Probabilistic Situations for Reasoning

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Abstract—One of the most substantial advantages that human analysts have over machine algorithms is the ability to seamlessly integrate sensed data into a situation-based internal narrative. Replicating an analogous internal representation algorithmically has proved to be a challenging problem that is the focus of much current research. For a machine to more accurately make complex decisions over a stable, consistent and useful representation, situations must be inferred from prior experience and corroborated by incoming data. We believe that a common mathematical framework for situations that addresses varying levels of complexity and uncertainty is essential to meeting this goal. In this paper, we present work in progress on developing the mathematics for probabilistic situations.

I. INTRODUCTION

The goal of any machine system is to ingest data, process the data to a form that can be reasoned over, and then make decisions based on the objectives of that system. In many cases, a primary objective of a system is to provide information in some form to a human operator necessitating a focus on human/machine interactions. Situation-based reasoning is attractive because of the immediate connection to human-relevant decisions. Our belief is that a machine that is able to reason at the level of situations will inherently be more adept at interfacing with human operators given that situations are fundamental to human cognition (see, for example, [1], [2], [3], [4], [5], [6]). In addition, uncertainty plays an essential role in any realistic system, originating both from natural randomness and unavoidable unkowns. In order to have a useful model, we must be able to account for noise of various kinds and the inability of any system to measure the entire world with infinite precision. A mathematical theory for probabilistic situations is a key step in the broader goal of developing a theory of knowledge which can be used to model both human and machine cognition.

II. BACKGROUND

In this section we give a short literature review of definitions of situation, then some background on the relevant mathematics that will be used in the sequel.

A. Situation

Although the meaning of the word *situation* is intuitively clear, a precise standard definition has proved elusive. We briefly discuss four semantic definitions that have been used previously and then provide our modification.

In his work, Barwise [7] envisions situations as atomic relational structures, defining a situation to be "a part of reality that can be comprehended in its own right—one that interacts with other things. By interacting with other things, we mean that they have properties or relate to other things." In the situation calculus work of Shanahan—which is based on earlier work by McCarthy and Hayes—the focus is on actions with a common definition for situation being "a finite sequence of actions" (for instance, see [8]). Barsalou (see [9]) defines a situation as "a region of perceived space that surrounds a focal entity over some temporal duration, perceived from the subjective perspective of an agent," bringing an emphasis on the temporal and subjective natures of situations into the definition. Finally, Pew [10] defines situation as a set of environmental conditions and system states with which the participant is interacting that can be characterized uniquely by a set of information, knowledge, and response options. We see again here a focus on the internal states of a system which can be captured in some way. We mainly follow Barwise, but view reality in the subjective sense of Barsalou, in that a situation is inherently relational, yet only exists in the internal representation of a particular agent. Clearly there are connections between the situations that interacting agents perceive, and indeed these connections are precisely the focus of this work. Combining these definitions, our working definition for situation is roughly the following.

Definition 1: A *situation* is any part the subjective internal representation of an agent which can be comprehended as a whole by that agent through defining how it interacts with or is related to other parts of the representation in that agent.

B. Mathematics

In order to discuss the mathematics of probabilistic situations, we must first lay the ground-work for the mathematics that we will use. Readers unfamiliar with category theory or probability theory based on measures can find basic definitions and results in the standard references such as [11], [12], [13].

Recall that a monoidal category is category $\mathcal{C}$ along with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative up to natural isomorphism and a distinguished object $I \in \mathcal{C}$ which functions as a left and right unit up to natural isomorphism. These are also required to satisfy a natural coherence condition (see [11] for the explicit diagrams). If one also has natural isomorphisms $X \otimes Y \simeq Y \otimes X$ for all $X, Y \in \mathcal{C}$ satisfying a coherence condition, then the monoidal structure is called symmetric. If for each fixed $Y \in \mathcal{C}$ the functor $- \otimes Y$ has a right adjoint $-^Y$, then the category is said to be closed.

Let $\mathcal{M}eas$ be the category of measurable spaces and measurable functions. For $X, Y \in \mathcal{M}eas$, let $Y^X = \mathrm{Hom}(X, Y)$ endowed with the smallest σ-algebra such that all of the evaluation maps $\mathrm{ev}_x : Y^X \to Y$ are measurable. We will denote this *evaluation σ-algebra* by $\Sigma_{\mathrm{ev}(X,Y)}$ or just Σ_{ev} when no confusion may occur.

If $f : X \to Y$, let $\Gamma_f : X \to X \times Y$ be the set function which sends $x \mapsto (x, f(x))$, *i.e.*, the graph of f. Define $X \otimes Y$ to be the set $X \times Y$ with the largest σ-algebra such that the graphs of all constant functions $X \to Y$ and $Y \to X$ are measurable. This then gives a symmetric monoidal structure on $\mathcal{M}eas$. Note that as sets, $X \otimes Y = X \times Y$ and that the projections $X \otimes Y \to X$ and $X \otimes Y \to Y$ are still measurable with this modified σ-algebra.

With these definitions, the evaluation morphism $X^Y \otimes Y \xrightarrow{\mathrm{ev}} X$ (sending $(f, y) \mapsto f(y)$) and the graph morphism $X \to (X \otimes Y)^Y$ (sending $x \mapsto \Gamma_x$) are measurable. Hence the usual adjunction $- \otimes Y \dashv -^Y$ extends to $\mathcal{M}eas$ and the category is closed with respect to the symmetric monoidal structure.

If we denote by $\mathcal{P}X$ the set of probability measures on a measurable set (X, Σ_X), then we can give this set the weakest σ-algebra such that for each $B \in \Sigma_X$, the evaluation map

$$\mathrm{ev}_B : \mathcal{P}X \to [0, 1]$$

sending $P \mapsto P(B)$ is measurable. Then we

can define an endofunctor on $\mathcal{M}eas$ which sends $X \to \mathcal{P}X$ and a measurable function $f : X \to Y$ to the measurable function $\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$ which sends a probability measure P on X to its pushforward measure f_*P on Y. It is straightforward to show that this defines a monad G on $\mathcal{M}eas$ called the Giry monad (see [14]).

In the category $\mathcal{S}et$, a relation is a subset of a cartesian product $R \subset X \times Y$ where $x \sim y$ if $(x, y) \in R$. This idea has been extended to regular categories by defining a categorical relation to be any monic into a (categorical) product (for full details on the conditions needed to compose relations, see [15]).

The notion of probabilistic functions has been around for decades, and has been thought of from a categorical perspective since at least [16]. Intuitively, the idea is clear: a function $f : X \to Y$ that sends a given x not to a particular y, but instead gives the probability that x will be sent to a given y. For finite (or even countably infinite) sets with the powerset (discrete) algebra, this is exactly what one can do. When we want to consider more general spaces, however, we need to find a suitable generalization of this concept. One solution is the Kleisli category $\mathcal{K}(G) = \mathcal{P}$ of the Giry monad which Lawvere called the category of probabilistic mappings. This category has been well studied (see [17], [14], [16], for example), but viewing its morphisms as probabilistic mappings has a serious flaw. Namely, the restriction that the probability measures on the codomain vary measurably is too weak and only allows one to specify certain types of dependencies between these measures.

For example, given a finite set X with the discrete σ-algebra, a probabilistic mapping in the sense of $\mathcal{P}$ is just a collection of arbitrary probability measures on Y parameterized by X. If, on the other hand, we defined a probabilistic function to be a distribution over all measurable functions, we would have a more general description that forces one to describe the dependencies.

Definition 2: The category $\mathcal{P}\mathcal{M}eas$ of *probabilistic measurable spaces* has all measurable spaces (X, Σ_X) as objects, and a morphism $P : X \to Y$ is a probability measure on the space of measurable functions $\mathrm{Hom}_{\mathcal{M}eas}(X, Y)$.

Note that given a $\mathcal{P}\mathcal{M}eas$-morphism, we can "marginalize" to obtain a $\mathcal{P}$-morphism.

III. PROBABILISTIC SITUATIONS

There have been many previous mathematical definitions of situations, most notably the situational calculus of Shanahan–McCarthy–Hayes (see [8], [18]) and the situation theory of Barwise–Perry–Devlin (see [19], [20]). The authors are not aware of previous work on the mathematical definition of a situation which incorporates probabilities fundamentally. To do so would require an extensive theory of probabilistic relations, which is currently lacking.

There have been several attempts to generalize the notion of relation to the probabilistic setting. Among these, [21], [17], [22] are recent examples of taking the Kleisli category of the Giry monad as the basis for probabilistic relations. Another approach by Jakobsen and Lychagin [23], attempts to describe probabilistic relations in terms of probability measures on product spaces. Categorically, relations are defined to be monics into products since set-theoretic relations can be realized as subsets of products. While this leads to interesting mathematics (see [15], for instance), it is not applicable to the category $\mathcal{P}$ because of the lack of structure of being a regular category (*e.g.*, only weak limits exist). Moreover, it is not clear how to use a family of probability measures on a product (*i.e.*, a $\mathcal{P}$-morphism into a product) to model some simple types of probabilistic relations.

We propose an alternate intuitive definition for a probabilistic relation which then leads to an obvious generalization. Intuitively, a probabilistic relation should just be a probability distribution on the set of all relations. Recall that in $\mathcal{S}et$, subsets of a given set are in bijection with characteristic functions. Hence we can also view relations as characteristic functions on products. With this in mind, a reasonable definition for a probabilistic relation is a probabilistic characteristic function on a product. That is, a $\mathcal{P}\mathcal{M}eas$ morphism $r : X \times Y \to \{0, 1\}$. We would like to be able to account not only for uncertainty in the relationships, but also uncertainty in our characterization of the relationships. To this end,

we can replace the set $\{0,1\}$ with $[0,1]$ and obtain the following definition.

Definition 3: Define a (binary) *generalized probabilistic relation* from X to Y to be a $\mathcal{PMeas}$ morphism $X \otimes Y \rightarrow [0,1]$—*i.e.*, a probability space

$$\left(\mathrm{Hom}_{\mathcal{Meas}}(X \otimes Y, [0,1]), \Sigma_{\mathrm{ev}}, P\right).$$

Intuitively, this roughly corresponds to assigning a probability distribution on $[0,1]$ for each pair $(x,y) \in X \times Y$. We will denote a generalized probabilistic relation $(\mathrm{Hom}_{\mathcal{Meas}}(X \otimes Y, [0,1]), \Sigma_{\mathrm{ev}}, P)$ between X and Y as an arrow $X \xrightarrow{P} Y$. This definition of probabilistic relation allows one to specify not only a measure of certainty that two elements are related, but also a confidence in this certainty. Now we are in a position to define probabilistic situations mathematically. Recall that our definition of situation is based on the subjective internal representation of a particular agent. Thus we wish to construct a mathematical definition of situation that is independent of the particular model representing an agent.

Definition 4: Suppose that we have a fixed agent X and let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be sets of parts of the internal representation of the agent (not necessarily disjoint), each with a given σ-algebra which captures the ability of the agent to reason over the various parts. An *X-probabilistic situation* is a family of generalized probabilistic relations $\{\mathcal{O}_i \rightarrow \mathcal{O}_j\}$.

Although the level of generality needed to work with a wide variety of implementations is still a work in progress, we believe that with a sufficiently broad understanding of what constitutes a "part" of an internal representation, our mathematical definition of a situation can handle any type of particular implementation.

We close this section with an example which we believe helps to illustrate the way that we envision this set of mathematics being used.

Example 5: Consider the setting of a system T consisting of a surveillance camera in an urban area that is attempting to track all vehicles which pass through its field of view. One traditional approach to this problem would include developing a method to detect which pixels change

from frame to frame and then associate these changes using various kinematic and physical models to form tracklets. These tracklets can then be stitched together to form entire tracks, which are stored as a sequence of positions.

In our approach, we would first develop a relevant representation of the scene. If $\mathcal{O}_b$ is the set of buildings and $\mathcal{O}_s$ the set of streets (each given the discrete σ-algebra), then we would begin by learning the relationships such as intersections, buildings which are neighbors, and which buildings are on which streets. From these relationships, we could construct probabilistic relations such as

$$\left(\mathrm{Hom}_{\mathcal{Meas}}(\mathcal{O}_s \otimes \mathcal{O}_s, [0,1]), \Sigma_{\mathrm{ev}}, I\right),$$

where the probability measure is supported on relations where pairs of intersecting streets are mapped to numbers close to 1. Then given video of moving vehicles, we would construct relations between sets of detected vehicles and other elements in the scene (in this example, buildings, streets, and other vehicles), thus situating the track of the vehicle in the scene. A T-probabilistic situation then, would be any diagram of relations between the sets $\mathcal{O}_v, \mathcal{O}_s$, and $\mathcal{O}_b$ such as
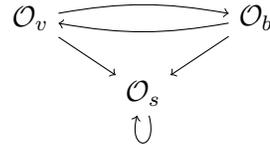
IV. FUTURE WORK

The two areas being developed are the modelling aspects of probabilistic situations and determination of the mathematical properties this requires of the category $\mathcal{PMeas}$. The properties of the category $\mathcal{PMeas}$ need to be further developed and verified to insure various constructions necessary for analysis are well defined. Furthermore we seek to understand whether these probabilistic relations are enough to quantitatively characterize the qualitative properties of situations. Situations are characterized not only by probabilistic relationships between objects but also properties of the individual objects. In the deterministic setting properties of a set X are

characterized by characteristic functions to the set $2 = \{0,1\}$ (false/true) which corresponds to a (classical) relation on X. This is a special instance of a relation which in our context, using the fact $X \otimes 1 \cong X$, corresponds to a $\mathcal{PM}eas$ arrow $X \rightarrow 1$. These desired properties are reminiscent of requiring a topos which is an area that we have already approached the problem from. The conditions of a topos however are too strong (stronger than even regularity) and have led us to looking at the monoidal closed structure as an alternative.

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