to date concentrates on channels that are either constant or quasi-static block fading, where each codeword sees one channel realization. This leaves open many interesting problems, including those where each codeword sees many channel realizations. The present paper addresses a

class of these problems, specifically the ergodic rates available via lattice codes.¹

We assume the receiver knows the realization of the channel and the transmitter knows the channel statistics. Similar to [2], the encoder emits a dithered signal. The receiver uses a time-varying MMSE scaling, a generalization of the fixed MMSE attenuation used in [2]. The decoder departs from the AWGN case because the effective noise is no longer identically distributed across time nor can be approximated as Gaussian, unlike [2]. Instead, we apply the *ambiguity lattice decoder* proposed by Loeliger [11]. In order to reduce the decoding complexity, the decision regions are both spherical and independent of the instantaneous channel realizations. With spherical decision regions, computationally efficient decoding algorithms can be used such as *sphere decoding* algorithms [12].

The proposed scheme achieves rates whose gap with the capacity of the ergodic channel can be upper bounded by a constant that is independent of the SNR when $\text{SNR} > 1$. The gap decreases quadratically with SNR when $\text{SNR} < 1$.

This paper uses the following notation. Boldface lowercase letters denote column vectors and boldface uppercase letters denote matrices. The set of real and complex numbers are denoted $\mathbb{R}$, $\mathbb{C}$, respectively. $\mathbf{A}^T$ denotes the transpose of matrix $\mathbf{A}$. The complex conjugate is denoted with superscript $*$. a_i denotes element i of vector $\mathbf{a}$. Unless otherwise stated, all logarithms are in base 2. $\mathcal{B}_n(q)$ is an n -dimensional ball of radius q and the volume of shape $\mathcal{A}$ is denoted $\text{Vol}(\mathcal{A})$. Real and imaginary parts of a complex number are denoted with superscripts, e.g., $a = a^R + j a^I$. $\mathbf{0}_l$ and $\mathbf{1}_l$ denote the $l \times l$ matrices containing zeros and ones, respectively.

II. PRELIMINARIES: SYSTEM MODEL, LATTICES, PROBLEM SETTING

A. System model

Consider a single antenna point-to-point link. Each codeword spans significantly many coherence intervals of the channel. Real-valued channels is considered. The extension

¹Although the work of [10] comes under the title of ergodic rates of compute-and-forward, it assumes each codeword sees one channel realization, and then the average performance across many codeword transmissions is quantified. Therefore [10] does not address the issue of short coherence lengths, which is driving our interest in ergodic rates.

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to complex-valued channels will appear in Section III. The received signal is given by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}, \quad (1)$$

where $\mathbf{H} \in \mathbb{R}^n$ is an $n \times n$ diagonal matrix denoting the channel matrix with coherence length m and k i.i.d complex entries such that $n = km$. The corresponding vector $\mathbf{h}$ containing the diagonal elements of $\mathbf{H}$ is zero mean with a block diagonal covariance matrix of the form

$$\mathbf{C}_h = \sigma_h^2 \begin{bmatrix} \mathbf{1}_m & \mathbf{0}_m & \dots & \mathbf{0}_m \\ \mathbf{0}_m & \mathbf{1}_m & \dots & \mathbf{0}_m \\ & & \ddots & \\ \mathbf{0}_m & \mathbf{0}_m & \dots & \mathbf{1}_m \end{bmatrix}. \quad (2)$$

For the channel to satisfy the ergodic assumption then $k = n/m$ must be sufficiently large. The vector $\mathbf{x} \in \mathbb{R}^n$ is the transmitted lattice codeword with average power P_x . We will elaborate more on the properties and structure of $\mathbf{x}$ in the sequel. The vector $\mathbf{w} \in \mathbb{R}^n$ is a zero mean i.i.d Gaussian noise vector with covariance $P_w \mathbf{I}_n$, and independent of $\mathbf{h}$. We define $\rho \triangleq P_x/P_w$ as the average received SNR when $\sigma_h^2 = 1$.

The transmitter has no instantaneous channel knowledge: it only knows the statistics of the channels as well as ρ . On the other hand the receiver perfectly knows the channel in addition to ρ .

B. Lattice Codes

A lattice Λ is a discrete subgroup of $\mathbb{R}^n$ which is closed under reflection and real addition. The lattice Λ is said to be nested in Λ_1 if $\Lambda \subseteq \Lambda_1$. The transmitter constructs a codebook $\mathcal{L}_1 = \Lambda_1 \cap \mathcal{V}$, whose rate is given by

$$R = \frac{1}{n} \log \frac{\text{Vol}(\mathcal{V})}{\text{Vol}(\mathcal{V}_1)}, \quad (3)$$

and $\mathcal{V}, \mathcal{V}_1$ are the fundamental Voronoi regions of the lattices Λ, Λ_1 , respectively. One can consider more general fundamental regions for the lattice. Let Ω be any fundamental region of Λ , i.e., every $\mathbf{s} \in \mathbb{R}^n$ can be uniquely written as $\mathbf{s} = \lambda + \mathbf{e}$ where $\lambda \in \Lambda, \mathbf{e} \in \Omega$. The quantizer is then defined by

$$Q_\Omega(\mathbf{s}) = \lambda, \quad \text{if } \mathbf{s} \in \lambda + \Omega. \quad (4)$$

Define the modulo- Λ operation corresponding to Ω as follows:

$$[\mathbf{s}] \bmod \Lambda \triangleq \mathbf{s} - Q_\Omega(\mathbf{s}). \quad (5)$$

Hereafter we omit the subscript Ω for simplicity. We employ a class of nested lattice codes originally proposed in [2] that approach the AWGN capacity. The coarse lattice Λ is good for covering, quantization and the fine lattice Λ_1 is good for AWGN coding.

Definition 1: A lattice Λ is good for covering if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\text{Vol}(\mathcal{B}_n(\sqrt{R_c}))}{\text{Vol}(\mathcal{B}_n(\sqrt{R_f}))} = 0, \quad (6)$$

where R_c is the radius of the smallest sphere spanning $\mathcal{V}$ and R_f is the radius of the sphere whose volume is equal to $\text{Vol}(\mathcal{V})$. For a good nested lattice code with average codeword

power P_x , the Voronoi region $\mathcal{V}$ approaches a sphere of radius $\sqrt{nP_x}$. For a more comprehensive review on nested lattice codes see [2], [3].

C. Problem Setting

We present a modified version of the problem setting in [2] for the time varying channel model. Given a lattice point $\mathbf{t}$, the transmitter generates a dither signal $\mathbf{d}$ (known at the receiver) that is uniformly distributed over $\mathcal{V}$, and emits the lattice codeword

$$\mathbf{x} = [\mathbf{t} - \mathbf{d}] \bmod \Lambda. \quad (7)$$

The following two lemmas elaborate more on the properties of the dither $\mathbf{d}$.

Lemma 1: If $\mathbf{t} \in \mathcal{V}$ is independent of $\mathbf{d}$, then $\mathbf{x}$ is also uniformly distributed over $\mathcal{V}$ and is independent of $\mathbf{t}$.

Proof: See [2, Lemma 1]. ■

Lemma 2: For a uniform vector $\mathbf{d}$ defined over a fundamental Voronoi region $\mathcal{V}$ of a covering good lattice, $\mathbf{d}$ is uncorrelated with $\mathbb{E}[\mathbf{d}\mathbf{d}^T] = P_x \mathbf{I}_n$ when $n \rightarrow \infty$.

Proof: See [13, Theorem 1]. ■

The receiver multiplies the received vector in (1) by a matrix and adds the dither as follows

$$\begin{aligned} \mathbf{y}' &= [\mathbf{U}\mathbf{y} + \mathbf{d}] \bmod \Lambda \\ &= [\mathbf{U}\mathbf{H}\mathbf{x} + \mathbf{U}\mathbf{w} + \mathbf{d}] \bmod \Lambda \\ &= [\mathbf{U}\mathbf{H}\mathbf{t} - \mathbf{U}\mathbf{H}\mathbf{d} + \mathbf{U}\mathbf{w} + \mathbf{d}] \bmod \Lambda \\ &= [\mathbf{t} + (\mathbf{I}_n - \mathbf{U}\mathbf{H})\mathbf{x} + \mathbf{U}\mathbf{w}] \bmod \Lambda \\ &= [\mathbf{t} + (\mathbf{I}_n - \mathbf{U}\mathbf{H})\mathbf{d}' + \mathbf{U}\mathbf{w}] \bmod \Lambda \\ &= [\mathbf{t} + \mathbf{z}] \bmod \Lambda, \end{aligned} \quad (8)$$

where

$$\mathbf{d}' \triangleq \mathbf{x}, \quad (9)$$

$$\mathbf{z} \triangleq (\mathbf{I}_n - \mathbf{U}\mathbf{H})\mathbf{d}' + \mathbf{U}\mathbf{w}, \quad (10)$$

and (9) follows from Lemma 2. The receiver then decides the point $\hat{\mathbf{t}}$ was transmitted through the following mapping

$$\hat{\mathbf{t}} = [Q(\mathbf{t} + \mathbf{z})] \bmod \Lambda. \quad (11)$$

III. ERGODIC LATTICE DECODING

We now discuss the choice of $\mathbf{U}$ and the decoding procedure. $\mathbf{U}$ is the MMSE matrix that minimizes $\mathbb{E}[\mathbf{z}^T \mathbf{z}]$,

$$\mathbf{U} = \rho \mathbf{H}^T (\rho \mathbf{H} \mathbf{H}^T + \mathbf{I}_n)^{-1}. \quad (12)$$

$\mathbf{U}$ is diagonal, with the following diagonal elements

$$u_i = \frac{\rho h_i}{h_i^2 \rho + 1}. \quad (13)$$

From (10) and (13)

$$z_i = \frac{-1}{h_i^2 \rho + 1} x_i + \frac{\rho h_i}{h_i^2 \rho + 1} w_i. \quad (14)$$

Thus elements of $\mathbf{z}$ are zero mean and uncorrelated but not identically distributed, since h_i are known at the receiver, and therefore elements of $\mathbf{z}$ have been scaled by unequal constants. On the other hand, if h_i were unknown at the receiver,

then elements of $\mathbf{z}$ would have been both uncorrelated and identically distributed, leading to simplifications. Therefore, for tractability we ignore the instantaneous channel knowledge at the receiver at all points subsequent to MMSE scaling. The covariance of $\mathbf{z}$ is then given by

$$\begin{aligned}\mathbb{E}[\mathbf{z}\mathbf{z}^T] &= \mathbb{E}_h[\mathbb{E}[z_i^2|h]]\mathbf{I}_n \\ &= \mathbb{E}_h\left[\frac{\rho}{\rho h^2 + 1}\right]\mathbf{I}_n.\end{aligned}\quad (15)$$

The following two theorems are the main results of the paper.

Theorem 1: For the ergodic point-to-point channel with real-valued channel coefficients, the rate

$$R < -\frac{1}{2} \log \left(\mathbb{E}_h \left[\frac{1}{\rho h^2 + 1} \right] \right) \quad (16)$$

is achievable using nested lattice codes.

Proof: We apply the ambiguity decoder of [11] defined by a spherical decision region $\Omega \subset \mathbb{R}^n$ as follows

$$\Omega \triangleq \left\{ \mathbf{v} \in \mathbb{R}^n : \mathbf{v}^T \mathbf{v} \leq (1 + \epsilon)n\rho \mathbb{E}_h \left[\frac{1}{\rho h^2 + 1} \right] \right\}, \quad (17)$$

where ϵ is an arbitrary positive constant. The decoder chooses $\hat{\mathbf{t}} \in \Lambda_1$ if and only if $\hat{\mathbf{t}}$ is the only lattice point within the decision region, i.e., $\mathbf{y}' \in \hat{\mathbf{t}} + \Omega$.

In the following we replace the effective noise $[\mathbf{z}] \bmod \Lambda$ with $\mathbf{z}$ to obtain a bound on error. This bounding is valid because we use a bounded distance decoder, and the modulo lattice operation can only decrease the magnitude, i.e. $\|[\mathbf{z}] \bmod \Lambda\| \leq \|\mathbf{z}\|$.

As shown in [11, Theorem 6], there exists a nested lattice code with at least M codewords whose probability of error can be bounded by

$$P_e < P(\mathbf{z} \notin \Omega) + (1 + \delta)M \frac{\text{Vol}(\Omega)}{\text{Vol}(\mathcal{V})}, \quad (18)$$

for any $\delta > 0$. This is a union bound, involving the union of two events: The event that the noise vector is outside the decision region $\{\mathbf{z} \notin \Omega\}$ and the event that the noise vector is in the intersection of two decision regions $\{\mathbf{y}' \in \{\mathbf{t}_1 + \Omega\} \cap \{\mathbf{t}_2 + \Omega\}\}$, where $\mathbf{t}_1, \mathbf{t}_2 \in \Lambda_1$ are two lattice points. The overlap of decision regions is then bounded by the union of all decision regions. Details can be found in [11].

From (15), $P(\mathbf{z} \notin \Omega) < \gamma$ for any $\gamma > 0$ as $n \rightarrow \infty$ by law of large numbers. Consequently, the error probability can be bounded by

$$P_e < \gamma + (1 + \delta)M \frac{\text{Vol}(\Omega)}{\text{Vol}(\mathcal{V})}, \quad (19)$$

for any $\gamma, \delta > 0$. The volume of Ω is given by

$$\text{Vol}(\Omega) = (1 + \epsilon)^{\frac{n}{2}} \left(\mathbb{E}_h \left[\frac{1}{\rho h^2 + 1} \right] \right)^{\frac{n}{2}} \text{Vol}(\mathcal{B}_n(\sqrt{n\rho})). \quad (20)$$

The second term in (19) is bounded by

$$(1 + \delta)2^{nR}(1 + \epsilon)^{n/2} \left(\mathbb{E}_h \left[\frac{1}{\rho h^2 + 1} \right] \right)^{\frac{n}{2}} \frac{\text{Vol}(\mathcal{B}_n(\sqrt{n\rho}))}{\text{Vol}(\mathcal{V})}$$

$$= (1 + \delta)2^{-n \left(-\frac{1}{n} \log \left(\frac{\text{Vol}(\mathcal{B}_n(\sqrt{n\rho}))}{\text{Vol}(\mathcal{V})} \right) - \xi \right)}, \quad (21)$$

where

$$\xi \triangleq -\frac{1}{2} \log(1 + \epsilon) - \frac{1}{2} \log \left(\mathbb{E}_h \left[\frac{1}{\rho h^2 + 1} \right] \right) - R, \quad (22)$$

and $R \triangleq \frac{1}{n} \log M$. From (6), since the lattice Λ is good for covering,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{\text{Vol}(\mathcal{B}_n(\sqrt{n\rho}))}{\text{Vol}(\mathcal{V})} \right) = 0. \quad (23)$$

Then by (21), whenever ξ is a positive constant we have $\lim_{n \rightarrow \infty} P_e = 0$. Since ϵ is arbitrary, positive ξ can be achieved when

$$R < -\frac{1}{2} \log \left(\mathbb{E}_h \left[\frac{1}{\rho h^2 + 1} \right] \right).$$

Note that the decision regions are independent of the instantaneous channel values and only depend on the channel statistics. Moreover, the decision regions are spherical, opening the door for using computationally efficient sphere decoding algorithms [12].

The following Theorem extends the results of Theorem 1 for the complex-valued channel.

Theorem 2: For the point-to-point ergodic channel with complex-valued channel coefficients, the rate

$$R < -\log \left(\mathbb{E}_h \left[\frac{1}{\rho |h|^2 + 1} \right] \right) \quad (24)$$

is achievable using nested lattice codes.

Proof: Since the achievable scheme follows that of the real-valued channel, we only provide a rough sketch of the proof. A similar technique for the extension to complex-valued channels can be found in [8, Theorem 6]. Since the channel is complex-valued, two independent codewords are transmitted on the real and imaginary domains each with average power $P_x/2$ as follows

$$\tilde{\mathbf{x}} = [\tilde{\mathbf{t}}^R - \tilde{\mathbf{d}}^R] \bmod \tilde{\Lambda} + j[\tilde{\mathbf{t}}^I - \tilde{\mathbf{d}}^I] \bmod \tilde{\Lambda}, \quad (25)$$

with $\mathbb{E}[\|\tilde{\mathbf{x}}\|^2] = nP_x$. The receive MMSE coefficients are given by

$$\tilde{u}_i = \frac{\rho h_i^*}{|h_i|^2 \rho + 1}. \quad (26)$$

Similar to (8), the real and imaginary equivalent channels at the receiver can be written as follows

$$\begin{aligned}\tilde{\mathbf{y}}'^R &= [(\tilde{\mathbf{t}}^R - \tilde{\mathbf{t}}^I) + \tilde{\mathbf{z}}^R] \bmod \tilde{\Lambda}, \\ \tilde{\mathbf{y}}'^I &= [(\tilde{\mathbf{t}}^R + \tilde{\mathbf{t}}^I) + \tilde{\mathbf{z}}^I] \bmod \tilde{\Lambda},\end{aligned} \quad (27)$$

where

$$\begin{aligned}\tilde{z}_i^R &\triangleq \frac{-1}{|h_i|^2 \rho + 1} \tilde{x}_i^R + \frac{\rho h_i^R}{|h_i|^2 \rho + 1} \tilde{w}_i^R + \frac{\rho h_i^I}{|h_i|^2 \rho + 1} \tilde{w}_i^I, \\ \tilde{z}_i^I &\triangleq \frac{-1}{|h_i|^2 \rho + 1} \tilde{x}_i^I + \frac{\rho h_i^R}{|h_i|^2 \rho + 1} \tilde{w}_i^I - \frac{\rho h_i^I}{|h_i|^2 \rho + 1} \tilde{w}_i^R.\end{aligned} \quad (28)$$

Fig. 1. Random codes vs. Lattice codes for the fast fading channel

Fig. 2. The actual gap between random coding and lattice coding

Since the lattice Λ_1 is closed under integer addition, the decoder decodes the new lattice points $\tilde{\mathbf{t}}^R - \tilde{\mathbf{t}}^I$ and $\tilde{\mathbf{t}}^R + \tilde{\mathbf{t}}^I$ from which the original transmitted lattice points $\tilde{\mathbf{t}}^R$ and $\tilde{\mathbf{t}}^I$ can be recovered. Hence, the receiver decodes two distinct messages on the real and imaginary domains and thus the achievable rate is twice the expression from Theorem 1. ■

IV. GAP ANALYSIS

In this Section we upper bound the gap Δ between the complex-valued ergodic channel capacity and the rates achieved in Theorem 2 by nested lattice codes. The gap for the real-valued case follows directly. The capacity of the complex-valued ergodic channel is given by [14]

$$C < \mathbb{E}_h [\log (1 + \rho |h|^2)]. \quad (29)$$

Without loss of generality $\mathbb{E}_h [|h|^2] = 1$.

Theorem 3: For the complex-valued ergodic channel, the gap (in bits/s/Hz) between the ergodic channel capacity and the achievable rate with nested lattice codes is bounded by

$$\Delta < \begin{cases} 1 + \log \left(\mathbb{E}_h \left[\frac{1}{|h|^2} \right] \right) & \text{for } \rho \geq 1 \\ 1.45 \mathbb{E}_h [|h|^4] \rho^2 & \text{for } \rho < 1. \end{cases} \quad (30)$$

Proof: See Appendix A. ■

V. NUMERICAL RESULTS

Through simulations, we evaluate the performance for i.i.d. complex Gaussian channel and noise vectors $\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_n)$ and $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_n)$. In Fig. 1, we plot the rates achieved using lattice codes in (24) vs. the capacity (achieved by random coding) in (29), which reveal that nested lattice codes approach random codes at the high SNR regime. The gap between the random coding and lattice coding rates is plotted in Fig. 2.

Fig. 3 plots the loss in rates incurred by using lattice codes. As expected the loss vanishes quadratically as SNR goes to zero.

Fig. 3. The fraction of loss in rates on using lattice codes

VI. CONCLUSION

We evaluate the performance of lattice codes for the point-to-point ergodic channel with receive channel state information. The rates achieved with lattice codes are within a constant gap that is independent of the SNR in the moderate and high SNR regimes, and within a gap that decreases quadratically with the SNR at the low SNR regime. Simulation results show that the rate loss incurred by using lattice codes vanishes in both high and low SNR regimes when the channel is i.i.d. complex Gaussian.

APPENDIX

A. Case 1: $\rho \geq 1$:

$$\begin{aligned} \Delta &= C - R \\ &= \mathbb{E}_h [\log (1 + \rho |h|^2)] + \log \left(\mathbb{E}_h \left[\frac{1}{\rho |h|^2 + 1} \right] \right) \\ &\leq \log (1 + \rho \mathbb{E}_h [|h|^2]) + \log \left(\mathbb{E}_h \left[\frac{1}{\rho |h|^2 + 1} \right] \right) \end{aligned}$$

$$\begin{aligned}
&< \log(1 + \rho \mathbb{E}[|h|^2]) + \log\left(\mathbb{E}_h\left[\frac{1}{\rho|h|^2}\right]\right) \\
&= \log\left(\left(1 + \frac{1}{\rho}\right) \mathbb{E}_h\left[\frac{1}{|h|^2}\right]\right) \tag{31}
\end{aligned}$$

$$\begin{aligned}
&\leq \log\left(2 \mathbb{E}_h\left[\frac{1}{|h|^2}\right]\right) \\
&= 1 + \log\left(\mathbb{E}_h\left[\frac{1}{|h|^2}\right]\right), \tag{32}
\end{aligned}$$

where the first inequality follows from Jensen's inequality. The second and third inequalities follow since $\frac{1}{x+1} < \frac{1}{x}$ and $1 + \frac{1}{x} \leq 2$ for all $x \geq 1$, respectively.

B. Case 2: $\rho < 1$:

$$\begin{aligned}
\Delta &= C - R \\
&= \mathbb{E}_h[\log(1 + \rho|h|^2)] + \log\left(\mathbb{E}_h\left[\frac{1}{\rho|h|^2 + 1}\right]\right) \\
&\leq \log(1 + \rho \mathbb{E}_h[|h|^2]) + \log\left(\mathbb{E}_h\left[\frac{1}{\rho|h|^2 + 1}\right]\right) \\
&< (\log e) \mathbb{E}_h[|h|^2] \rho + (\log e) \mathbb{E}_h\left[\frac{-\rho|h|^2}{\rho|h|^2 + 1}\right] \\
&= (\log e) \mathbb{E}_h\left[\rho|h|^2 - \frac{\rho|h|^2}{\rho|h|^2 + 1}\right] \\
&= (\log e) \mathbb{E}_h\left[\frac{|h|^2}{\rho + 1/|h|^2}\right] \rho^2 \\
&< 1.45 \mathbb{E}_h\left[\frac{|h|^2}{\rho + 1/|h|^2}\right] \rho^2 \\
&< 1.45 \mathbb{E}_h[|h|^4] \rho^2, \tag{33}
\end{aligned}$$

where the second and third inequalities follow since $\log_e(1+x) < x$ for all $|x-1| < 1$, and $\log_e$ denotes the natural logarithm. The last inequality follows since $\frac{y}{x+1/y} < y^2$ for all $x, y > 0$.

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