Feedback stabilizability of infinite dimensional neutral systems

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Abstract—A new functional analytic approach to the concept of feedback stabilizability of infinite dimensional linear neutral systems is presented. We first reformulate this systems as an infinite dimensional open-loop systems with appropriate semigroups and unbounded control operators. We introduce conditions for which such semigroups are eventually compact. In addition, when the image of the control operator is finite dimensional, we give necessary and sufficient conditions for the feedback stabilizability of neutral system. Our approach is based on the concept of regular linear systems in the Salamon–Weiss sens.

Index Terms— Neutral systems, feedback stabilizability, infinite dimensional systems, compact semigroup, Hilbert space

I. INTRODUCTION

In this paper, we study the concept of feedback stabilizability of partial differential equations of neutral type in infinite dimensional spaces. Before going into details we first discuss the concept of stabilizability of infinite dimensional distributed linear systems. Let A be the generator of a strongly continuous semigroup $T := (T(t))_{t \geq 0}$ on a Hilbert space X . This semigroup (which is the solution of the Cauchy problem $\dot{x}(t) = Ax(t)$) is exponentially stable if and only if the growth bound of T satisfies $\omega_0(A) < 0$. In finite dimensional case, this is equivalent to the fact that all eigenvalues of A are located in the set $\mathbb{C}_- := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$. In the infinite dimensional case and if the semigroup is eventually norm continuous (so that compact semigroup) T is exponentially stable if and only if the spectral growth bound of A which is $s(A) := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A), \sigma(A) \text{ is the spectrum of } A\} < 0$. This shows that regularity of the semigroup T has an effect of the stability of Cauchy problems. On the other hand, the eventual compactness of the semigroup T gives more information on the spectrum $\sigma(A)$ since in this case $\sigma(A)$ is countable and the set $\Lambda_\alpha := \{\lambda \in \sigma(A) : \operatorname{Re} \lambda \geq \alpha\}$ is finite for all $\alpha \in \mathbb{R}$. If, in addition, we assume that Λ_0 is not empty (which means that T is unstable) then in practice it is important to shift the unstable set Λ_0 to $\mathbb{C}_-$, hence the concept of stabilizability. More generally, assume now that T

is unstable and let $B : U \rightarrow X$ be a linear bounded operator (U is another Hilbert space). The stabilization problem of the open-loop system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0,$$

consists in determining a (feedback) linear bounded operator $\Pi : X \rightarrow U$ such that the operator $A + B\Pi$ generates an exponentially stable semigroup on X , see [7, Chap.5]. This concept of stabilization can also be extended to unbounded control operator, that is, linear bounded operators B from U to $(\mathcal{D}(A^*))'$, the dual of the domain of the adjoint operator A^* , see [40, Def.2], [30].

In this paper, we consider the neutral system

$$\begin{cases} \frac{d}{dt}[Dx_t] = ADx_t + Lx_t + Bu(t), & t \geq 0, \\ \lim_{t \rightarrow 0} Dx_t = z, \\ x(t) = \varphi(t), & -r \leq t \leq 0, \end{cases} \quad (1)$$

where A is the generator of a strongly continuous semigroup T on a Hilbert space X , $x : [-r, \infty) \rightarrow X$, $r > 0$, and its history function $x_t : [-r, 0] \rightarrow X$ is defined by $x_t(s) = x(t + s)$ for $t \geq 0$ and $s \in [-r, 0]$, the operators $D, L : W^{1,2}([-r, 0], X) \rightarrow X$ are defined by

$$D\varphi = \varphi(0) - \int_{-r}^0 d\mu(s)\varphi(s) \quad L\varphi = \int_{-r}^0 d\eta(s)\varphi(s)$$

for $\varphi \in W^{1,2}([-r, 0], X)$. The control function $u : [0, \infty) \rightarrow U$ (U is a Hilbert space) is locally 2-integrable, and the control operator $B : U \rightarrow X$ are linear bounded.

The stabilization of (1) in the case when $\mu(s) = 0$ for $s \in [-r, 0]$, i.e. the standard delay system has been extensively studied in the literature, see e.g. [26], [22], [23], [25] in the case of $X = \mathbb{R}^n$ and [8], [18], [21], [24] in the case of infinite dimensional spaces.

In the case of $X = \mathbb{R}^n$ with generator $A \equiv 0$ and particular delay operators, see e.g. [20], [27], [29], [31]. The technique used in the so-called finite pole shifting [26]. However, the stabilizability of the neutral partial differential system (1) seems to have not yet been investigated if the space X is infinite dimensional and for general delays operators (the delays are not necessary commensurable). In the present paper, we shall introduce a new approach for the stabilizability problem of (1). First, start with the stabilizability of the difference equation associated with the

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system (1). This equation can be reformulated as an open-loop system with strictly unbounded control operator, see [19]. By using regular linear systems in the Salamon–Weiss sense and the definition of feedback stabilizability of open-loop systems with unbounded control operators [40], we introduce a rank condition for the feedback stabilizability of the difference equation. Second, we reformulate the system (1) as an open-loop system with a linear and bounded control operator. We shall use this new system to introduce necessary and sufficient conditions for stabilizability of the neutral system (1). The necessary condition does not require any regularity on the semigroup of the transformed system. However, the sufficient conditions requires that this semigroup is eventually compact and the control operator B has finite dimension.

The organization of the paper is as follows. For the reader's convenience, in Section II we briefly recall the relevant background of regular linear system from [32], [33], [39]. In Section III we present the reformulation of the neutral system (1) as a distributed linear system. In Section IV we study feedback stabilizability of the difference equation associated with (1). Section V is devoted to the stabilizability of the neutral system (1). In the last section we summarize the results of the paper.

II. An overview of Salamon–Weiss systems

For the reader's convenience, we briefly recall the concept of infinite dimensional well-posed and regular linear systems in the Salamon–Weiss sense. See [32], [33], [38], [39] for more details.

Throughout this section, X, U and Y are Hilbert spaces, $A : \mathcal{D}(A) \subset X \rightarrow X$ be the generator of a strongly continuous semigroup $(e^{tA})_{t \geq 0}$ on X and $1 \leq p < \infty$. We let X_1^A the domain $\mathcal{D}(A)$ endowed with the graph norm and by X_{-1}^A the completion of X with respect to the norm $\|x\|_{-1} := \|(\mu - A)^{-1}x\|$ for $x \in X$ and μ in the resolvent set $\rho(A)$. We have the following continuous injections

$$X_1^A \hookrightarrow X \hookrightarrow X_{-1}^A.$$

We note that $(e^{tA})_{t \geq 0}$ can be extended to strongly continuous semigroup on X_{-1}^A whose generator $A_{-1} : X \rightarrow X_{-1}^A$ is an extension of A . Define the set

$$\begin{aligned} \mathfrak{B}^p(A, U, X) = \left\{ \mathbb{B} \in \mathcal{L}(U, X_{-1}^A) : \right. \\ \left. \int_0^t e^{(t-s)A_{-1}} \mathbb{B}u(s) ds \in X, \quad (2) \right. \\ \left. t > 0, u \in L^p([0, \infty), U) \right\}. \end{aligned}$$

The elements of this set are called *admissible control oper-*

ators for A. Define the set

$$\begin{aligned} \mathcal{O}_Y^p(A) = \left\{ C \in \mathcal{L}(X_1^A, Y) : \right. \\ \left. \int_0^\alpha \|Ce^{tA}x\|^p \leq \gamma^p \|x\|^p \quad \forall x \in \mathcal{D}(A), \quad (3) \right. \\ \left. \text{for constants } \alpha > 0, \gamma := \gamma(\alpha) > 0 \right\}. \end{aligned}$$

The elements of this set are called *admissible observation operator for A*.

Consider the input-output linear system (A, B, C) described by

$$\begin{aligned} \dot{z}(t) &= Az(t) + \mathbb{B}u(t), \quad z(0) = z^0, \quad t > 0, \\ y(t) &= Cz(t), \quad t \geq 0, \end{aligned} \quad (4)$$

where $A : \mathcal{D}(A) \rightarrow X$ is a generator on X , $\mathbb{B} \in \mathfrak{B}^p(A, U, X)$ and $C \in \mathcal{O}_Y^p(A)$. The functions $z : [0, \infty) \rightarrow X$, $u : [0, \infty) \rightarrow U$ and $y : [0, \infty) \rightarrow Y$ are, respectively, the state trajectory, the control function and observation function of the system (4).

Define

$$\Phi_t u = \int_0^t e^{(t-s)A_{-1}} \mathbb{B}u(s) ds \quad (5)$$

for $t \geq 0$ and $u \in L^p([0, \infty), U)$. The state trajectory of the system (4) is given by

$$z(t) = e^{tA} z^0 + \Phi_t u, \quad t \geq 0.$$

As $C \in \mathcal{O}_Y^p(A)$, then the following operator

$$\Psi_\infty : X_1^A \rightarrow L_{loc}^p(\mathbb{R}^+, Y), \quad x \mapsto Ce^{tA}x \quad (6)$$

can be extended to a linear bounded operator from X to $L_{loc}^p(\mathbb{R}^+, Y)$, called the *extended output map* of the system (4). The *Yosida extension* of C for A is the following operator

$$C_\Lambda x := \lim_{\mu \rightarrow +\infty} C\mu(\mu - A)^{-1}x \quad (7)$$

$$\mathcal{D}(C_\Lambda) := \{x \in X : \text{the above limit exists in } Y\}.$$

Then, $\mathcal{D}(A) \subset \mathcal{D}(C_\Lambda)$ and that $C_\Lambda x = Cx$ for any $x \in \mathcal{D}(A)$. Further more it is shown in [35, Theorem 4.5] that

$$e^{tA}x \in \mathcal{D}(C_\Lambda) \quad \text{and} \quad (\Psi_\infty x)(t) = C_\Lambda e^{tA}x$$

for all $x \in X$ and almost every $t \geq 0$. Now, define

$$(\Psi_t x)(\tau) = C_\Lambda e^{\tau A}x \quad (8)$$

for $x \in X$ and $0 \leq \tau \leq t$. Hence, $(\Psi_t)_{t \geq 0}$ is a family of linear bounded operators from X to $L_{loc}^p([0, \infty), Y)$.

Definition 2.1: We say that the system (4) is well-posed (on X, U, Y) if there exists a family $(\mathbb{F}_t)_{t \geq 0}$ of linear bounded operators from $L_{loc}^p([0, \infty), U)$ to $L_{loc}^p([0, \infty), Y)$ such that

$$\begin{aligned} [\mathbb{F}_{t+s}u](\tau) &= [\mathbb{F}_t(u(\cdot + s))][0, t](\tau - s) \\ &\quad + [\Psi_t \Phi_s(u[0, s])](\tau - s) \end{aligned} \quad (9)$$

for $\tau \in [s, s+t]$, $t, s \geq 0$, and $u \in L^p([0, s+t], U)$. In this case we also say that the quadruple $\Sigma := (e^{tA}, \Phi_t, \Psi_t, \mathbb{F}_t)$ is well-posed on X, U, Y . The operators $\mathbb{F}_t$, $t \geq 0$, are called input-output operators of Σ .

Let $\mathbf{P}_\tau$ be the operator of truncation to $[0, \tau]$, that is $(\mathbf{P}_\tau f)(t) = f(t)$ for $t \in [0, \tau]$ and zero otherwise. The operators $\mathbb{F}_t$ are compatible in the sense that $\mathbf{P}_\tau \mathbb{F}_t = \mathbb{F}_\tau$ for $t > \tau$. This property provides a unique operator $\mathbb{F}_\infty : L^p_{loc}(\mathbb{R}_+, U) \rightarrow L^p_{loc}(\mathbb{R}_+, Y)$, called the *extended input-output map*, which satisfies $\mathbb{F}_\tau = \mathbf{P}_\tau \mathbb{F}_\infty = \mathbf{P}_\tau \mathbb{F}_\infty \mathbf{P}_\tau$ for $\tau \geq 0$.

Definition 2.2: Let Σ be a well-posed system on X, U, Y and $\mathbb{F}_\infty$ its extended input-output map. We say that Σ is *regular* (with zero feedthrough) if the limit

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t (\mathbb{F}_\infty u_0)(\tau) d\tau = 0$$

exists in Y for the constant input $u_0(t) = z$, $z \in U$, $t \geq 0$.

The following definitions will be used throughout this paper.

Definition 2.3: We say that the triple $(A, \mathbb{B}, C)$ generates a regular system Σ if there exists a bounded operator $\mathbb{F}_\infty : L^p_{loc}(\mathbb{R}_+, U) \rightarrow L^p_{loc}(\mathbb{R}_+, Y)$ such that $\Sigma := (e^{tA}, \Phi_t, \Psi_t, \mathbb{F}_t)$ is regular on Z, U, Y .

Definition 2.4: Let Σ be a regular system with input-output operators $\mathbb{F}_t$. An operator $\Pi \in \mathcal{L}(Y, U)$ is called an *admissible feedback* for Σ if $I - \mathbb{F}_t \Pi$ has uniformly bounded inverse.

The following theorem is due to Weiss in Hilbert spaces [39] and to Staffans in general Banach spaces [33, Chap.7].

Theorem 2.5: Assume that $(A, \mathbb{B}, C)$ generates a regular system Σ with admissible feedback operator Π . Then the operator defined by

$$A^\Pi = A_{-1} + \mathbb{B}\Pi C_\Lambda$$

$$\mathcal{D}(A^\Pi) := \{x \in \mathcal{D}(C_\Lambda) : (A_{-1} + \mathbb{B}\Pi C_\Lambda)x \in X\}$$

generates a strongly continuous semigroup on X .

Now we give the definition of feedback stabilizability of open-loop systems with unbounded control operators.

Definition 2.6: Assume that $\mathbb{B} \in \mathfrak{B}^p(A, U, X)$. We say that the system

$$\dot{x}(t) = Ax(t) + \mathbb{B}u(t), \quad t \geq 0,$$

is feedback stabilizable if there exists an operator $C : \mathcal{D}(A) \rightarrow U$ such that

- (i) The triple $(A, \mathbb{B}, C)$ generates a regular linear system Σ and the identity operator $I : U \rightarrow U$ is an admissible feedback operator for Σ .
- (ii) The operator

$$A^I = A_{-1} + \mathbb{B}C_\Lambda$$

$$\mathcal{D}(A^I) := \{x \in \mathcal{D}(C_\Lambda) : (A_{-1} + \mathbb{B}C_\Lambda)x \in X\}$$

generates an exponentially stable semigroup on X .

III. Reformulation of the neutral system

In this section we recall the reformulation of the system (1) as an infinite dimensional distributed linear system on a product state space. To that purpose, introduce the new state space

$$\mathcal{X} = X \times L^2([-r, 0], X).$$

Define the operator

$$\mathcal{A} = \begin{pmatrix} A & L \\ 0 & \frac{d}{ds} \end{pmatrix}$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} z \\ \varphi \end{pmatrix} \in \mathcal{D}(A) \times W^{1,2}([-r, 0], X) : \right. \quad (10)$$

$$\left. \varphi(0) = z + \int_{-r}^0 d\mu(s)\varphi(s) \right\}.$$

This operator generates a strongly continuous semigroup $\mathcal{T} := (\mathcal{T}(t))_{t \geq 0}$ on $\mathcal{X}$, see [16]. On the other hand, the following operator

$$Qf = f',$$

$$\mathcal{D}(Q) = \{f \in W^{1,2}([-r, 0], X) : f(0) = \} \quad (11)$$

generates a strongly continuous semigroup $S := (S(t))_{t \geq 0}$ on $L^2([-r, 0], X)$. Let Q_{-1} be the extension of Q in the extrapolation sense. Now define the operator

$$\mathcal{B}u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}, \quad u \in U. \quad (12)$$

The neutral system (1) can be reformulated as the following well-posed open-loop system

$$\begin{cases} \dot{w}(t) = \mathcal{A}w(t) + \mathcal{B}u(t), & t \geq 0, \\ w(0) = \begin{pmatrix} z \\ \varphi \end{pmatrix}, \end{cases} \quad (13)$$

see [6].

Remark 3.1: Define the operator

$$\mathcal{A}_0 = \begin{pmatrix} A & L \\ 0 & Q \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(Q). \quad (14)$$

This operator is a generator of a strongly continuous semigroup on $\mathcal{X}$. Denote by $\mathcal{A}_{0,-1}$ its extension in the extrapolation sense. On the other hand define the operator

$$\mathcal{B}_0 = \begin{pmatrix} 0 \\ (\lambda - Q_{-1})e_\lambda \end{pmatrix}, \quad \mathcal{C}_0 = (I \ K) : \mathcal{D}(\mathcal{A}_0) \rightarrow \mathcal{X},$$

where

$$K\varphi = \int_{-r}^0 d\mu(s)\varphi(s), \quad \varphi \in W^{1,2}([-r, 0], X)$$

and $e_\lambda : X \rightarrow L^2([-r, 0], X)$ is defined by $(e_\lambda z)(s) = e^{\lambda s} z$ for $\lambda \in \mathbb{C}$, $z \in X$ and $s \in [-r, 0]$.

We denote by $\mathcal{C}_{0,\Lambda}$ the Yosida extension of $\mathcal{C}_0$ for $\mathcal{A}_0$. It is shown in [14] and [16] that the triple $(\mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0)$ is a regular triple and the identity $I : X \rightarrow X$ is a feedback operator of its associated regular linear system Σ . Moreover, the operator $\mathcal{A}$ coincides with the generator of the closed-loop system associated with Σ and the feedback operator I . This means that

$$\mathcal{A} = \mathcal{A}_{0,-1} + \mathcal{B}_0 \mathcal{C}_{0,\Lambda}, \quad \mathcal{D}(\mathcal{A}) = \left\{ \xi \in \mathcal{D}(\mathcal{C}_{0,\Lambda}) : \mathcal{A}_{0,-1} \xi + \mathcal{B}_0 \mathcal{C}_{0,\Lambda} \xi \in \mathcal{X} \right\}. \quad (15)$$

The decomposition of the generator $\mathcal{A}$ as shown in Remark 3.1 has been used in [17] to prove the following result.

Theorem 3.2: If $T(t)$ is compact for $t > 0$ then $\mathcal{T}(t)$ is compact for $t > 2r$.

IV. Feedback stabilizability of difference equations

Here we characterize the feedback stabilizability of the following difference equation associated with the neutral system (1) (in the case $A \equiv 0$ and $L \equiv 0$)

$$x(t) = \int_{-r}^0 d\mu(s)x(t+s) + Bu(t), \quad t \geq 0. \quad (16)$$

This is associated with the operator

$$\begin{aligned} Q_\mu f &= f' \\ \mathcal{D}(Q_\mu) &= \left\{ \varphi \in W^{1,2}([-r, 0], X) : \right. \\ &\quad \left. \varphi(0) = \int_{-r}^0 d\mu(s)\varphi(s) \right\}. \end{aligned} \quad (17)$$

This operator generates a strongly continuous semigroup $S_\mu := (S_\mu(t))_{t \geq 0}$ in $L^2([-r, 0], X)$ and its resolvent set is characterized by

$$\rho(Q_\mu) = \left\{ \lambda \in \mathbb{C} : 1 \in \rho\left(\int_{-r}^0 d\mu(s)e^{\lambda s}\right) \right\}. \quad (18)$$

As in [19], one can see that the equation (16) can be reformulated as the well-posed open-loop system

$$\dot{v}(t) = Q_\mu v(t) + (-Q_{-1})e_0 Bu(t), \quad t \geq 0. \quad (19)$$

Now we state the main result of this section.

Theorem 4.1: The difference equation (16) is feedback stabilizable if and only if there exists $\delta > 0$ such that

$$\text{Ran}[\Delta_0(\lambda) \mid B] = X \quad \text{for any } \text{Re} \lambda \geq -\delta, \quad (20)$$

where

$$\Delta_0(\lambda)z = z - \int_{-r}^0 d\mu(s)e^{\lambda s}z, \quad z \in X. \quad (21)$$

Proof: First, assume that the difference equation (16) is feedback stabilizable. By using (19) and Definition 2.6, this

means that there exists an operator $N : W^{1,2}([-r, 0], X) \rightarrow U$ such that the following operator

$$\begin{aligned} Q_{\mu,N} f &= f' \\ \mathcal{D}(Q_{\mu,N}) &= \left\{ \varphi \in W^{1,2}([-r, 0], X) : \right. \\ &\quad \left. \varphi(0) = \int_{-r}^0 d\mu(s)\varphi(s) + BN\varphi \right\} \end{aligned}$$

generates an exponentially stable semigroup $S_{\mu,N}$ on $L^2([-r, 0], X)$. Hence $s(Q_{\mu,N}) \leq \omega_0(Q_{\mu,N}) < 0$, by [9, p.281]. Now due to (18) this is equivalent to the existence of $\delta > 0$ such that $\int_{-r}^0 d\mu(s)e^{\lambda s} + BN$ is invertible for any $\text{Re} \lambda \geq -\delta$. In particular, for any $x \in X$ there exists $\vartheta \in X$ such that $\Delta_0(\lambda)\vartheta + B(-Ne_\lambda\vartheta) = x$. This proves that the stabilizability of the difference equation (16) implies that $\text{Ran}[\Delta_0(\lambda) \mid B] = X$ for $\text{Re} \lambda \geq -\delta$. ■

V. Feedback stabilizability of neutral systems

Here, we present a necessary condition and a sufficient condition for the feedback stabilizability of the neutral system (1). This is equivalent to prove feedback stabilizability of the open-loop system (13). To that purpose, we need some information on the spectrum $\sigma(\mathcal{A})$. It is shown in [16] that for $\lambda \in \rho(Q_\mu)$ we have

$$\sigma(\mathcal{A}) = \sigma\left(A + Le_\lambda\left(I - \int_{-r}^0 d\mu(s)e^{\lambda s}\right)^{-1}\right). \quad (22)$$

Define the operator

$$\begin{aligned} \Delta(\lambda) &:= (\lambda - A)\Delta_0(\lambda) - Le_\lambda, \\ \mathcal{D}(\Delta(\lambda)) &:= \{z \in X : \Delta_0(\lambda)z \in \mathcal{D}(A)\} \end{aligned} \quad (23)$$

for $\lambda \in \mathbb{C}$, where $\Delta_0(\lambda)$ is defined by (21). Observe that for $\lambda \in \rho(Q_\mu)$ we have $\lambda \in \sigma(\mathcal{A})$ if and only if $\Delta(\lambda)$ is not bijective.

Proposition 5.1: If the neutral system (1) is feedback stabilizable then there exists $\delta > 0$ such that

$$\text{Ran}[\Delta(\lambda) \mid B] = X \quad \text{and} \quad \text{Ran}[\Delta_0(\lambda) \mid B] = X \quad (24)$$

for any $\text{Re} \lambda > -\delta$.

Proof: As we have seen instead of the neutral system (1) one can consider the open-loop system $(\mathcal{A}, \mathcal{B})$ described by (13), which we assume that it is feedback stabilizable. According to [5], there exists $\delta > 0$ such that

$$\text{Ran}[\lambda - \mathcal{A} \mid \mathcal{B}] = \mathcal{X} \quad \text{for } \text{Re} \lambda > -\delta. \quad \blacksquare$$

Let us now show how the condition (24) together with some extra conditions on A and B are also sufficient for the feedback stabilizability of the neutral system (1). We assume that A generates an immediately compact semigroup T (i.e. $T(t)$ is compact for $t > 0$). Theorem 3.2 shows that

$\mathcal{A}$ generates an eventually compact semigroup $\mathcal{T}$ (i.e. there exists $t_0 > 0$ such that $\mathcal{T}(t)$ is compact for $t \geq t_0$). Hence, the spectrum $\sigma(\mathcal{A})$ is countable and the following unstable set

$$\Omega = \{\lambda \in \sigma(\mathcal{A}) : \operatorname{Re} \lambda \geq 0\}$$

is finite, see [9, Cor.3.2, Chap.V, page 330].

Theorem 5.2: The neutral system (1) is feedback stabilizable if the following conditions are met:

- (i) A generates an immediately compact semigroup,
- (ii) the dimension of the image $\operatorname{Im} B$ is finite, and
- (iii) the condition (24) is satisfied for all $\lambda \in \Omega$.

Proof: Let $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{X}$ and put

$$\Omega = \{\lambda_1, \dots, \lambda_k\}.$$

First, we assume that $\Omega \subset \rho(Q_\mu)$. We shall use a result of [5], which gives condition for stabilizability of systems having compact semigroups. For $i = 1, \dots, k$, there exists $\varphi \in \rho(Q_\mu)$ such that $\lambda_i \varphi - \varphi' = f$. From (24), there exists $z_0 \in X$ and $u_0 \in U$ such that

$$\begin{aligned} (\lambda_i - A)z - Le_{\lambda_i} z_0 + Bu_0 \\ = x + LR(\lambda_i, Q)f + Le_{\lambda_i} \varphi(0), \end{aligned} \quad (25)$$

where $z = \Delta_0(\lambda_i)z_0 \in \mathcal{D}(A)$. We now introduce

$$\psi = e_{\lambda_i} z_0 + \varphi, \quad (26)$$

then we have

$$\begin{pmatrix} z \\ \psi \end{pmatrix} \in \mathcal{D}(\mathcal{A}) \quad \text{and} \quad \lambda_i \psi - \psi' = f.$$

From (25)–(26) and the fact that $\varphi = e_{\lambda_i} \varphi(0) + R(\lambda_i, Q)f$ we have

$$\begin{aligned} (\lambda_i - A)z - L\psi + Bu_0 \\ = x + Le_{\lambda_i} Ke_{\lambda_i} z_0. \end{aligned} \quad (27)$$

On the other hand, there exists $y_0 \in X$ and $u_1 \in U$ such that

$$(\lambda_i - A)y - Le_{\lambda_i} y_0 + Bu_1 = Le_{\lambda_i} Ke_{\lambda_i} z_0, \quad (28)$$

where $y = \Delta_0(\lambda_i)y_0 \in \mathcal{D}(A)$. If we put $g := \psi - e_{\lambda_i} y_0$, $w := z - y$ and $u = u_0 - u_1$ then by (27) and (28) we have

$$(\lambda_i - A)w - Lg + Bu = x \quad \text{and} \quad \lambda_i g - g' = f.$$

Observe that $\begin{pmatrix} w \\ g \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ and then $(\lambda_i - A)\begin{pmatrix} w \\ g \end{pmatrix} + Bu = \begin{pmatrix} x \\ f \end{pmatrix}$. Since, by Theorem 3.2, $\mathcal{A}$ generates an eventually compact semigroup on $\mathcal{X}$, then by [5] the open-loop system $(\mathcal{A}, \mathcal{B})$ is feedback stabilizable.

Second, let us assume that there exists a set $\Theta \subsetneq \Omega$ such that $\Theta \cap \rho(Q_\mu) = \{\emptyset\}$. As in the proof of Theorem 4.1, the second condition in (24) implies that there exists an operator $N : W^{1,2}([-r, 0], X) \rightarrow X$ such that the semigroup $S_{\mu, N}$ solution of the difference equation $x(t) = (K + BN)x_t$ is

exponentially stable. If we denote by $Q_{\mu, N}$ the generator of $S_{\mu, N}$ then we have $\Theta \subset \rho(Q_{\mu, N})$. Now we consider the feedback law $u(t) = \frac{d}{dt}Nx_t + v(t)$, where $v(t)$ is a new input. It suffices then to stabilize the system

$$\begin{cases} \frac{d}{dt}[Dx_t - BNx_t] = ADx_t + Lx_t + Bu(t), & t \geq 0, \\ \lim_{t \rightarrow 0} Dx_t = z, \\ x(t) = \varphi(t), & -r \leq t \leq 0. \end{cases} \quad (29)$$

We then proceed as in the first step to shift the unstable set Θ to $\mathbb{C}_-$. This ends the proof. $\blacksquare$

Example 5.3: Let A be the generator of an immediately compact semigroup on a Hilbert space X , $(M_i)_{1 \leq i \leq m}$ be a family of bounded linear operators on X , and $C : X \rightarrow X$ be a linear and bounded operator. Consider the neutral system

$$\begin{cases} \dot{x}(t) - \sum_{i=1}^m M_i \dot{x}(t-r) = \\ A[x(t) - \sum_{i=1}^m M_i x(t-r)] \\ + Cx(t-r) + Bu(t), & t \geq 0, \\ \lim_{t \rightarrow 0} [x(t) - \sum_{i=1}^m M_i x(t-r)] = z, \\ x(t) = \varphi(t), & a.e. \quad -r \leq t \leq 0, \end{cases} \quad (30)$$

with $0 < r_1 < r_2 < \dots < r_m \leq r$. After defining the difference operator $D\varphi = \varphi(0) - \sum_{i=1}^m M_i \varphi(r_i)$ and the operators $K, L : W^{1,2}([-r, 0], X) \rightarrow X$ by $K\varphi = \sum_{i=1}^m M_i \varphi(r_i)$ and $L\varphi = C\varphi(-r)$, the system (30) corresponds to the neutral system (1). Now by Proposition 5.1 and Theorem 5.2, the neutral system (30) is feedback stabilizable if and only if

$$\operatorname{Ran} \left[I - \sum_{i=1}^m e^{\lambda r_i} M_i \mid B \right] = X,$$

and there exist $x \in X$ and $u \in U$ such that $x - \sum_{i=1}^m e^{\lambda r_i} M_i x \in \mathcal{D}(A)$ and

$$(\lambda_A)(x - \sum_{i=1}^m e^{\lambda r_i} M_i x) - e^{-\lambda r} Cx + Bu = z$$

for all $z \in X$ and $\lambda \in \Omega$.

VI. Conclusion

In this paper, we have presented a new functional analytic approach to stabilize a class of neutral systems. We first presented a necessary and sufficient condition for the feedback stabilizability of difference systems (see Theorem 4.1). Such systems are reformulated as distributed linear systems with strictly unbounded control operators. We have then used Salamon–Weiss systems to prove Theorem 4.1. On the other hand, to stabilize the neutral system (1) we have reformulated this system as a linear system $(\mathcal{A}, \mathcal{B})$ with a linear bounded control operator. In order to use a well-known result on the

stabilization of the evolution systems we have introduced conditions for which the operator $\mathcal{A}$ generates an eventually compact semigroup.

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