Multi-Agent Deployment in the Plane Using Stochastic Extremum Seeking

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Abstract—We consider the problem of deployment of a group of N autonomous fully actuated vehicles (agents) in a non-cooperative manner in a planar signal field using the recently introduced method of stochastic extremum seeking. The spatial distribution of the signal is unknown to the vehicles but known to be convex. The vehicles are not able to sense their own positions but are capable of sensing the distance between their neighbors and themselves. Each vehicle employs a stochastic extremum seeking control law whose goal is to minimize the value of the measured signal, namely to be as close as possible to the bottom of the signal field, as well as to simultaneously minimizing a function of the distances between neighboring agents. Such a seemingly conflicting and mutually competitive nature of the agents' control laws produces a Nash equilibrium that depends on the agents' control parameters and the unknown signal distribution. We prove local exponential convergence, both almost surely and in probability, to a small neighborhood near the Nash equilibrium. The theoretical results are illustrated with simulations.

I. INTRODUCTION

In past years, extremum seeking [1] has proved to be a powerful tool in real-time non-model based control and optimization for *single* unmanned autonomous vehicles without position information [2], [3], [4], [5], [6]. Recently, extremum seeking has been considered for distributed control of vehicles in a network with each vehicle having limited local information in [7], [8], [9]. The applications include groups of vehicles operating under water, under ice, in caves or in urban environments where GPS is unavailable, or where inertial navigation systems are too costly. Other applications include scenarios where communication or interaction among all agents is not feasible.

Several other works have advanced or employed in applications the theory of extremum seeking [10], [11], [12], [13], [14]. A hybrid strategy for solving the source seeking problem was developed in [15]. In [16], [17], [18] the proposed problem in this paper is considered as a GPS-enabled game problem where each agent is trying to maximize its own cost function, but in these algorithms the agents also require the cost information of their neighbors. The work [8] considers a non-cooperative problem where each agent is trying to maximize or minimize their local cost function, which results in the convergence of the group of agents to a Nash equilibrium. In [8], similar to the one agent case [3], each agent employs two out-of-phase sinusoidal

perturbations in order to generate gradient estimates in the x and y directions for the extremum seeking algorithm.

We investigate a stochastic version of non-cooperative source seeking by navigating the autonomous vehicles with the help of a random perturbation. We use stochastic extremum seeking and apply an extra forcing to some of the vehicles, which we refer to as *anchor* agents, to increase the deployment area. The remaining agents, which we refer to as *follower* agents, achieve deployment over a source by using stochastic extremum seeking to maximize or minimize their local costs. The vehicles have no knowledge of their own position, nor the position of the source, and are only required to sense the distance between their neighbors and themselves. In an application, the signal could be the concentration of a chemical or biological agent, or it could be an electromagnetic, acoustic, or thermal source. The strength of the signal is assumed to decay away from the source through diffusion or other physical processes, but the spatial distribution of the signal is *not* available to the vehicles.

We consider two cases of excitation for the group of N vehicles. Case 1 uses an independent Brownian motion on a unit circle for every vehicle, and Case 2 uses only one Brownian motion on a unit circle for all vehicles, but with limited interaction between neighbors. We provide a stability analysis for both cases based on stochastic averaging theorems recently developed in [19]. The choice of using random processes for perturbation was motivated by [20] and [21], where it is observed that the bacterium *Escherichia coli* (*E. coli*) is able to move up chemical gradients towards higher densities of nutrients by using what appears to be random searching from time to time. In the works [22], [6], also motivated by *E. coli*, the problem of stochastic source seeking was considered for vehicles with unicycle dynamics.

In Section II, we give a description of the vehicle model and the cost function used by each agent. Section III presents the control scheme applied according to Case 1, which allows interaction among all agents, and Case 2, which allows limited interaction between the agents. We prove convergence results of a group of vehicles to a Nash equilibrium in probability and almost surely for the control law in Case 1 and in Case 2 in Sections IV. Simulation results in Section V illustrate the distinct behavior exhibited using both cases for control of the agents.

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II. VEHICLE MODEL AND LOCAL AGENT COST

We consider vehicles modeled as a velocity-actuated point mass

$$\frac{dx_i}{dt} = v_{xi}, \quad \frac{dy_i}{dt} = v_{yi}, \quad (1)$$

where (x_i, y_i) is the position of the vehicle in the plane, and v_{xi}, v_{yi} are the vehicle velocity inputs. The subscript i is used to denote the i^{th} vehicle.

We assume that the nonlinear map defining the distribution of the signal field is quadratic and takes the form

$$f_i(x_i, y_i) = f^* + q_x(x_i - x^*)^2 + q_y(y_i - y^*)^2 \quad (2)$$

where (x^*, y^*) is the minimizer, $f^* = f(x^*, y^*)$ is the minimum, and (q_x, q_y) are unknown positive constants. To account for the interactions between the vehicles we assume that each vehicle can sense the distance,

$$d_{ij}(x, y) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}, \quad (3)$$

between itself and other vehicles. The cost function

$$J_i = f_i + \sum_{j \in N} q_{ij} d_{ij}^2 \quad (4)$$

includes inter-vehicle interactions, where $q_{ij} \geq 0$ is the weighting that vehicle i puts on its distance to vehicle j .

III. CONTROL DESIGN

To deploy the agents about the source position, we propose a control scheme that utilizes Brownian motion on the unit circle as the excitation signal to perform stochastic extremum seeking. Brownian motion on the unit circle has the following form:

$$Y = e^{jB} = [Y_1, Y_2]^T = [\cos(B), \sin(B)]^T, \quad (5)$$

where j is the imaginary unit, B is a 1-dimensional Brownian motion.

First, for clarity, we introduce the control scheme for a single vehicle i , which does not impose any constraints on the excitation signal. Then, we discuss the deployment on N vehicles, which utilizes excitation signals from two cases. The two types of excitation are as follows, one where each vehicle uses an independent Brownian motion and the other where every vehicle uses the same Brownian motion process but the initial conditions of the processes differ by $k\pi/2$, $k \in \mathbb{Z}$, between neighboring vehicles.

We propose the following stochastic control algorithm for vehicle i :

$$v_{xi} = -a\dot{\eta}_{1i} + c_x \xi_i \eta_{1i} + \nu_{xi}, \quad (6)$$

$$v_{yi} = -a\dot{\eta}_{2i} + c_y \xi_i \eta_{2i} + \nu_{yi}, \quad (7)$$

$$\xi_i = \frac{s}{s+h} [J_i], \quad (8)$$

$$\eta_{1i} = \cos(W_i(t/\epsilon)), \quad (9)$$

$$\eta_{2i} = \sin(W_i(t/\epsilon)), \quad (10)$$

where ξ_i is the output of the washout filter for the cost J_i , η_{1i} , and η_{2i} are used as perturbations in the stochastic extremum

seeking scheme, $a, c_x, c_y, \epsilon, h > 0$ are extremum seeking design parameters, and $\nu_{xi}, \nu_{yi} \in \mathbb{R}$. In (8) s represents frequency domain in the transfer function acting on the cost J_i . We consider vehicles with $\nu_{xi}, \nu_{yi} \neq 0$ to be the anchor agents and those with $\nu_{xi} = \nu_{yi} = 0$ to be the follower agents. The signal $(W_i(t), t \geq 0)$ is a standard Brownian motion defined in a complete probability space $(\Omega, \mathcal{F}, P)$ with sample space Ω , the σ -field $\mathcal{F}$, and the probability measure P .

Using Ito's formula, η_{1i} and η_{2i} can be written as the solution to the differential equations,

$$d\eta_{1i} = -\frac{1}{2\epsilon} \cos(W_i(t/\epsilon))dt - \sin(W_i(t/\epsilon))dW_i(t/\epsilon), \quad (11)$$

$$d\eta_{2i} = -\frac{1}{2\epsilon} \sin(W_i(t/\epsilon))dt + \cos(W_i(t/\epsilon))dW_i(t/\epsilon), \quad (12)$$

which are equivalent to the stochastic differential equations

$$d\eta_{1i} = -\frac{1}{2\epsilon} \eta_{1i}dt - \eta_{2i}dW_i, \quad (13)$$

$$d\eta_{2i} = -\frac{1}{2\epsilon} \eta_{2i}dt + \eta_{1i}dW_i, \quad (14)$$

with initial condition $W_i(0) = 0$ and $[\eta_{1i}(0), \eta_{2i}(0)]^T = [\cos(\phi_i), \sin(\phi_i)]^T$, $\phi_i \in \mathbb{R}$, i.e., $\eta_{1i}^2(0) + \eta_{2i}^2(0) = 1$. This equivalence is shown with more detail in Section II of [19]. Hence, the control signals (6) and (7) become

$$v_{xi} = -\frac{a}{2\epsilon} \eta_{1i} - a\eta_{2i}\dot{W}_i + c_x \xi_i \eta_{1i} + \nu_{xi}, \quad (15)$$

$$v_{yi} = -\frac{a}{2\epsilon} \eta_{2i} + a\eta_{1i}\dot{W}_i + c_y \xi_i \eta_{2i} + \nu_{yi}, \quad (16)$$

where $\dot{W}_i$ is a white noise signal for all $i \in \{1, 2, \dots, N\}$. The vehicle dynamics in closed-loop with control laws (6)–(10) are rewritten as

$$dx_i = -\frac{a}{2\epsilon} \eta_{1i}dt - a\eta_{2i}dW_i + c_x \xi_i \eta_{1i}dt + \nu_{xi}dt, \quad (17)$$

$$dy_i = -\frac{a}{2\epsilon} \eta_{2i}dt + a\eta_{1i}dW_i + c_y \xi_i \eta_{2i}dt + \nu_{yi}dt, \quad (18)$$

where η_{1i} and η_{2i} are given in (13) and (14), respectively.

Remark 1: It is not necessary to choose the Brownian motion on the unit circle as the probing signal in the stochastic design for one vehicle. It is only required that the excitation signals in the x and y directions are uncorrelated and bounded. Note that the Brownian motion on the unit circle was primarily chosen for the ease that it provides in the stochastic averaging and the ability to use one Brownian motion per vehicle or, as it will be shown in the next Subsection, use only one Brownian motion for the entire group of vehicles.

For the stable deployment of N vehicles, with dynamics (13)–(14), (17)–(18), we impose additional constraints on the Brownian motion W_i and the initial condition of the Brownian motion on a unit circle $(\cos(\phi_i), \sin(\phi_i))$ according to the two types of excitation that we consider.

Case 1: For this case, we require the Brownian motion used by i^{th} agent, $W_i(t)$, to be uncorrelated with Brownian motion used by j^{th} agent, $W_j(t)$, for $i \neq j$ and allow $\phi_i \in \mathbb{R}$. Under these constraints, each vehicle is allowed to interact with any of the other vehicles.

Case 2: This case allows every vehicle to use the same Brownian motion, $W_i(t) = W(t), \forall i \in \{1, 2, \dots, N\}$, but requires the initial condition of the Brownian motion on a unit circle to satisfy

$$\begin{aligned} \phi_i - \phi_j &= \frac{k\pi}{2}, & i \in \Omega_{\text{odd}}, j \in \Omega_{\text{even}}, \\ \phi_i - \phi_j &= k\pi, & i, j \in \Omega_{\text{odd}} \text{ or } i, j \in \Omega_{\text{even}}, \end{aligned} \quad (19)$$

$\forall i, j \in \{1, 2, \dots, N\}, k \in \mathbb{Z}$, where Ω_{odd} and Ω_{even} are nonempty sets chosen such that

$$\Omega_{\text{even}} \cup \Omega_{\text{odd}} = \{1, 2, \dots, N\} \text{ and } \Omega_{\text{even}} \cap \Omega_{\text{odd}} = \emptyset. \quad (20)$$

With these constraints, a vehicle in the set Ω_{odd} cannot gather distance information about vehicles in the the same set because their perturbation signals are correlated. Therefore a vehicle in the set Ω_{odd} indirectly interacts with another vehicle in the same set by influencing vehicles in the set Ω_{even} . The same is true for vehicles in the set Ω_{even} .

Remark 2: Besides dealing with N vehicles converging to a Nash equilibrium, the main difference between [3], where $\cos(\omega t)$ and $\sin(\omega t)$ were used as probing signals, and this work is that we use components of Brownian motion on the unit circle as a probing signal.

IV. STABILITY ANALYSIS

In this section, we present and prove the local stability, in a specific probabilistic sense, for a group of vehicles with the two excitation cases presented in Section III.

We define an output error variable $e_i = \frac{h}{s+h}[J_i(t)] - f^*$ where $\frac{h}{s+h}$ is a low-pass filter applied to the cost J , which allows us to express $\xi_i(t)$, the signal from the washout filter, as $\xi_i(t) = \frac{s}{s+h}[J_i(t)] = J_i(t) - f^* - e_i(t)$, noting also that $\dot{e}_i = h\xi_i$.

A. Case 1

Here we show stability for a group of fully actuated vehicles with control laws (6)–(10) using the Case 1 perturbations.

Theorem 1: Consider the closed-loop system,

$$dx_i = -\frac{a}{2\epsilon}\eta_{1i}dt - a\eta_{2i}dW_i + c_x\xi_i\eta_{1i}dt + \nu_{xi}dt, \quad (21)$$

$$dy_i = -\frac{a}{2\epsilon}\eta_{2i}dt + a\eta_{1i}dW_i + c_y\xi_i\eta_{2i}dt + \nu_{yi}dt, \quad (22)$$

$$de_i = h\xi_i dt, \quad (23)$$

$$d\eta_{1i} = -\frac{1}{2\epsilon}\eta_{1i}dt - \eta_{2i}dW_i, \quad (24)$$

$$d\eta_{2i} = -\frac{1}{2\epsilon}\eta_{2i}dt + \eta_{1i}dW_i, \quad (25)$$

$$\begin{aligned} \xi_i &= q_x(x_i - x^*)^2 + q_y(y_i - y^*)^2 \\ &+ \sum_{j=1}^N q_{ij}d_{ij}^2 - e_i, \end{aligned} \quad (26)$$

$\forall i \in \{1, 2, \dots, N\}$, where the parameters $\nu_x, \nu_y \in \mathbb{R}^N$, $a, c_x, c_y, h, q_x, q_y > 0$ and $q_{ij} \geq 0, \forall i, j \in \{1, 2, \dots, N\}$, and the signal W_i is a Brownian motion with $W_i(0) = 0, W_i(t) \neq W_j(t)$. If the initial conditions are $[\eta_1(0), \eta_2(0)]^T = [\cos(\phi_i), \sin(\phi_i)]^T$ with $\phi_i \in \mathbb{R}$ and $x(0), y(0), e(0)$ are such that the quantities, $|x_i(0) - x^* - \tilde{x}_i^{\text{eq}}|, |y_i(0) - y^* - \tilde{y}_i^{\text{eq}}|, |e_i(0) - e^{\text{eq}}|$, are sufficiently small, where (x^*, y^*) is the minimizer of (2),

$$\tilde{x}^{\text{eq}} = \frac{1}{c_x a} Q_x^{-1} \nu_x, \quad (27)$$

$$\tilde{y}^{\text{eq}} = \frac{1}{c_y a} Q_y^{-1} \nu_y, \quad (28)$$

$$\begin{aligned} e_i^{\text{eq}} &= q_x \tilde{x}_i^{\text{eq}^2} + q_y \tilde{y}_i^{\text{eq}^2} + \frac{a^2}{2}(q_x + q_y) \\ &+ \sum_{j \in N} q_{ij} [(\tilde{x}_i^{\text{eq}} - x_j^{\text{eq}})^2 + (\tilde{y}_i^{\text{eq}} - y_j^{\text{eq}})^2], \end{aligned} \quad (29)$$

and the matrices Q_x and Q_y , given by

$$Q_{xij} = \begin{cases} -q_x - \sum_{k=1}^N q_{ik} & i = j \\ q_{ij} & i \neq j \end{cases}, \quad (30)$$

$$Q_{yij} = \begin{cases} -q_y - \sum_{k=1}^N q_{ik} & i = j \\ q_{ij} & i \neq j \end{cases}, \quad (31)$$

are invertible, then there exist constants $C_x, C_y, \gamma_x, \gamma_y > 0$ and a function $T(\epsilon) : (0, \epsilon_0) \rightarrow \mathbb{N}$ such that for any $\delta > 0$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \inf \{t \geq 0 : |x_i(t) - x^* - x_i^{\text{eq}}| > C_x e^{-\gamma_x t} + \delta + a\} \\ = \infty, \text{ a.s.}, \end{aligned} \quad (32)$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \inf \{t \geq 0 : |y_i(t) - y^* - y_i^{\text{eq}}| > C_y e^{-\gamma_y t} + \delta + a\} \\ = \infty, \text{ a.s.}, \end{aligned} \quad (33)$$

and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} P\{|x_i(t) - x^* - \tilde{x}_i^{\text{eq}}| \\ \leq C_x e^{-\gamma_x t} + \delta + a, \forall t \in [0, T(\epsilon)]\} = 1, \end{aligned} \quad (34)$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} P\{|y_i(t) - y^* - \tilde{y}_i^{\text{eq}}| \\ \leq C_y e^{-\gamma_y t} + \delta + a, \forall t \in [0, T(\epsilon)]\} = 1, \end{aligned} \quad (35)$$

$\forall i \in \{1, 2, \dots, N\}$ with the $\lim_{\epsilon \rightarrow 0} T(\epsilon) = \infty$. The constants C_x and C_y are dependent on both the initial condition $(x(0), y(0), e(0))$ and the parameters $a, c_x, c_y, \nu_x, \nu_y, h, q_x, q_y$. The constants γ_x, γ_y are dependent on the parameters $a, c_x, c_y, \nu_x, \nu_y, h, q_x, q_y$.

Proof: We start by defining the error variables

$$\tilde{x} = x - x^* - a\eta_1, \quad (36)$$

$$\tilde{y} = y - y^* - a\eta_2, \quad (37)$$

and define $[\chi_1(t), \chi_2(t)] = [\eta_1(\epsilon t), \eta_2(\epsilon t)]$. Since

$$d\tilde{x} = dx - \frac{a}{2}\chi_1(t/\epsilon)dt - \chi_2(t/\epsilon)dW, \quad (38)$$

$$d\tilde{y} = dy - \frac{a}{2}\chi_2(t/\epsilon)dt + \chi_1(t/\epsilon)dW, \quad (39)$$

we obtain the following dynamics for the error variables:

$$\frac{d\tilde{x}}{dt} = c_x \xi \chi_1(t/\epsilon) + \nu_x, \quad (40)$$

$$\frac{d\tilde{y}}{dt} = c_y \xi \chi_2(t/\epsilon) + \nu_y, \quad (41)$$

$$\frac{de}{dt} = h\xi, \quad (42)$$

$$\begin{aligned} \xi_i = & q_x (\tilde{x}_i + a\chi_{1i}(t/\epsilon))^2 + q_y (\tilde{y}_i + a\chi_{2i}(t/\epsilon))^2 \\ & + \sum_{j \in N} q_{ij} \left[(\tilde{x}_i + a\chi_{1i}(t/\epsilon) - \tilde{x}_j - a\chi_{1j}(t/\epsilon))^2 \right. \\ & \left. + (\tilde{y}_i + a\chi_{2i}(t/\epsilon) - \tilde{y}_j - a\chi_{2j}(t/\epsilon))^2 \right] - e_i, \end{aligned} \quad (43)$$

$$d\chi_1(t) = -\frac{1}{2}\chi_1(t) - \chi_2(t)dW, \quad (44)$$

$$d\chi_2(t) = -\frac{1}{2}\chi_2(t) + \chi_1(t)dW. \quad (45)$$

We use general stochastic averaging given in Theorem 2 of [19] to analyze the error system. We first calculate the average system of (40)–(42). The signals χ_1 and χ_2 are both components of the Brownian motion on a unit circle, which is known to be exponentially ergodic with invariant distribution $\mu(S) = \frac{l(S)}{2\pi}$ for any set $S \subset \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ where $l(S)$ denotes the length (Lebesgue measure) of S [23]. The integral over the entire space of functions of Brownian motion on a unit circle can be reduced to the integral from 0 to 2π . Since

$$\int_{\mathbb{R}} \cos^{2k+1}(s) \mu(ds) = \int_0^{2\pi} \cos^{2k+1}(s) \frac{1}{2\pi} ds = 0, \quad (46)$$

$$\int_{\mathbb{R}} \cos^2(s) \mu(ds) = \int_0^{2\pi} \cos^2(s) \frac{1}{2\pi} ds = \frac{1}{2}, \quad (47)$$

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \cos(s) \cos(r) \mu(ds) \mu(dr) = \\ \int_0^{2\pi} \int_0^{2\pi} \cos(s) \cos(r) \frac{1}{4\pi^2} ds dr = 0, \end{aligned} \quad (48)$$

(note that the same applies to the sine function) and

$$\int_{\mathbb{R}} \cos(s) \sin(s) \mu(ds) = \int_0^{2\pi} \cos(s) \sin(s) \frac{1}{2\pi} ds = 0, \quad (49)$$

we obtain the average error system

$$\frac{d\tilde{x}^{\text{ave}}}{dt} = c_x a Q_x \tilde{x}^{\text{ave}} + \nu_x \quad (50)$$

$$\frac{d\tilde{y}^{\text{ave}}}{dt} = c_y a Q_y \tilde{y}^{\text{ave}} + \nu_y \quad (51)$$

$$\begin{aligned} \frac{de_i^{\text{ave}}}{dt} = & h \left(-e_i^{\text{ave}} + q_x \tilde{x}^{\text{ave}^2} + q_y \tilde{y}^{\text{ave}^2} + \frac{a^2}{2} (q_x + q_y) \right) \\ & + h \sum_{j \in N} q_{ij} \left[(\tilde{x}^{\text{ave}} - \tilde{x}_j^{\text{ave}})^2 + (\tilde{y}^{\text{ave}} - \tilde{y}_j^{\text{ave}})^2 \right]. \end{aligned} \quad (52)$$

Using the fact that Q_x and Q_y have the special form, shown in (30) and (31), with Gershgorin Circle Theorem (Theorem 7.2.1 in [24]) we get that as long as the $q_x, q_y > 0$, the

matrices Q_x, Q_y have all of their eigenvalues on the left hand side (*i.e.* they are Hurwitz and invertible).

The average error system has equilibria (27), (28), and (29) with the Jacobian,

$$A = \begin{bmatrix} \frac{c_x a}{2\pi} Q_x & 0 & 0 \\ 0 & \frac{c_y a}{2\pi} Q_y & 0 \\ 0 & 0 & -hI \end{bmatrix}. \quad (53)$$

Matrices Q_x and Q_y are Hurwitz, which implies that A is Hurwitz and that the equilibria (27), (28), and (29) are exponentially stable.

Using Theorem 2 in [19] there exist constants $c > 0, r > 0, \gamma > 0$ and functions $T(\epsilon) : (0, \epsilon_0) \rightarrow \mathbb{N}$, such that for any $\delta > 0$, and any initial conditions $|\Lambda_\epsilon(0)| < r$

$$\lim_{\epsilon \rightarrow 0} \inf \{t \geq 0 : |\Lambda^\epsilon(t)| > c|\Lambda^\epsilon(0)|e^{-\gamma t} + \delta\} = \infty, \text{ a.s.}, \quad (54)$$

and

$$\lim_{\epsilon \rightarrow 0} P \{|\Lambda^\epsilon(t)| > c|\Lambda^\epsilon(0)|e^{-\gamma t} + \delta, t \in [0, T(\epsilon)]\} = 1, \quad (55)$$

with $\lim_{\epsilon \rightarrow 0} T(\epsilon) = \infty$, where

$$\Lambda^\epsilon(t) = \begin{bmatrix} \tilde{x} - x^{\text{eq}} \\ \tilde{y} - y^{\text{eq}} \\ e - e^{\text{eq}} \end{bmatrix}. \quad (56)$$

The results (54) and (55) state that the norm of the error vector $\Lambda^\epsilon(t)$ exponentially converges, both almost surely and in probability, to a point below an arbitrarily small residual value δ over an arbitrarily long time interval, which tends to infinity as ϵ goes to zero. In particular, each $\tilde{x}_i$ -component and $\tilde{y}_i$ -component for all $i \in \{1, 2, \dots, N\}$ of the error vector converges to below δ , which gives us (32)–(35). ■

B. Case 2

Here we show stability for a group of fully actuated vehicles with control laws (6)–(10) using the Case 2 perturbations.

Theorem 2: Consider the closed-loop system (21)–(26) where the parameters $\nu_x, \nu_y \in \mathbb{R}^N$, $a, c_x, c_y, h, q_x, q_y > 0$, and $q_{ij} \geq 0 \forall i, j \in \{1, 2, \dots, N\}$. If the initial conditions are $W_i(0) = 0, [\eta_{1i}(0), \eta_{2i}(0)]^T = [\cos(\phi_i), \sin(\phi_i)]^T$ with ϕ_i chosen such that (19)–(20) holds and $x(0), y(0), e(0)$ are such that the following quantities

$$|x_i(0) - x^* - \tilde{x}_i^{\text{eq}}|, |y_i(0) - y^* - \tilde{y}_i^{\text{eq}}|, |e_i(0) - e^{\text{eq}}|, \quad (57)$$

are sufficiently small, where

$$\begin{bmatrix} \tilde{x}^{\text{eq}} \\ \tilde{y}^{\text{eq}} \end{bmatrix} = A_{xy}^{-1} \begin{bmatrix} \nu_x \\ \nu_y \end{bmatrix}, \quad (58)$$

$$\begin{aligned} e_i^{\text{eq}} = & q_x \tilde{x}_i^{\text{eq}^2} + q_y \tilde{y}_i^{\text{eq}^2} + \frac{a^2}{2} (q_x + q_y) \\ & + \sum_{j \in N} q_{ij} \left[(x_i^{\text{eq}} - x_j^{\text{eq}})^2 + (y_i^{\text{eq}} - y_j^{\text{eq}})^2 \right], \end{aligned} \quad (59)$$

and the matrix A_{xy} is invertible and is given by

$$A_{xy} = \begin{bmatrix} c_x a(Q_\Omega - Iq_x) & -c_x a Q_\Omega \\ -c_y a Q_\Omega & c_y a(Q_\Omega - Iq_y) \end{bmatrix}, \quad (60)$$

$$Q_{\Omega ij} = \begin{cases} -\sum_{k \in \Omega_{\text{odd}}} q_{ik} & i \in \Omega_{\text{even}}, i = j \\ -\sum_{k \in \Omega_{\text{even}}} q_{ik} & i \in \Omega_{\text{odd}}, i = j \\ q_{ij} & i \in \Omega_{\text{even}}, j \in \Omega_{\text{odd}} \\ q_{ij} & i \in \Omega_{\text{odd}}, j \in \Omega_{\text{even}} \\ 0 & \text{otherwise} \end{cases}, \quad (61)$$

then there exist constants $C_x, C_y, \gamma_x, \gamma_y > 0$ and a function $T(\epsilon) : (0, \epsilon_0) \rightarrow \mathbb{N}$ such that for any $\delta > 0$

$$\lim_{\epsilon \rightarrow 0} \inf \{t \geq 0 : |x_i(t) - x^* - x_i^{\text{eq}}| > C_x e^{-\gamma_x t} + \delta + a\} = \infty, \text{ a.s.} \quad (62)$$

$$\lim_{\epsilon \rightarrow 0} \inf \{t \geq 0 : |y_i(t) - y^* - y_i^{\text{eq}}| > C_y e^{-\gamma_y t} + \delta + a\} = \infty, \text{ a.s.} \quad (63)$$

and

$$\lim_{\epsilon \rightarrow 0} P\{|x_i(t) - x^* - x_i^{\text{eq}}| \leq C_x e^{-\gamma_x t} + \delta + a, \forall t \in [0, T(\epsilon)]\} = 1 \quad (64)$$

$$\lim_{\epsilon \rightarrow 0} P\{|y_i(t) - y^* - y_i^{\text{eq}}| \leq C_y e^{-\gamma_y t} + \delta + a, \forall t \in [0, T(\epsilon)]\} = 1 \quad (65)$$

$\forall i \in [1, N]$ with the $\lim_{\epsilon \rightarrow 0} T(\epsilon) = \infty$

Proof: Similar to the proof for Theorem 1 we start by applying (36), (37) and defining $[\chi_1(t), \chi_2(t)] = [\eta_1(\epsilon t), \eta_2(\epsilon t)]$. By employing stochastic averaging to compute the average system and then, linearizing the average system about the equilibrium $[x^{\text{eq}}, y^{\text{eq}}, e^{\text{eq}}]^T$, we obtain the Jacobian,

$$A = \begin{bmatrix} A_{xy} & 0 \\ 0 & -hI \end{bmatrix}, \quad (66)$$

which is block diagonal and Hurwitz since A_{xy} and $-hI$ are both Hurwitz. By applying Theorem 2 in [19], similar to Theorem 1, we can obtain the results (62)–(65). ■

V. SIMULATION

In this section, we show numerical results for a group of vehicles with the control scheme presented in Section III. For the following simulations, without loss of generality, we let the unknown location of the signal field be at the origin $(x^*, y^*) = (0, 0)$, and let the unknown signal field parameters be $(q_x, q_y) = (1, 1)$.

In Figure 1 we consider 13 vehicles with Case 1 perturbations. We choose the design parameters as $a = 0.01, c_x = c_y = 150, h = 10$, and define agents 1 through 6 as the anchor agents with the forcing terms,

$$(\nu_{xi}, \nu_{yi}) = 0.05 \left(\cos\left(\frac{i\pi}{3}\right), \sin\left(\frac{i\pi}{3}\right) \right), \quad (67)$$

where $i = 1, \dots, 6$. In addition to the design parameters, we picked the interaction gains q_{ij} such that

$$q_{ij} = \begin{cases} q_{i,i+1} = q_{i+1,i} = 0.5, & i \in \{1, \dots, 12\}, i \neq 6 \\ q_{i,13} = 0.5, & i \in \{7, \dots, 12\} \\ q_{i,i-6} = q_{i-6,i} = 1, & i \in \{7, \dots, 12\} \\ q_{i,j} = 0, & \text{otherwise} \end{cases} \quad (68)$$

Figure 1 shows the ability of the control algorithm to produce a circular distribution around the source with a higher density of vehicles near the source. In this plot, the trajectories of the vehicles are not shown to avoid obscuring the final vehicle formation.

Fig. 1. This plot shows a group of vehicles using the stochastic extremum seeking algorithm with Case 1 perturbations and interaction gains given by (68). The anchors agents are denoted by red triangles and the follower agents are denoted by blue dots. The agents start inside the dashed black line and converge to a circular formation around the source.

In Figure 2 we consider 5 vehicles with Case 2 constraints. We pick agent 1 and agent 5 as the anchor agents with $(\nu_{x1}, \nu_{y1}) = (-0.1, 0.1)$, $(\nu_{x5}, \nu_{y5}) = (0.1, -0.1)$, and choose the other design parameters to be the same as in the previous simulation. We assumed that each vehicle interacts with only the closest indexed agents with a weighting of 0.5, i.e., $q_{i,i+1} = q_{i+1,i} = 0.5, i = \{1, \dots, 4\}$. Figure 2 shows a line formation centered at the source, with a higher density of agents near the source, and generated by agents using a single Brownian motion signal.

Illustrated in these figures is the effect of the forcing terms (ν_{xi}, ν_{yi}) assigned to the anchor agents. By carefully selecting these forcing terms, other geometric deployments can be made, which will be distorted by the signal field. For instance, if ν_{xi} and ν_{yi} (67) were defined as

$$\nu_{xi} = 0.05 \left(a \cos\left(\frac{i\pi}{3}\right) - b \sin\left(\frac{i\pi}{3}\right) \right) \quad (69)$$

$$\nu_{yi} = 0.05 \left(a \cos\left(\frac{i\pi}{3}\right) + b \sin\left(\frac{i\pi}{3}\right) \right), \quad (70)$$

where a is the semimajor axis and b is the semiminor axis, an elliptical deployment will result.

Fig. 2. Shows a group of vehicles using the stochastic extremum seeking algorithm with Case 2 perturbations. The agents start inside the dashed black line and converge to a line formation centered around the source with the anchor agents at the end of the line formation.

VI. CONCLUSIONS

In this paper, we presented a stochastic extremum seeking algorithm for a group of agents, with two different constraints on the agents, to achieve stable deployment over a source. We present a stability proof that shows convergence of the vehicles, to a Nash equilibrium, both in the almost sure sense and in probability when using two kinds of excitation signals. We show simulation results for the control algorithm applied to agents on a static source.

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