

Active–Passive Networked Multiagent Systems

Tansel Yucelen and John Daniel Peterson

Abstract—This paper introduces an active–passive networked multiagent system framework, which consists of agents subject to exogenous inputs (active agents) and agents without any inputs (passive agents), and analyze its convergence using Lyapunov stability. Apart from the existing relevant literature, where either none of the agents are subject to exogenous inputs (i.e., average consensus problem) or all agents are subject to these inputs (i.e., dynamic average consensus problem), the key feature of our approach is that the states of all agents converge to the average of the exogenous inputs applied only to the active agents, where these inputs may or may not overlap within the active agents.

I. INTRODUCTION

The objective of this paper is to develop a fundamental distributed control approach to unify and generalize the results applied to two important classes of networked multiagent systems, namely leaderless networks and leader–follower networks. The agreement of networked agents upon certain initial quantities of interest is called average consensus, which is a well studied class of leaderless networks (see, for example, [1]–[6] and references therein). Since this class of leaderless networks is insufficient for applications in dynamic environments, such as average consensus of distance measurements between static sensors and a moving target, [7]–[12] consider the dynamic average consensus problem – a problem that deals with the agreement of networked agents, where each agent is subject to an exogenous input. Throughout this paper, we say that an agent is active when it is subject to an exogenous input, otherwise we say that an agent is passive. From this point of view, the agents considered for the average consensus problem are all passive, whereas the agents considered for the dynamic average consensus problem are all active.

It should be noted that one may be interested in reaching the average of the exogenous inputs only applied to a specific set of agents in the network (e.g., see the underwater exploration scenario in Figure 1). For such cases the networked multiagent system is in fact heterogenous and consists of both active and passive agents. Motivating from this point of view, this paper studies an active–passive networked multiagent system framework and analyze its convergence

T. Yucelen is an Assistant Professor of the Mechanical and Aerospace Engineering Department and the Director of the Advanced Systems Research Laboratory at the Missouri University of Science and Technology, Rolla, MO 65409, USA (e-mail: yucelen@mst.edu).

J. D. Peterson is a Graduate Research Assistant of the Mechanical and Aerospace Engineering Department and a member of the Advanced Systems Research Laboratory at the Missouri University of Science and Technology, Rolla, MO 65409, USA (e-mail: jdp6q5@mst.edu).

This research was supported by the University of Missouri Research Board and the Missouri Space Grant Consortium.

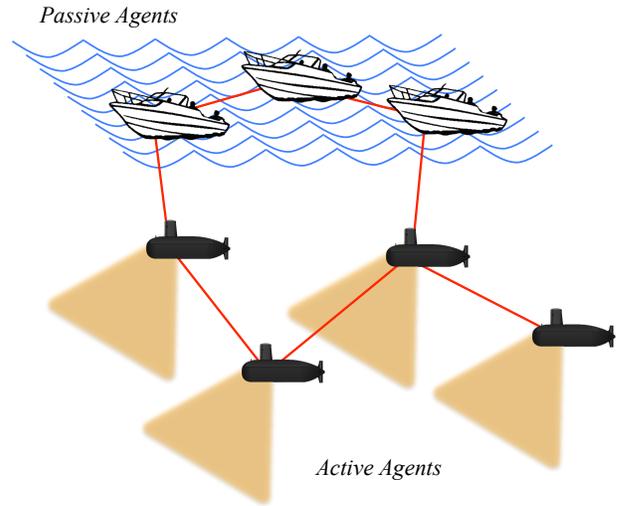


Fig. 1. An underwater exploration scenario, where only submarines are subject to exogenous inputs from the ocean surface, and hence, are active agents in the network formed by both submarines and boats (lines denote communication links).

using Lyapunov stability. Apart from the existing relevant literature, the key feature of our approach is that the states of all agents converge to the average of the exogenous inputs applied only to the active agents, where these inputs may or may not overlap within the active agents. We further show that the proposed algorithm acts as an average consensus algorithm only when there are no active agents, and as a dynamic average consensus algorithm only when all agents are active.

A. Standard Notation

In this paper, \mathbb{R} denotes the set of real numbers, \mathbb{R}^n denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, \mathbb{R}_+ denotes the set of positive real numbers, $\mathbb{R}_+^{n \times n}$ (resp., $\overline{\mathbb{R}}_+^{n \times n}$) denotes the set of $n \times n$ positive-definite (resp., nonnegative-definite) real matrices, $\mathbb{S}_+^{n \times n}$ (resp., $\overline{\mathbb{S}}_+^{n \times n}$) denotes the set of $n \times n$ symmetric positive-definite (resp., symmetric nonnegative-definite) real matrices, \mathbb{Z} denotes the set of integers, \mathbb{Z}_+ (resp., $\overline{\mathbb{Z}}_+$) denotes the set of positive (resp., nonnegative) integers, $\mathbf{0}_n$ denotes the $n \times 1$ vector of all zeros, $\mathbf{1}_n$ denotes the $n \times 1$ vector of all ones, $\mathbf{0}_{n \times n}$ denotes the $n \times n$ zero matrix, and \mathbf{I}_n denotes the $n \times n$ identity matrix. In addition, we write $(\cdot)^T$ for transpose, $(\cdot)^{-1}$ for inverse, $(\cdot)^\dagger$ for generalized inverse, $\|\cdot\|_2$ for the Euclidian norm, $\lambda_{\min}(A)$ (resp., $\lambda_{\max}(A)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix A , $\lambda_i(A)$ for the i -th eigenvalue of A (A is symmetric and the eigenvalues are ordered from least to greatest value),

$\text{diag}(a)$ for the diagonal matrix with the vector a on its diagonal, and $[A]_{ij}$ for the entry of the matrix A on the i -th row and j -th column.

B. Graph-Theoretical Notation

We now recall some of the basic notions from graph theory, where we refer to [1], [13] for further details. In the multiagent literature, graphs are broadly adopted to encode interactions in networked systems. An *undirected* graph \mathcal{G} is defined by a set $\mathcal{V}_{\mathcal{G}} = \{1, \dots, n\}$ of *nodes* and a set $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$ of *edges*. If $(i, j) \in \mathcal{E}_{\mathcal{G}}$, then the nodes i and j are *neighbors* and the neighboring relation is indicated with $i \sim j$. The *degree* of a node is given by the number of its neighbors. Letting d_i be the degree of node i , then the *degree* matrix of a graph \mathcal{G} , $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{n \times n}$, is given by $\mathcal{D}(\mathcal{G}) \triangleq \text{diag}(d)$, $d = [d_1, \dots, d_n]^T$. A *path* $i_0 i_1 \dots i_L$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \dots, L$, and a graph \mathcal{G} is *connected* if there is a path between any pair of distinct nodes. The *adjacency* matrix of a graph \mathcal{G} , $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{n \times n}$, is given by

$$[\mathcal{A}(\mathcal{G})]_{ij} \triangleq \begin{cases} 1, & \text{if } (i, j) \in \mathcal{E}_{\mathcal{G}}, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

The *Laplacian* matrix of a graph, $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{S}}_+^{n \times n}$, playing a central role in many graph theoretic treatments of multiagent systems, is given by

$$\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G}). \quad (2)$$

Throughout this paper, we model a given multiagent system by a connected, undirected graph \mathcal{G} , where nodes and edges represent agents and inter-agent communication links, respectively, and use the following lemmas.

Lemma 1 [1]. The spectrum of the Laplacian of a connected, undirected graph can be ordered as

$$0 = \lambda_1(\mathcal{L}(\mathcal{G})) < \lambda_2(\mathcal{L}(\mathcal{G})) \leq \dots \leq \lambda_n(\mathcal{L}(\mathcal{G})), \quad (3)$$

with $\mathbf{1}_n$ as the eigenvector corresponding to the zero eigenvalue $\lambda_1(\mathcal{L}(\mathcal{G}))$ and $\mathcal{L}(\mathcal{G})\mathbf{1}_n = \mathbf{0}_n$ and $e^{\mathcal{L}(\mathcal{G})}\mathbf{1}_n = \mathbf{1}_n$.

Lemma 2. Let $K = \text{diag}(k)$, $k = [k_1, k_2, \dots, k_n]^T$, $k_i \in \overline{\mathbb{Z}}_+$, $i = 1, \dots, n$, and assume that at least one element of k is nonzero. Then, for the Laplacian of a connected, undirected graph,

$$\mathcal{F}(\mathcal{G}) \triangleq \mathcal{L}(\mathcal{G}) + K \in \mathbb{S}_+^{n \times n}, \quad (4)$$

and $\det(\mathcal{F}(\mathcal{G})) \neq 0$.

Proof. We prove this lemma by relating $\mathcal{F}(\mathcal{G})$ to a partition of an expanded graph. For this purpose, let the expanded graph $\bar{\mathcal{G}}$ be constructed by adding m nodes to the graph \mathcal{G} having n nodes. In addition, let the degree of node 1, node 2, \dots , node n of \mathcal{G} are increased by k_1, k_2, \dots, k_n , respectively, due to this expansion ($m = k_1 + k_2 + \dots + k_n$ in this case) and let the expanded graph $\bar{\mathcal{G}}$ consisting of $\bar{n} = n + m$ nodes to be connected and undirected. Note that this expanded graph exists, which satisfies the properties stated in Lemma 1, since it is assumed that at least one element of k is nonzero, and hence, $m \geq 1$, or equivalently, $\bar{n} > n$. Furthermore, let

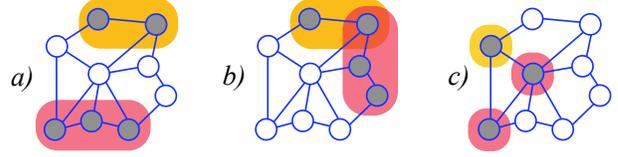


Fig. 2. An active-passive networked multiagent system with a) two non-overlapping non-isolated inputs, b) two overlapping non-isolated inputs, and c) two non-overlapping inputs, where one of them is isolated and the other one is non-isolated (lines denote communication links, gray circles denote active agents, white circles denote passive agents, and shaded areas denote the exogenous inputs interacting with this system).

the nodes of the original graph \mathcal{G} be indexed first in the expanded graph $\bar{\mathcal{G}}$. Define $\mathcal{B}(\bar{\mathcal{G}}) \in \mathbb{R}^{\bar{n} \times \bar{p}}$ to be the (node-edge) incidence matrix of graph $\bar{\mathcal{G}}$, where $[\mathcal{B}(\bar{\mathcal{G}})]_{ij} = 1$ if node i is the head of edge j , $[\mathcal{B}(\bar{\mathcal{G}})]_{ij} = -1$, if node i is the tail of edge j , and $[\mathcal{B}(\bar{\mathcal{G}})]_{ij} = 0$ otherwise, and let this incidence matrix be partitioned as

$$\mathcal{B}(\bar{\mathcal{G}}) = [\mathcal{B}_1^T(\bar{\mathcal{G}}), \mathcal{B}_2^T(\bar{\mathcal{G}})]^T, \quad (5)$$

where $\mathcal{B}_1(\bar{\mathcal{G}}) \in \mathbb{R}^{n \times \bar{p}}$ and $\mathcal{B}_2(\bar{\mathcal{G}}) \in \mathbb{R}^{m \times \bar{p}}$ with \bar{p} denoting the number of edges of the expanded graph $\bar{\mathcal{G}}$. Since the graph Laplacian of the expanded graph $\bar{\mathcal{G}}$ can be defined as

$$\mathcal{L}(\bar{\mathcal{G}}) = \mathcal{B}(\bar{\mathcal{G}})\mathcal{B}^T(\bar{\mathcal{G}}), \quad (6)$$

it follows that

$$\mathcal{L}(\bar{\mathcal{G}}) = \begin{bmatrix} \mathcal{P}(\bar{\mathcal{G}}) & \mathcal{Q}^T(\bar{\mathcal{G}}) \\ \mathcal{Q}(\bar{\mathcal{G}}) & \mathcal{R}(\bar{\mathcal{G}}) \end{bmatrix}, \quad (7)$$

where $\mathcal{P}(\bar{\mathcal{G}}) = \mathcal{B}_1(\bar{\mathcal{G}})\mathcal{B}_1^T(\bar{\mathcal{G}})$, $\mathcal{Q}(\bar{\mathcal{G}}) = \mathcal{B}_2(\bar{\mathcal{G}})\mathcal{B}_1^T(\bar{\mathcal{G}})$, and $\mathcal{R}(\bar{\mathcal{G}}) = \mathcal{B}_2(\bar{\mathcal{G}})\mathcal{B}_2^T(\bar{\mathcal{G}})$. Since $\mathcal{P}(\bar{\mathcal{G}}) \in \mathbb{S}_+^{n \times n}$ (and $\mathcal{R}(\bar{\mathcal{G}}) \in \mathbb{S}_+^{m \times m}$) holds [1], [14] and $\mathcal{P}(\bar{\mathcal{G}}) = \mathcal{F}(\mathcal{G}) + K$ based on the above construction of the expanded graph, then it follows that $\mathcal{F}(\mathcal{G})$ is a nonsingular, symmetric, and positive-definite real matrix. \square

Lemma 3 [15]. The Laplacian of a connected, undirected graph satisfies

$$\mathcal{L}(\mathcal{G})\mathcal{L}^\dagger(\mathcal{G}) = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T. \quad (8)$$

II. PROBLEM FORMULATION

In this section, we present the problem formulation for active-passive networked multiagent systems. Specifically, consider a system of n agents exchanging information among each other using their local measurements according to a connected, undirected graph \mathcal{G} . In addition, consider that there exists $m \geq 1$ exogenous inputs that interact with this system.

Definition 1. If agent i , $i = 1, \dots, n$, is subject to one or more exogenous inputs (resp., no exogenous inputs), then it is an active agent (resp., passive agent).

Definition 2. If an exogenous input interacts with only one agent (resp., multiple agents), then it is an isolated input (resp., non-isolated input).

An illustration of Definition 2 is given in Figure 2, where it also highlights that the exogenous inputs may or may not overlap within the active agents.

In this paper, we are interested in the problem of driving the states of all (active and passive) agents to the average

of the applied exogenous inputs. Motivating from this standpoint, we propose an integral action-based distributed control approach given by

$$\begin{aligned} \dot{x}_i(t) &= -\sum_{i \sim j} (x_i(t) - x_j(t)) + \sum_{i \sim j} (\xi_i(t) - \xi_j(t)) \\ &\quad - \sum_{i \sim h} (x_i(t) - c_h(t)), \quad x_i(0) = x_{i0}, \end{aligned} \quad (9)$$

$$\dot{\xi}_i(t) = -\sum_{i \sim j} (x_i(t) - x_j(t)), \quad \xi_i(0) = \xi_{i0}, \quad (10)$$

where $x_i(t) \in \mathbb{R}$ and $\xi_i(t) \in \mathbb{R}$ denote the state and the integral action of agent i , $i = 1, \dots, n$, respectively, and $c_h(t) \in \mathbb{R}$, $h = 1, \dots, m$, denotes an exogenous input applied to this agent. Similar to the $i \sim j$ notation indicating the neighboring relation between agents, we use $i \sim h$ to indicate the exogenous inputs that an agent is subject to. To elucidate this point, we introduce the following example.

Example 1. Consider the active-passive networked multiagent system given in Figure 3. It follows from (9) and (10) that three agents implement the following distributed control approaches locally

$$\dot{x}_1(t) = -(x_1(t) - x_2(t)) + (\xi_1(t) - \xi_2(t)), \quad (11)$$

$$\dot{\xi}_1(t) = -(x_1(t) - x_2(t)), \quad (12)$$

$$\begin{aligned} \dot{x}_2(t) &= -(x_2(t) - x_1(t)) - (x_2(t) - x_3(t)) \\ &\quad + (\xi_2(t) - \xi_1(t)) + (\xi_2(t) - \xi_3(t)) \\ &\quad - (x_2(t) - c_1(t)) - (x_2(t) - c_2(t)), \end{aligned} \quad (13)$$

$$\dot{\xi}_2(t) = -(x_2(t) - x_1(t)) - (x_2(t) - x_3(t)), \quad (14)$$

$$\begin{aligned} \dot{x}_3(t) &= -(x_3(t) - x_2(t)) + (\xi_3(t) - \xi_2(t)) \\ &\quad - (x_3(t) - c_2(t)), \end{aligned} \quad (15)$$

$$\dot{\xi}_3(t) = -(x_3(t) - x_2(t)), \quad (16)$$

since agent 1 is passive, agent 2 is active and subject to $c_1(t)$ and $c_2(t)$, and agent 3 is active and subject to $c_2(t)$. \triangle

Remark 1. The proposed integral action-based distributed control approach in (9) is applied to agents having dynamics of the form $\dot{x}_i(t) = u_i(t)$, where $u_i(t) \in \mathbb{R}$ denotes the control input of agent i , $i = 1, \dots, n$, satisfying the right hand side of (9) along with (10). For agents having complex dynamics, one can design low-level feedback controllers (or assume their existence) for suppressing existing dynamics and enforcing $\dot{x}_i(t) = u_i(t)$ (see, for example, Example 6.3 of [17]). Therefore, (9) and (10) are considered without loss of generality and to focus the main results of this paper.

Remark 2. In discrete-time, the proposed integral action-based distributed control approach is given by

$$\begin{aligned} x_i^+ &= x_i + \delta \left[-\sum_{i \sim j} (x_i - x_j) + \sum_{i \sim j} (\xi_i - \xi_j) \right. \\ &\quad \left. - \sum_{i \sim h} (x_i - c_h) \right], \end{aligned} \quad (17)$$

$$\xi_i^+ = \xi_i + \delta \left[-\sum_{i \sim j} (x_i - x_j) \right], \quad (18)$$

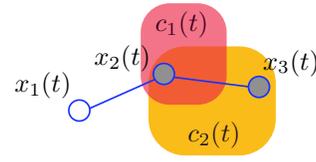


Fig. 3. The active-passive networked multiagent system in Example 1 (lines denote communication links, two gray circles denote active agents $x_2(t)$ and $x_3(t)$, white circle denotes the passive agent $x_1(t)$, and two shaded areas denote applied two exogenous inputs $c_1(t)$ and $c_2(t)$).

where x_i^+ and ξ_i^+ (resp., x_i and ξ_i) are the next state and integral action (resp., the current state and integral action) of agent i , $i = 1, \dots, n$, respectively, and δ is the step size of iterations. We will conduct all of our analysis in continuous-time.

Next, let

$$x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n, \quad (19)$$

$$\xi(t) = [\xi_1(t), \xi_2(t), \dots, \xi_n(t)]^T \in \mathbb{R}^n, \quad (20)$$

$$c(t) = [c_1(t), c_2(t), \dots, c_m(t), 0, \dots, 0] \in \mathbb{R}^n, \quad (21)$$

where we assume $m \leq n$ for the ease of the following notation and without loss of generality. We can now write (9) and (10) in a compact form as

$$\begin{aligned} \dot{x}(t) &= -\mathcal{L}(\mathcal{G})x(t) + \mathcal{L}(\mathcal{G})\xi(t) - K_1x(t) + K_2c(t), \\ x(0) &= x_0, \end{aligned} \quad (22)$$

$$\dot{\xi}(t) = -\mathcal{L}(\mathcal{G})x(t), \quad \xi(0) = \xi_0, \quad (23)$$

where $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{S}}_+^{n \times n}$ satisfies Lemma 1,

$$K_1 \triangleq \text{diag}([k_{1,1}, k_{1,2}, \dots, k_{1,n}]^T) \in \overline{\mathbb{S}}_+^{n \times n}, \quad (24)$$

with $k_{1,i} \in \overline{\mathbb{Z}}_+$ denoting the number of the exogenous inputs applied to agent i , $i = 1, \dots, n$, and

$$K_2 \triangleq \begin{bmatrix} k_{2,11} & k_{2,12} & \cdots & k_{2,1n} \\ k_{2,21} & k_{2,22} & \cdots & k_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{2,n1} & k_{2,n2} & \cdots & k_{2,nn} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (25)$$

with $k_{2,ih} = 1$ if the exogenous input $c_h(t)$, $h = 1, \dots, m$, is applied to agent i , $i = 1, \dots, n$, and $k_{2,ih} = 0$ otherwise. Note that

$$k_{1,i} = \sum_{j=1}^n k_{2,ij}. \quad (26)$$

To visualize this representation, we introduce the following example.

Example 2. For the active-passive networked multiagent system given Figure 3, one can write

$$K_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad (27)$$

since $n = 3$, $m = 2$, and $c(t) = [c_1(t), c_2(t), 0]^T$. \triangle

Since we are interested in driving the states of all (active and passive) agents to the average of the applied exogenous

inputs, let

$$\delta(t) \triangleq x(t) - \epsilon(t)\mathbf{1}_n \in \mathbb{R}^n, \quad (28)$$

$$\epsilon(t) \triangleq \frac{\mathbf{1}_n^\top K_2 c(t)}{\mathbf{1}_n^\top K_2 \mathbf{1}_n} \in \mathbb{R}, \quad (29)$$

be the error between $x_i(t)$, $i = 1, \dots, n$, and the average of the applied exogenous inputs $\epsilon(t)$. Based on (29), $\epsilon(t)$ can be equivalently written as

$$\begin{aligned} \epsilon(t) = & \left(k_{2,11}c_1(t) + k_{2,12}c_2(t) + \dots + k_{2,21}c_1(t) \right. \\ & \left. + k_{2,22}c_2(t) + \dots \right) / \left(k_{2,11} + k_{2,12} \right. \\ & \left. + \dots + k_{2,21} + k_{2,22} + \dots \right), \end{aligned} \quad (30)$$

which is the average of the applied exogenous inputs. Furthermore, note for the special case of isolated exogenous inputs that

$$\epsilon(t) = \left(c_1(t) + c_2(t) + \dots + c_m(t) \right) / m. \quad (31)$$

Based on the problem formulation introduced in this section, the next section analyze the stability properties of the error given by (28) in the presence of constant inputs. Furthermore, we also discuss how the proposed distributed control approach given by (9) and (10) can be generalized to allow time-varying inputs without stating formal proofs due to space limitations.

III. CONSENSUS IN THE PRESENCE OF ACTIVE AGENTS

Let $c_h(t) = c_h$ be constant exogenous inputs for $h = 1, \dots, m$ (i.e., constant inputs applied to active agents are considered). Since $c(t) = c$, and hence, $\epsilon(t) = \epsilon$ from (29), it follows from (28) and $\mathcal{L}(\mathcal{G})\mathbf{1}_n = \mathbf{0}_n$ of Lemma 1 that

$$\begin{aligned} \dot{\delta}(t) &= -\mathcal{L}(\mathcal{G})[\delta(t) + \epsilon\mathbf{1}_n] + \mathcal{L}(\mathcal{G})\xi(t) - K_1[\delta(t) \\ & \quad + \epsilon\mathbf{1}_n] + K_2c(t) \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})\xi(t) - [K_1\mathbf{1}_n\epsilon - K_2c] \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})\xi(t) - \left[\frac{K_1\mathbf{1}_n\mathbf{1}_n^\top K_2c}{\mathbf{1}_n^\top K_2\mathbf{1}_n} - K_2c \right] \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})\xi(t) - L_c K_2c, \end{aligned} \quad (32)$$

where

$$\mathcal{F}(\mathcal{G}) \triangleq \mathcal{L}(\mathcal{G}) + K_1, \quad (33)$$

and

$$L_c \triangleq \frac{K_1\mathbf{1}_n\mathbf{1}_n^\top}{\mathbf{1}_n^\top K_2\mathbf{1}_n} - I_n. \quad (34)$$

Note that $\mathcal{F}(\mathcal{G}) \in \mathbb{S}_+^{n \times n}$ from Lemma 2 and

$$\begin{aligned} \mathbf{1}_n^\top L_c &= \mathbf{1}_n^\top \left[\frac{K_1\mathbf{1}_n\mathbf{1}_n^\top}{\mathbf{1}_n^\top K_2\mathbf{1}_n} - I_n \right] \\ &= \frac{\mathbf{1}_n^\top K_1\mathbf{1}_n}{\mathbf{1}_n^\top K_2\mathbf{1}_n} - \mathbf{1}_n^\top \\ &= 0, \end{aligned} \quad (35)$$

since $(\mathbf{1}_n^\top K_1\mathbf{1}_n) / (\mathbf{1}_n^\top K_2\mathbf{1}_n) = 1$ from $k_{1,i} = \sum_{j=1}^n k_{2,ij}$.

Next, letting

$$e(t) \triangleq \xi(t) - \mathcal{L}^\dagger(\mathcal{G})L_c K_2c, \quad (36)$$

and using this in (32) along with Lemma 3 yields

$$\begin{aligned} \dot{\delta}(t) &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})[e(t) + \mathcal{L}^\dagger(\mathcal{G})L_c K_2c] - L_c K_2c \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})e(t) + \left[I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top \right] L_c K_2c \\ & \quad - L_c K_2c \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})e(t), \end{aligned} \quad (37)$$

since

$$\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top L_c K_2c = 0, \quad (38)$$

as a direct consequence of (35). In addition, differentiating $e(t)$ with respect to time yields

$$\begin{aligned} \dot{e}(t) &= -\mathcal{L}(\mathcal{G})[\delta(t) + \epsilon\mathbf{1}_n] \\ &= -\mathcal{L}(\mathcal{G})\delta(t), \end{aligned} \quad (39)$$

where $\mathcal{L}(\mathcal{G})\mathbf{1}_n = \mathbf{0}_n$ of Lemma 1 is used. The next theorem shows that the state of all agents $x_i(t)$, $i = 1, \dots, n$ asymptotically converge to ϵ .

Theorem 1. Consider the networked multiagent system given by (9) and (10), where agents exchange information using local measurements and with \mathcal{G} defining a connected, undirected graph topology. Then, the closed-loop error dynamics defined by (37) and (39) are Lyapunov stable for all initial conditions and $\delta(t)$ asymptotically vanishes.

Proof. Consider the Lyapunov function candidate given by

$$V(\delta, e) = \frac{1}{2}\delta^\top \delta + \frac{1}{2}e^\top e, \quad (40)$$

and note that $V(0, 0) = 0$ and $V(\delta, e) > 0$ for all $(\delta, e) \neq (0, 0)$. Differentiating (40) along the trajectories of (37) and (39) yields

$$\dot{V}(\delta(t), e(t)) = -\delta^\top(t)\mathcal{F}(\mathcal{G})\delta(t) \leq 0. \quad (41)$$

Hence, the closed-loop error dynamics given by (37) and (39) are Lyapunov stable for all initial conditions. Because $\dot{V}(\delta(t), e(t))$ is bounded for all $t \in \mathbb{R}_+$, it follows from Barbalat's lemma [16] that

$$\lim_{t \rightarrow \infty} \dot{V}(\delta(t), e(t)) = 0, \quad (42)$$

which consequently shows that $\delta(t)$ asymptotically vanishes since $\mathcal{F}(\mathcal{G}) \in \mathbb{S}_+^{n \times n}$. \square

In Theorem 1, we establish the convergence characteristics of the proposed integral action-based distributed control approach, i.e., we show that all $x_i(t)$, $i = 1, \dots, n$ asymptotically converge to ϵ . The results of this theorem is illustrated in the next example.

Example 3. Consider a networked multiagent system represented by the graph shown in Figure 4 with 5 active agents and 20 passive agents. Let the active agents be subject to random and isolated constant inputs and let all agents have arbitrary initial conditions and $\xi_i(0) = 0$, $i = 1, \dots, n$. Figure 5 shows the response with the proposed

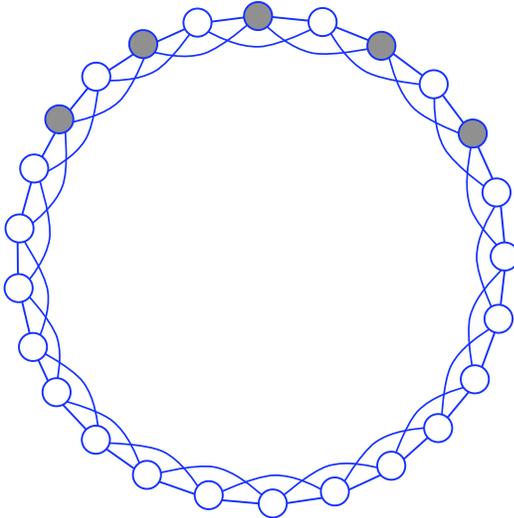


Fig. 4. Graph used in Figure 5 of Example 3 (lines denote communication links, gray circles denote active agents, and white circles denote passive agents).

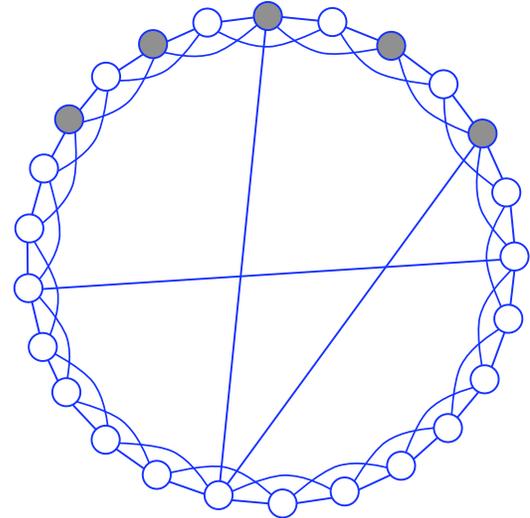


Fig. 6. Graph used in Figure 7 of Example 3 (lines denote communication links, gray circles denote active agents, and white circles denote passive agents).

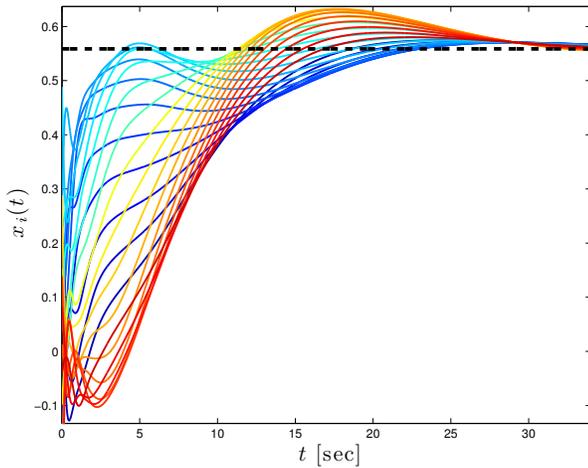


Fig. 5. Response of the networked multiagent system in Example 3 with graph topology in Figure 4 and with the distributed controller given by (9) and (10) (solid lines denote agent positions and dashed line denote the average of applied constant inputs).

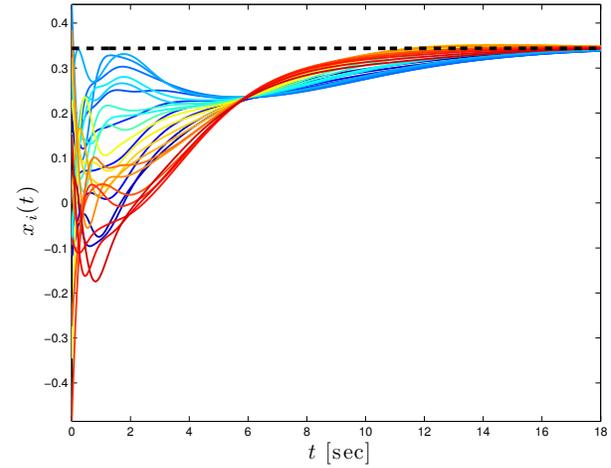


Fig. 7. Response of the networked multiagent system in Example 3 with graph topology in Figure 6 and with the distributed controller given by (9) and (10) (solid lines denote agent positions and dashed line denote the average of applied constant inputs).

distributed controller given by (9) and (10), where all agents converge to the average of the applied exogenous inputs as expected. In order to illustrate the effect of graph topology on convergence, also consider the graph shown in Figure 6. Figure 7 shows the response with the proposed distributed controller given by (9) and (10), where the convergence time is faster due to the increased connectivity of the graph. As a future research direction, we will generalize the proposed distributed controller given by (9) and (10) by multiplying the terms on the left hand side of these equations by appropriate constants to control the convergence time independently of the graph topology. \triangle

In order to deal with the case when active agents are subject to time-varying exogenous inputs $c_h(t)$, $h = 1, \dots, m$

(subject to the assumption that both $c_h(t)$ and $\dot{c}_h(t)$ are bounded for each input h , $h = 1, \dots, m$), one can modify the distributed controller given by (9) and (10) as

$$\begin{aligned} \dot{x}_i(t) = & - \sum_{i \sim j} (x_i(t) - x_j(t)) + \sum_{i \sim j} (\xi_i(t) - \xi_j(t)) \\ & - \sum_{i \sim h} (x_i(t) - c_h(t)), \quad x_i(0) = x_{i0}, \end{aligned} \quad (43)$$

$$\begin{aligned} \dot{\xi}_i(t) = & - \left[\sum_{i \sim j} (x_i(t) - x_j(t)) + \sigma \xi_i(t) \right], \\ & \xi_i(0) = \xi_{i0}, \end{aligned} \quad (44)$$

where $\sigma \in \mathbb{R}_+$, a sufficiently small constant, is introduced to guarantee boundedness in the presence of time-varying

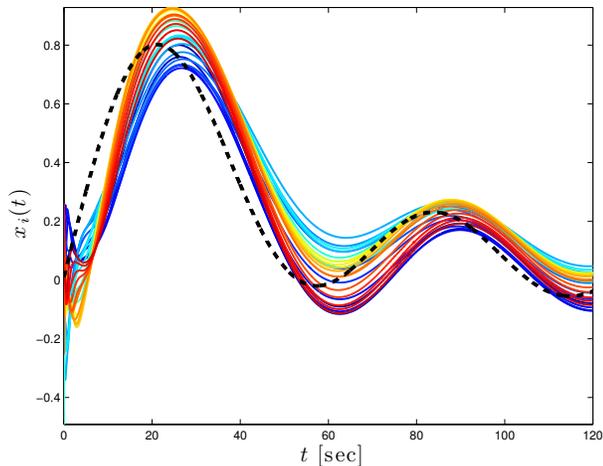


Fig. 8. Response of the networked multiagent system in Example 4 with graph topology in Figure 6 and with the distributed controller given by (43) and (44) subject to $\sigma = 0.1$ (solid lines denote agent positions and dashed line denote the average of applied time-varying sinusoidal inputs).

exogenous inputs. This controller is illustrated in the next example.

Example 4. Consider a networked multiagent system represented by the graph shown in Figure 6 with 5 active agents (subject to time-varying sinusoidal inputs) and 20 passive agents. Let all agents have arbitrary initial conditions and $\xi_i(0) = 0$, $i = 1, \dots, n$. Figure 8 shows the response with the proposed distributed controller given by (43) and (44), where all agents approximately converge to the average of the applied exogenous inputs. \triangle

Remark 3. We now draw connections between the proposed distributed control approach given by (9) and (10) and other approaches to highlight why our methodology presents a generalization among those approaches. Specifically, assume that there does not exist any active agent in the network as the first special case. In this case, the proposed algorithm given by (9) and (10) acts as an integral-based average consensus algorithm, where all passive agents converge to the average of their initial conditions (this proof is omitted due to space limitations). As the second special case, consider that all agents in the network are active and they are subject to isolated exogenous inputs only. In this case, it follows from (9) and (10) that we recover a dynamic average consensus algorithm [7]–[12].

Remark 4. The authors of [18] and [19] propose a selective gossip algorithm in the context of efficiently computing sparse approximations of network data via considering nodes that contain significant energy, which is similar in spirit to the problem considered in this paper. However, our results go beyond [18] in that the proposed framework of this paper is applicable to situations where applied exogenous inputs may overlap within the active agents and such inputs can be time-varying, which is important for applications in dynamic environments.

IV. CONCLUSION

We investigated a system consisting of agents subject to exogenous inputs (active agents) and agents without any inputs (passive agents). Specifically, we introduced and analyzed a new integral action-based distributed control approach that drives the states of all agents to the average of the exogenous inputs applied only to the active agents, where these inputs may or may not overlap within the active agents. Several illustrative numerical examples illustrated the proposed approach.

REFERENCES

- [1] M. Mesbahi and M. Egerstedt, "Graph Theoretic Methods in Multi-agent Networks," *Princeton University Press*, 2010.
- [2] R. Olfati-Saber, J. Fax, and R. M. Murray, "Consensus and Cooperation in Networked Multiagent Systems," *Proc. of the IEEE*, vol. 95, pp. 215–233, 2007.
- [3] W. Ren and R. W. Beard, "Distributed Consensus in Multivehicle Cooperative Control: Theory and Applications," *Springer*, 2010.
- [4] G. Antonelli, "Interconnected Dynamic Systems: An Overview on Distributed Control," *IEEE Contr. Sys. Mag.*, vol. 33, pp. 76–88, 2013.
- [5] Y. Cao, W. Yu, W. Ren, and G. Chen, "An Overview of Recent Progress in the Study of Distributed Multiagent Coordination," *IEEE Trans. on Ind. Inf.*, vol. 9, pp. 427–438, 2013.
- [6] R. M. Murray, "Control in an Information Rich World: Report of the Panel on Future Directions in Control, Dynamics, and Systems," *SIAM*, 2003.
- [7] D. P. Spanos, R. Olfati-Saber, and R. M. Murray, "Dynamic Consensus for Mobile Networks," *IFAC World Congress*, 2005.
- [8] R. Olfati-Saber and J. Shamma, "Consensus Filters for Sensor Networks and Distributed Sensor Fusion," *IEEE Conference on Decision and Control*, 2005.
- [9] R. A. Freeman, P. Yang, and K. M. Lynch, "Stability and Convergence Properties of Dynamic Average Consensus Estimators," *IEEE Conference on Decision and Control*, 2006.
- [10] H. Bai, R. A. Freeman, and K. M. Lynch, "Robust Dynamic Average Consensus of Time-Varying Inputs," *IEEE Conference on Decision and Control*, 2010.
- [11] C. N. Taylor, R. W. Beard, and J. Humpherys, "Dynamic Input Consensus using Integrators," *American Control Conference*, 2011.
- [12] F. Chen, Y. Cao, and W. Ren, "Distributed Average Tracking of Multiple Time-Varying Reference Signals with Bounded Derivatives," *IEEE Transactions on Automatic Control*, vol. 57, pp. 3169–3174, 2012.
- [13] C. Godsil and G. Royle, *Algebraic graph theory*. Springer, 2001.
- [14] Y. Cau, W. Ren, and M. Egerstedt, "Distributed Containment Control with Multiple Stationary or Dynamic Leaders in Fixed and Switching Directed Networks," *Automatica*, vol. 48, pp. 1586–1597, 2012.
- [15] I. Gutman and W. Xiao, "Generalized Inverse of the Laplacian Matrix and Some Applications," *Bulletin T.CXXIX de l'Academie serve des sciences et des arts*, vol. 29, pp. 15–23, 2004.
- [16] H. K. Khalil, *Nonlinear systems*. Upper Saddle River, NJ: Prentice Hall, 2002.
- [17] T. Yucelen and W. M. Haddad, "Consensus Protocols for Networked Multiagent Systems with a Uniformly Continuous Quasi-Resetting Architecture," *International Journal of Control* (to appear, available online).
- [18] D. Ustebay, R. Castro, and M. Rabbat, "Selective Gossip," *IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, 2009.
- [19] D. Ustebay, R. Castro, and M. Rabbat, "Efficient Decentralized Approximation via Selective Gossip," *IEEE Journal of Selected Topics in Signal Processing*, vol. 5, pp. 805–816, 2011.