Pointing*

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Abstract

In this work we treat the following problem. Given two vectors of same length, find an orthogonal transformation that transforms one to the other. This problem arises in different engineering categories. In particular, this problem arises in aerospace engineering where it is called *pointing problem*. In this paper we establish the theoretical background of the pointing problem in n and thus also in 3-D.

We give a straightforward solution to this problem, but since it is not unique we widen the scope of the problem, define the notions of *minimal pointing* and *optimal pointing*, and require that the sought matrix be, not only orthogonal, but also a minimal, or an optimal, pointing. We then give an illustrative solution in 3-D and then extend the solution to n -D using two different approaches which we present and prove. Several examples are given in 3-D and 4-D. The 3-D examples are used to illustrate the characteristics of the solution.

1 Introduction

There are cases where a multi-dimensional vector is given by its components in a certain Cartesian coordinate system, but we wish to perform an orthogonal transformation on the vector in a way which will change its components into some specified values. In particular, we may want to transform the vector such that all its resulting components be equal¹.

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Orthogonal transformations are carried out by orthogonal matrices. Orthogonal matrices whose determinant is equal to +1 are known as *proper* orthogonal matrices, and those whose determinant is equal to -1 are called *improper* orthogonal matrices². Proper orthogonal matrices can express Cartesian coordinate transformation and thus are also known as *rotation* matrices or *attitude* matrices.

If we limit ourselves to proper orthogonal matrices, then the transformation problem described above can be conceived as a rotation of the coordinate system in which the vector is resolved initially. This coordinate rotation is equivalent to a rotation of the *vector* in the original coordinate system to a direction in which the new vector components meet the desired requirement on its components. We call the latter operation *pointing*.

For simplicity and with no loss of generality, we will assume that the two vectors are of unit length, thus the problem that we wish to solve is posed as follows. *Given an n -dimensional unit vector $\mathbf{b}$, find an orthogonal transformation T , which transforms $\mathbf{b}$ into a given unit vector, $\mathbf{d}$. Note that since T is orthogonal, it is obvious that $\mathbf{d}$ has to have the same length as $\mathbf{b}$.*

We can solve the problem of pointing in the following manner. Let us write

$$[\mathbf{d} \ D] = T[\mathbf{b} \ B],$$

where D and B are any $n \times (n-1)$ matrices such that $[\mathbf{d} \ D]$ and $[\mathbf{b} \ B]$ are nonsingular. The use of the Gram-Schmidt orthogonalization algorithm on $[\mathbf{d} \ D]$ and $[\mathbf{b} \ B]$ starting with $\mathbf{d}$ and $\mathbf{b}$ (a numerically reliable way to do that is by using the Householder QR algorithm³), that transforms the matrices $[\mathbf{d} \ D]$ and $[\mathbf{b} \ B]$ into orthogonal matrices $[\mathbf{d} \ D_0]$ and $[\mathbf{b} \ B_0]$ (with the same $\mathbf{d}$ and $\mathbf{b}$). Consequently,

$$T = [\mathbf{d} \ D_0] [\mathbf{b} \ B_0]' \quad (1)$$

is an orthogonal transformation which transforms $\mathbf{b}$ and $\mathbf{d}$. Here and elsewhere in the paper A' denotes the transpose.

It is quite easy to show that there are infinitely many solutions to this problem unless the vectors $\mathbf{d}$ and $\mathbf{b}$ are 2-dimensional. Therefore we are free to select additional requirements on the transformation. In particular, we may require that the transformation be a

minimal one, or an optimal one. (An exact definition of what constitutes a minimal or an optimal transformation will be given in the ensuing.)

To formulate precisely the problem of minimal pointings, we need a measure of deviation of a given transformation T from the target transformation, which in our case will be the identity matrix I . We use the Frobenius norm $\|A\| = [\text{trace}(AA')]^{\frac{1}{2}}$ to define such a measure. (Another measure of deviation is discussed in Ref. 1.) Thus, the deviation of T from the target transformation is defined by $\|T - I\| = (\text{trace}(T - I)(T - I)')^{\frac{1}{2}}$. We can now state the problems on hand as follows.

A. Statement of the Problems

Given two n -dimensional unit vectors $\mathbf{b}$ and $\mathbf{d}$.

- (a) Find the *minimal pointing* T_m , i.e., orthogonal matrix T_m , such that $T_m \mathbf{b} = \mathbf{d}$ and $\|T_m - I\|$ is minimal among all deviations of the orthogonal matrices T that satisfy $T\mathbf{b} = \mathbf{d}$. In other words, find T_m such that

$$\|T_m - I\| \leq \|T - I\|$$

for every orthogonal matrix T that satisfies $T\mathbf{b} = \mathbf{d}$.

- (b) Find the *minimal proper pointing* T_m , i.e., the proper orthogonal matrix T_m , such that $T_m \mathbf{b} = \mathbf{d}$ and $\|T_m - I\|$ is minimal among all deviations of the proper orthogonal matrices T that satisfy $T\mathbf{b} = \mathbf{d}$.

The optimal pointing problem requires a different measure of deviation: instead of minimal norm, as for minimal pointings, we now ask for maximal dimension of a subspace on which the pointing coincides with the identity transformation. The exact formulation is as follows:

- (c) Find the *optimal pointing* T_o , i.e., orthogonal matrix T_o such that $T_o \mathbf{b} = \mathbf{d}$ and the dimension of the subspace $\{\mathbf{x} : T_o \mathbf{x} = \mathbf{x}\}$ is maximal among all orthogonal matrices T that satisfy $T\mathbf{x} = \mathbf{x}$.
- (d) The problem (c) but limited to proper orthogonal matrices.

The vector spaces in this paper are $\mathbf{R}^n$, the vector space of n -dimensional columns with real components, equipped with the standard Euclidean metric (distance and norm).

2 Pointings in 3-Dimensional Case

In this section we consider only proper orthogonal transformations. By Euler's theorem⁴, such transformations

are accomplished by a single rotation around an axis (called Euler axis); in fact, this axis is determined by the eigenvector corresponding to the eigenvalue 1.

In a three-dimensional space we have one degree of freedom, and thus we have infinite orthogonal matrices which will transform $\mathbf{b}$ into $\mathbf{d}$. However, if we are considering only proper pointings, there is only one transformation that will be *minimal* in the sense that it is obtained from a single rotation about a fixed axis by the *smallest* possible angle which is the angle between the two vectors. Finding the minimal rotation is a straightforward procedure. What we have to do is find the angle, ϕ , from $\mathbf{b}$ to $\mathbf{d}$, find the normal to the plane defined by the two vectors, and then rotate the coordinate system about this normal by $-\phi$. Let us demonstrate it by the following example.

EXAMPLE 1. Let

$$\mathbf{b}' = [0.912871 \quad 0.182574 \quad -0.365148]$$

and

$$\mathbf{d}' = [0.57735 \quad 0.57735 \quad 0.57735].$$

(Note that they are of equal length). The unit vector $\hat{\phi}$ in the direction of the normal (also known as Euler axis) and the angle, ϕ , are found as follows:

$$\hat{\phi} = \frac{\mathbf{b} \times \mathbf{d}}{|\mathbf{b} \times \mathbf{d}|}; \quad \phi = \sin^{-1} \left[\frac{|\mathbf{b} \times \mathbf{d}|}{|\mathbf{b}| \cdot |\mathbf{d}|} \right],$$

the results are $\hat{\phi}' = [-0.348743 \quad 0.813733 \quad -0.464991]$ and $\phi = 1.13555$. Next define the quaternion of rotation, $\mathbf{q} = iq_1 + jq_2 + kq_3 + q_4$ for a rotation by $-\phi$ about $\hat{\phi}$:

$$\left. \begin{aligned} q_1 &= \sin\left(-\frac{\phi}{2}\right) \frac{\phi_x}{\phi} \\ q_2 &= \sin\left(-\frac{\phi}{2}\right) \frac{\phi_y}{\phi} \\ q_3 &= \sin\left(-\frac{\phi}{2}\right) \frac{\phi_z}{\phi} \\ q_4 &= \cos\left(-\frac{\phi}{2}\right) \end{aligned} \right\}$$

where ϕ_x, ϕ_y, ϕ_z are the components of $\Phi = \phi\hat{\phi}$. When we use q to compute T_m (see Ref. 1), the result is

$$T_m = \begin{bmatrix} 0.491978 & -0.585767 & -0.644076 \\ 0.257507 & 0.804607 & -0.535068 \\ 0.831653 & 0.097388 & 0.546688 \end{bmatrix}.$$

As mentioned, it will be shown later that T_m is the minimal proper pointing. Let us check this point. For T_m of the example, we find that

$$\|T_m - I\| = 1.521.$$

The following matrix,

$$T = \begin{bmatrix} 0.86038 & -0.227924 & 0.455848 \\ 0.36038 & -0.36038 & -0.86038 \\ 0.36038 & 0.904531 & -0.227924 \end{bmatrix},$$

too, is an orthogonal matrix which transforms $\mathbf{b}$ to $\mathbf{d}$ (T was found using the method described in the Introduction); however,

$$\|T - I\| = 2.33578.$$

Thus

$$\|T_m - I\| < \|T - I\|,$$

which stems from the fact that among the orthogonal matrices which transform $\mathbf{b}$ to $\mathbf{d}$, T_m is the closest to I .

After this discussion of the solution in 3-D, we are now ready to consider the general n -D case.

3 Pointing in n -Dimensional Spaces

A. Arbitrary Pointing

A construction of an arbitrary pointing in $\mathbf{R}^n$ has already been given in the introduction (see the formula in Eq. (1)). One might elaborate on this as follows: Write

$$\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}.$$

Let i and j be indices such that $b_j \neq 0$ and $d_i \neq 0$. Then take in Eq. (1),

$$D = [\mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_{i-1} \mathbf{e}_{i+1} \cdots \mathbf{e}_n],$$

and

$$B = [\mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_{j-1} \mathbf{e}_{j+1} \cdots \mathbf{e}_n],$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix.

To reduce the numerical errors in the subsequent Gram-Schmidt reduction, we propose to use the indices i and j determined by the properties that

$$|b_j| = \max(|b_1|, |b_2|, \dots, |b_n|),$$

$$|d_i| = \max(|d_1|, |d_2|, \dots, |d_n|).$$

B. Minimal Pointing

Let be given two unit vectors $\mathbf{b}$, $\mathbf{d}$ in $\mathbf{R}^n$. Recall that an orthogonal $n \times n$ matrix T is called a *pointing* if $T\mathbf{b} = \mathbf{d}$. A pointing T_0 is called *minimal* if the norm

$$\|T_0 - I\| = (\text{trace}(T_0 - I)(T_0 - I)')^{\frac{1}{2}}$$

is minimal among all pointings T . We denote by $\langle \mathbf{x}, \mathbf{y} \rangle$ the standard scalar product of the vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

where

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^n x_j y_j.$$

The solution of the minimal pointing problem is based on the following theorem:

THEOREM 3.1. Let $\mathbf{b}$, $\mathbf{d}$ be unit vectors in $\mathbf{R}^n$.

(a) The minimal deviation among all pointings,

$$Q = \min_T \|T - I\|,$$

subject to $TT' = I$, $T\mathbf{b} = \mathbf{d}$, is given by the formula

$$Q = \begin{cases} 2(1 - \langle \mathbf{d}, \mathbf{b} \rangle)^{\frac{1}{2}} & \text{if } \langle \mathbf{d}, \mathbf{b} \rangle > 0; \\ 2 & \text{if } \langle \mathbf{d}, \mathbf{b} \rangle \leq 0. \end{cases}$$

(b) If $\langle \mathbf{d}, \mathbf{b} \rangle \neq 0$ and $\mathbf{d} \neq \pm \mathbf{b}$, then the minimal pointing T_0 is unique and is uniquely determined by the equalities of Eqs. (2) and (3) where $\mathbf{c}$ is a fixed unit vector which is a linear combination of $\mathbf{b}$ and $\mathbf{d}$ and which is orthogonal to $\mathbf{b}$:

$$T_0 \mathbf{b} = \mathbf{d}; \quad T_0 \mathbf{c} = \pm(-\langle \mathbf{d}, \mathbf{c} \rangle \mathbf{b} + \langle \mathbf{d}, \mathbf{b} \rangle \mathbf{c}), \quad (2)$$

the sign in Eq. (2) is $+1$ if $\langle \mathbf{d}, \mathbf{b} \rangle > 0$, and -1 if $\langle \mathbf{d}, \mathbf{b} \rangle < 0$;

$$T_0 \mathbf{x} = \mathbf{x} \quad \text{if } \mathbf{x} \perp \mathbf{b}, \mathbf{x} \perp \mathbf{d}. \quad (3)$$

(c) If $\mathbf{d} = \mathbf{b}$, then I is the unique minimal pointing. If $\mathbf{d} = -\mathbf{b}$, then the unique minimal pointing T_0 is defined by the properties that

$$T_0 \mathbf{b} = -\mathbf{b}; \quad T_0 \mathbf{x} = \mathbf{x} \text{ for every } \mathbf{x} \perp \mathbf{b}.$$

If $\langle \mathbf{d}, \mathbf{b} \rangle = 0$, then there are exactly two minimal pointings T_0 given by

$$T_0 \mathbf{b} = \mathbf{d}; \quad T_0 \mathbf{d} = \pm \mathbf{b};$$

$$T_0 \mathbf{x} = \mathbf{x} \text{ for all } \mathbf{x} \text{ which are orthogonal to both } \mathbf{b} \text{ and } \mathbf{d}.$$

(4)

For the proof of Theorem 3.1 the reader is referred to Ref. 1.

An algorithm for computing the optimal pointing is proposed, as follows. We assume $\mathbf{d} \neq \pm \mathbf{b}$, and $\langle \mathbf{d}, \mathbf{b} \rangle \neq 0$. Let $\phi_1, \dots, \phi_{n-2}$ be n -dimensional vectors such that

$$\mathbf{b}, (\mathbf{d} - \langle \mathbf{d}, \mathbf{b} \rangle \mathbf{b}) \|\mathbf{d} - \langle \mathbf{d}, \mathbf{b} \rangle \mathbf{b}\|^{-1}, \phi_1, \dots, \phi_{n-2} \quad (5)$$

form an orthonormal basis in $\mathbf{R}^n$ (to obtain such vectors, apply the Gram-Schmidt orthonormalization to a linearly independent set $\mathbf{b}, \mathbf{d}, \psi_1, \dots, \psi_{n-2}$, where, for example, $\psi_1, \dots, \psi_{n-2}$ can be unit coordinate vectors, suitably chosen). Let A be the orthogonal $n \times n$ matrix having columns as in Eq. (5) (in this order). Then

$$T_0 = A \begin{bmatrix} \langle \mathbf{d}, \mathbf{b} \rangle & \pm(-q) & 0 \\ q & \pm \langle \mathbf{d}, \mathbf{b} \rangle & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix} A', \quad (6)$$

where $q = (1 - \langle \mathbf{d}, \mathbf{b} \rangle^2) \|\mathbf{d} - \langle \mathbf{d}, \mathbf{b} \rangle \mathbf{b}\|^{-1}$, and the sign in Eq. (6) is $+1$ if $\langle \mathbf{d}, \mathbf{b} \rangle > 0$ and -1 if $\langle \mathbf{d}, \mathbf{b} \rangle < 0$.

C. Minimal Proper Pointing

Recall that a pointing T is called *proper* if $\det(T) = 1$. We now present a solution to the minimal proper pointing problem (Problem (b) of the Introduction).

THEOREM 3.2. *Let $\mathbf{b}, \mathbf{d}$ be unit vectors in $\mathbf{R}^n$.*

(a) *The minimal deviation among all proper pointings,*

$$Q_p = \min_T \|T - I\|,$$

subject to $TT' = I$, $T\mathbf{b} = \mathbf{d}$, $\det(T) = 1$ is given by the formula

$$Q_p = 2(1 - \langle \mathbf{d}, \mathbf{b} \rangle)^{\frac{1}{2}}.$$

(b) *If $\mathbf{d} \neq -\mathbf{b}$, then a minimal proper pointing (i.e., a proper pointing T_0 such that $\|T_0 - I\| = 2(1 - \langle \mathbf{d}, \mathbf{b} \rangle)^{\frac{1}{2}}$) is unique and is given by the rotation that transforms $\mathbf{b}$ to $\mathbf{d}$ and leaves every vector in the orthogonal complement to $\text{span}\{\mathbf{b}, \mathbf{d}\}$ unchanged.*

(c) *Assume $\mathbf{d} = -\mathbf{b}$. Then all minimal proper pointings T_0 are given by the following recipe: Select any unit vector $\mathbf{v}$ which is orthogonal to $\mathbf{b}$, and define T_0 as the rotation through the angle π that transforms $\mathbf{b}$ to $\mathbf{d}$, $\mathbf{v}$ to $-\mathbf{v}$ and leaves every vector in the orthogonal complement to $\text{span}\{\mathbf{b}, \mathbf{v}\}$ unchanged.*

The result of Theorem 3.2 in case $\langle \mathbf{d}, \mathbf{b} \rangle \geq 0$ is essentially contained in Theorem 3.1. Indeed, the formula in Eq. (2) with the sign $+1$ gives the rotation described in the part (b) of Theorem 3.2. Since it is the minimal pointing by Theorem 3.1, it is also the minimal proper pointing.

An independent proof of Theorem 3.2 which is applicable to all cases is given in Ref. 1.

4 Optimal Pointing: Another Approach

In this section we suggest a different mathematical approach to select a pointing in the set of all pointings. Namely, the selection will be based on the maximal dimension of the subspace consisting of the vectors fixed by a pointing.

As before, let $\mathbf{b}$ and $\mathbf{d}$ be fixed unit length vectors in $\mathbf{R}^n$. As defined in the statement of the problem (see 1.A), a pointing T is said to be *optimal* if the subspace of all vectors $\mathbf{x}$ that are invariant under T (i.e., $T\mathbf{x} = \mathbf{x}$) has maximal dimension among all pointings.

It will be assumed throughout this section that $\mathbf{b} \neq \mathbf{d}$. We denote by W the orthogonal complement of the subspace spanned by $\mathbf{b}$ and $\mathbf{d}$. It is not difficult to see that the reflection with respect to the corresponding subspace that reflects $\mathbf{b}$ to $\mathbf{d}$ is an optimal pointing. It turns out that it is the only optimal pointing:

THEOREM 4.1. *The reflection with respect to $\text{span}\{\mathbf{b} + \mathbf{d}, W\}$ is the (only) optimal pointing.*

See Ref. 1 for the proof of Theorem 4.1.

Theorem 4.1 provides a simple procedure for the computation of an optimal pointing, as is demonstrated by the following example.

EXAMPLE 2. Let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

By solving the system $\langle \mathbf{b}, \mathbf{x} \rangle = 0$, $\langle \mathbf{d}, \mathbf{x} \rangle = 0$, we obtain that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for W . We have

$$\mathbf{b} + \mathbf{d} = [5 \ 5 \ 5 \ 5]'$$

The optimal pointing T is defined by $T\mathbf{v}_1 = \mathbf{v}_1$, $T\mathbf{v}_2 = \mathbf{v}_2$, $T(\mathbf{b} + \mathbf{d}) = \mathbf{b} + \mathbf{d}$, and $T\mathbf{b} = \mathbf{d}$. Therefore

$$T \begin{bmatrix} 1 & 2 & 5 & 1 \\ -2 & -3 & 5 & 2 \\ 1 & 0 & 5 & 3 \\ 0 & 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 & 4 \\ -2 & -3 & 5 & 3 \\ 1 & 0 & 5 & 2 \\ 0 & 1 & 5 & 1 \end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} .1 & -.3 & .3 & .9 \\ -.3 & .9 & .1 & .3 \\ .3 & .1 & .9 & -.3 \\ .9 & .3 & -.3 & .1 \end{bmatrix}.$$

A comparison between Theorems 3.1 and 4.1 reveals that if $\langle \mathbf{d}, \mathbf{b} \rangle \neq 0$, then the minimal and optimal pointings coincide. In other words, the pointing which has the least deviation from the identity matrix is also the pointing for which the dimension of the subspace of fixed vectors is maximal. In the case $\langle \mathbf{d}, \mathbf{b} \rangle = 0$, among the two minimal pointings T_0 given by Eq. (4), only one is optimal; namely, the pointing T_0 defined by $T_0 \mathbf{b} = \mathbf{d}$; $T_0 \mathbf{d} = \mathbf{b}$; $T_0 \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in W$.

Consider now *optimal proper pointings*, i.e., proper pointings T such that the subspace of vectors fixed by T has maximal dimension among all proper pointings.

THEOREM 4.2. *A pointing T is an optimal proper pointing if and only if it is a rotation around any $n-2$ dimensional subspace of $\text{span}\{\mathbf{b} + \mathbf{d}, W\}$ that rotates $\mathbf{b}$ to $\mathbf{d}$.*

The proof of Theorem 4.2 is found in Ref. 1.

In view of Theorem 4.2, we can now introduce the following definition. A pointing T is said to be a *perfect optimal proper pointing* if it is an optimal proper pointing with minimum rotation angle.

Note that, if $\mathbf{b} + \mathbf{d} \neq 0$, then $\mathbf{b}$ and $\mathbf{d}$ are linearly independent and $\dim(W) = n-2$. In this case we have

THEOREM 4.3. *If $\mathbf{b} \neq -\mathbf{d}$ then the rotation around W that rotates $\mathbf{d}$ to $\mathbf{b}$ (yielding a transformation from $\mathbf{b}$ to $\mathbf{d}$, see the third paragraph in Section I) is the (only) perfect optimal proper pointing.*

Proof. As is well known, the rotation angle that rotates $\mathbf{d}$ to $\mathbf{b}$ is greater than or equal to the angle between $\mathbf{b}$ and $\mathbf{d}$. The latter is the rotation angle if we rotate $\text{span}\{\mathbf{b}, \mathbf{d}\}$, that is a rotation around $\text{span}\{\mathbf{b}, \mathbf{d}\}^\perp = W$.

A comparison with Theorem 3.2 shows that (if $\mathbf{b} \neq -\mathbf{d}$) the perfect optimal proper pointing coincides with the minimal proper pointing. Theorem 4.3 provides a simple procedure for the computation of a perfect optimal proper pointing, as is demonstrated by the following example.

EXAMPLE 3. Let $\mathbf{b}, \mathbf{d}, \mathbf{v}_1, \mathbf{v}_2$ be as in Example 2. Our rotation T satisfies $T\mathbf{v}_1 = \mathbf{v}_1$ and $T\mathbf{v}_2 = \mathbf{v}_2$. Since T is an orthogonal transformation, it maps a vector orthogonal to $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{b}$ to a vector orthogonal to $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{b}$. Therefore, the vector

$$\mathbf{v}_3 = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ 1 & 2 & 3 & 4 \\ 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \\ 0 \\ -10 \end{bmatrix}$$

is mapped on

$$\mathbf{v}_4 = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ 4 & 3 & 2 & 1 \\ 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -10 \\ -20 \end{bmatrix}.$$

The perfect optimal proper pointing T is thus defined by $T\mathbf{v}_1 = \mathbf{v}_1$, $T\mathbf{v}_2 = \mathbf{v}_2$, $T\mathbf{b} = \mathbf{d}$, and $T\mathbf{v}_3 = \mathbf{v}_4$, and hence

$$T \begin{bmatrix} 1 & 2 & 1 & 20 \\ -2 & -3 & 2 & 10 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 10 \\ -2 & -3 & 3 & 0 \\ 1 & 0 & 2 & -10 \\ 0 & 1 & 1 & -20 \end{bmatrix}$$

$$\Rightarrow T = \frac{1}{30} \begin{bmatrix} 23 & 1 & 9 & 17 \\ -9 & 27 & 3 & 9 \\ -11 & -7 & 27 & 1 \\ -13 & -11 & -9 & 23 \end{bmatrix}.$$

5 Summary

In this work we addressed the problem of finding an orthogonal transformation of a *given* vector, $\mathbf{b}$, in $\mathbf{R}^n$ to another *given* vector, $\mathbf{d}$, of same length, in $\mathbf{R}^n$. We solved the general case of such transformations where the transformation matrix could be either proper or improper, and in addition certain optimality constraints are imposed (see below). In 3-D, and when considering only proper transformations without the optimality constraints, this problem is the well-known practical problem of *pointing*. In 2-D it is easy to see that we have only one proper and one improper such transformation matrices, but in n -D (where $n > 2$) we obviously have infinitely many such transformations. If we impose some constraints on the transformation, we limit the number of solutions. One such constraint is imposed by our search for the *minimal* pointing. Minimal pointing is defined as an orthogonal transformation T satisfying $T\mathbf{b} = \mathbf{d}$ that is also the closest (in the Euclidean, or Frobenius, norm) to the identity matrix. Another constraint is imposed by our search for *optimal* pointing, where optimal pointing is defined as the orthogonal transformation T satisfying $T\mathbf{b} = \mathbf{d}$ for which the subspace of all vectors $\mathbf{x}$, unaltered by T , is of maximal dimension among all transformations. When we limit these special transformations to proper transformations, then we obtain *minimal proper pointing* (MPP) and *optimal proper pointing*, respectively. If we further limit the latter to be of minimum rotation angle, then we obtain what we define as *perfect optimal proper pointing* (POPP).

In Section 2 we discussed the practical use of pointing in 3-D where, if the transformation is a proper one, the problem and its solution are easily visualized. Accordingly, in our geometric approach to the problem, we were looking for the axis of rotation about which the rotation of $\mathbf{b}$ into $\mathbf{d}$ was minimal; that is, the rotation angle was minimal. Obviously, this axis is the normal to the plane in which $\mathbf{d}$ and $\mathbf{b}$ lie, and the rotation angle is the angle between $\mathbf{d}$ and $\mathbf{b}$. We found the corresponding transformation matrix, T_m , using the quaternion of rotation. It is easy to see that we were finding the POPP (see its definition in n -D) in 3-D. It is not difficult to see that this minimal matrix, T_m , is also the MPP in 3-D. Let us now see, geometrically, how the problem solved in 3-D corresponds to POPP and MPP. Before doing so, let us be reminded that the minimal rotation problem in 3-D, the POPP and the MPP, all deal with proper transformations. Bearing in mind that this is the case, we omit, in the following discussion, the specific reference to proper rotations.

As shown in Section 4, barring the trivial case when $\mathbf{d}$ and $\mathbf{b}$ are orthogonal, for the POPP in n -D, the dimension of the subspace $W = \text{span}\{\mathbf{b}, \mathbf{d}\}^\perp$ is $n - 2$, and the rotation angle about W which brings $\mathbf{b}$ into $\mathbf{d}$ is minimal. Moreover, then the MPP and POPP are identical. Now, in 3-D, $\text{span}\{\mathbf{b}, \mathbf{d}\}$ is the plane defined by $\mathbf{b}$ and $\mathbf{d}$, the dimension of W is, of course, 1, and W is no other than the Euler axis about which the rotation is performed. The angle of rotation is, of course, the angle between $\mathbf{d}$ and $\mathbf{b}$. The correspondence between the MPP and the n -D case is the same, only that Theorem 3.1 treats the problem in a different basis. In the 3-D case this means that the Cartesian coordinates are defined such that $\mathbf{d}$ and $\mathbf{b}$ lie in the xy plane, and thus W , or the rotation axis, is along the z axis. Thus, I_{n-2} in Eq. (6) is just the number 1.

Finally, it should be noted that this paper lays out the theoretical background for the three dimensional minimal (optimal) pointing problem which is encountered in target acquisition, target designation, thrust vectoring, robotics, and in guidance problems in general. This problem is also encountered in coding theory and information theory, although there, the pointing is not necessarily proper pointing.

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