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# A NEW INTERPRETATION OF THE LQG/LTR TECHNIQUE USING OPTIMAL PROJECTION EQUATIONS

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**Abstract**

This paper extends the optimal projection equation (O.P.E.) robustness conditions of Bernstein [1] to a class of distributed parameter systems described in Section 1. The O.P.E. approach for reduced order controller design was extended to infinite dimensional systems by Bernstein in 1986 [2]. Using that work, Bernstein's conditions for robustness are extended to infinite-dimensional systems by following his development in [1]. Thus, Theorem 3.3 is a result of his earlier work.

Section 2 demonstrates that one cannot recover a loop transfer function perfectly when the order of the controller is less than the order of the design model. This points out that one cannot perform perfect asymptotic loop transfer function recovery using a reduced order controller. However, as a result of the O.P.E. robustness conditions of Theorem 3.3, a new interpretation of the LQG/LTR technique is given in Section 4. It is demonstrated that, although loop transfer function recovery does not occur when a finite-dimensional controller is used along with an infinite-dimensional design model, one can achieve robustness to bounded perturbations by using the LQG/LTR technique. This is an insight not mentioned in the literature since the LQG/LTR technique has been used to recover loop transfer functions to achieve robustness.

## 1 Introduction

The class of problems addressed in this paper is the set of infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t), \quad x_o = x(0) \in D(A) \tag{1}$$

$$y(t) = Cx(t) + \eta(t) \tag{2}$$

where the control vector $u$ is in the input space $U = L^2\{[0,\infty);\Re^N\}$ , $x$ is an element of a Hilbert space $\mathcal{H}$, $y$ is an observation vector which is an element the output space $Y = \Re^N$, and $w$ and $\eta$ are white Gaussian noise terms with realizations in the spaces $\mathcal{H}$ and $Y$ respectively. The strength of the dynamics noise term $w$ is described by the positive semi-definite operator $Q_o$, and the strength of the measurement noise term $\eta$ is described by the strictly positive operator $R_f$. The operator $Q_f$ used in the nominal Kalman filter design will be chosen so that $Q_f = GQ_oG^*$, where $G^*$ denotes the adjoint of $G$. $x(t)$ will be denoted simply as $x$ (and similarly for the other functions), and the following assumptions are made:

1. $A$ is the infinitesimal generator of a $C_o$ semigroup (i.e., strongly continuous) $T(t)$ on a real separable Hilbert space (Hilbert space with a countable orthonormal basis) $\mathcal{H}$ [3].
2. $B$ is a bounded linear operator from $\Re^N$ to $\mathcal{H}$.
3. $C$ is a bounded linear operator from $\mathcal{H}$ to $\Re^N$.
4. $G$ is a bounded linear operator from the Hilbert space $\mathcal{H}$ to $\mathcal{H}$.
5. The spectrum of $A$ (denoted $\sigma(A)$) is discrete.
6. The system is exponentially stabilizable and detectable [3].
7. The eigenvectors of $A$ are complete.
8. The system is minimum phase (i.e. no transmission zeros in the right-half plane).
9. $A$ satisfies the *spectrum decomposition assumption* [4]. This means that the spectrum of $A$ contains only a finite number of eigenvalues whose real part is greater than some $\omega$ which determines the exponential stability of the operator.
10. The restriction of $A$ to the stable subspace $\mathcal{H}_s$ satisfies the spectrum determined growth assumption [3] (i.e. the supremum of the real part of the spectrum of $A_s$ equals the growth constant of the semigroup $T(t)$ generated by $A_s$) and generates an exponentially stable semigroup.

An approach being currently taken to address the problem of robust reduced order controllers [2, 5, 6, 1, 7, 8, 9] is to fix the order of the compensator based on physical constraints, and determine the optimal robust controller using the optimal projection equation approach [5]. Ignoring the issue of robustness for the moment, Bernstein [2] gives a set of necessary conditions for a reduced order controller to be the "optimum controller". A controller will be considered "optimum" if it stabilizes the system at design conditions, and if it produces a feedback control law that minimizes a desired cost functional which characterizes the system's steady state performance. The cost functional to be minimized will be of the form

$$J = \lim_{t\to\infty} \mathcal{E}[< Q_c x(t), x(t) > + < R_c u(t), u(t) >]$$

where $\mathcal{E}$ denotes the expectation operator. This is discussed in more detail in Section 3.

Let $\Lambda$ and $\Gamma$ be bounded linear operators mapping $\mathcal{H} \to \Re^k$ (where $k$ is the dimension of the finite-dimensional controller) such that $\Gamma\Lambda^* = I_k$. $I_k$ is the identity operator on $\Re^k$. The conditions developed by [2] are a pair of modified algebraic Riccati equations (A.R.E.s) and a pair of coupled Lyapunov equations. The two modified A.R.E.s are given by

$$AQ + QA^* + Q_f - QC^*R_f^{-1}CQ + \tau_\perp QC^*R_f^{-1}CQ\tau_\perp^* = 0 \tag{3}$$

$$A^*P + PA + Q_c - PBR_c^{-1}B^*P + \tau_\perp^* PBR_c^{-1}B^*P\tau_\perp = 0 \tag{4}$$

and the two Lyapunov equations are given by

$$(A - BR_c^{-1}B^*P)\hat{Q} + \hat{Q}(A - BR_c^{-1}B^*P)^* + QC^*R_f^{-1}CQ - \tau_\perp QC^*R_f^{-1}CQ\tau_\perp^* = 0 \tag{5}$$

$$(A - QC^*R_f^{-1}C)^*\hat{P} + \hat{P}(A - QC^*R_f^{-1}C) + PBR_c^{-1}B^*P - \tau_\perp^* PBR_c^{-1}B^*P\tau_\perp = 0 \tag{6}$$

where the operator $\tau$ is defined by the the operators $\Lambda$ and $\Gamma$ ($\tau = \Lambda^*\Gamma$) which determine the projection of the full order compensator to a fixed order compensator. The projection operator $\tau_\perp$ satisfies the relation

$$\tau_\perp = I - \tau = I - \Lambda^*\Gamma \tag{7}$$

and using the operators $\Lambda$ and $\Gamma$, the compensator is defined by the equations

$$\dot{x}_c = A_c x_c + B_c y \tag{8}$$

$$u = C_c x_c \tag{9}$$

where $A_c$, $B_c$ and $C_c$ are given by

$$A_c = \Gamma(A - QC^*R_f^{-1}C - BR_c^{-1}B^*P)\Lambda^* \tag{10}$$

$$B_c = \Gamma QC^*R_f^{-1} \tag{11}$$

$$C_c = -R_c^{-1}B^*P\Lambda^* \tag{12}$$

Notice that the Lyapunov equations are coupled to the A.R.E.s by the projection operator $\tau_\perp$. If the order of the system equals the order of the compensator, then $\tau_\perp = 0$, and one gets the standard LQG A.R.E.s, and the Lyapunov equations become decoupled from the A.R.E.s. In that case the standard LQG results are obtained. For the case when the order of the compensator is less than the system order, one might wonder if the optimum projection equations (O.P.E.) can be modified not only to stabilize the system and minimize the associated cost functional $J$, but also to provide robustness as is done using the LQG/LTR technique [10, 11].

The next section will consider whether or not loop transfer functions can be recovered when the order of the compensator is less than or equal to the order of the system.

## 2 Loop Recovery

Let $\Lambda$ and $\Gamma$ be the bounded linear operators that define the projection of the full order compensator. Figure 1 represents the infinite-dimensional system using the finite-dimensional controller.

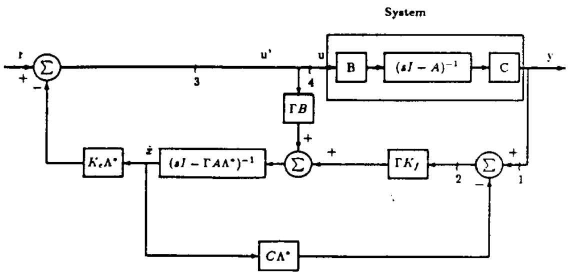


Figure 1: System with Finite-Dimensional LQG Based Controller

The infinite-dimensional system is described by the equations

$$\dot{x} = Ax + Bu \tag{13}$$

$$y = Cx \tag{14}$$

and the compensator is described by Equations (8)-(12). With $A_c, B_c$ and $C_c$ given as in Equations (10)-(12), the compensator state equations can be written as

$$\dot{x}_c = (\Gamma A \Lambda^* - \Gamma K_f C \Lambda^* - \Gamma B K_c \Lambda^*)x_c + \Gamma K_f y \tag{15}$$

or:

$$\dot{x}_c = \Gamma A \Lambda^* x_c + \Gamma B u + \Gamma K_f (y - C\Lambda^* x_c) \tag{16}$$

where $u = -K_c \Lambda^* x_c$.

The system input appears at point 4 in Figure 1. This point is physically important since it is one point where the compensator interfaces with the system being controlled. The other point that is important is point 1 where the system output interfaces with the compensator. The transfer function at point 4 is given by

$$G_4(s) = K(s)P(s) = C_c(sI - A_c)^{-1}B_c C(sI - A)^{-1}B \tag{17}$$

which can be written as

$$G_4(s) = K_c\Lambda^*(\Phi_c^{-1} + \Gamma K_f C \Lambda^* + \Gamma B K_c \Lambda^*)^{-1}\Gamma K_f C \Phi_p B \tag{18}$$

where the controller state transition operator is $\Phi_c = (sI - \Gamma A \Lambda^*)^{-1}$ and the plant state transition operator is $\Phi_p = (sI - A)^{-1}$. Since $A$ generates a $C_o$ semigroup, $(sI - A)^{-1}$ will exist. $(sI - \Gamma A \Lambda^*)^{-1}$ will exist since $\Gamma A \Lambda^*$ is a finite-dimensional operator. Note that when the order of the compensator is less than order of the system, $\Phi_c$ does not equal $\Phi_p$. For infinite-dimensional systems, the state transition operator is the semigroup $T(t)$ which is generated by the operator $A$.

In a similar fashion, one can write the transfer function at point 3 in the system. This is the transfer function that one would try to recover at point 4 using the LQG/LTR technique for robustness enhancement. This point is internal to the compensator, and has guaranteed robustness properties as discussed in Matson's dissertation [11]. Using the equations

$$u = -K_c \Lambda^* x_c \tag{19}$$

and

$$y = C\Phi_p B u' \tag{20}$$

where $u'$ is the input at point 3, the state $x_c$ can be expressed as

$$x_c = (sI - \Gamma A \Lambda^*)^{-1}\Gamma B u' + (sI - \Gamma A \Lambda^*)^{-1}\Gamma K_f y - (sI - \Gamma A \Lambda^*)^{-1}\Gamma K_f C \Lambda^* x_c \tag{21}$$

or equivalently

$$x_c = (I + \Phi_c \Gamma K_f C \Lambda^*)^{-1}[\Phi_c \Gamma B u' + \Phi_c \Gamma K_f y] \tag{22}$$

Substituting for $y$ via Equation (20), and rearranging terms, the state can be written as

$$x_c = (I + \Phi_c \Gamma K_f C \Lambda^*)^{-1}(\Phi_c \Gamma)(I + K_f C \Phi_p)Bu' \tag{23}$$

Substituting this into the expression for $u$, Equation (19), the loop transfer function at point 3 is given as

$$G_3(s) = -K_c\Lambda^*(I + \Phi_c \Gamma K_f C \Lambda^*)^{-1}(\Phi_c \Gamma)(I + K_f C \Phi_p)B \tag{24}$$

If the order of the compensator equals the order of the system, then $\tau$ is an identity operator (and $\tau_\perp$ is a zero operator), and from Equation (24) it is clear that the loop transfer function at point 3 can be expressed as

$$G_3(s) = -K_c \Phi_p B \tag{25}$$

Since $\Phi_c$ does not equal $\Phi_p$ when a reduced order controller is used, then one cannot perfectly recover a desired transfer function asymptotically via LTR methods if the problem involves an infinite-dimensional design model. Note that, since any model is an approximation of the true system, then the controller order will always be less than the true system order. The issue considered here is the case when the controller order is less than the design model order. The design model is the mathematical model one chooses to describe the physical system to be controlled, and as such, it is an approximation of the true system. In practice, one assumes that the design model is the true system so that a result can be synthesized. Robustness is needed due to the fact that the design model does not equal the true system being controlled. Thus, loop transfer recovery using a finite-dimensional controller can only be accomplished using a reduced order model.

## 3 O.P.E. Robustness

The O.P.E. approach allows one to achieve robustness by modifying the infinite-dimensional A.R.E., and this will give a new interpretation of LQG/LTR when the controller is finite-dimensional but the design model is infinite-dimensional. The O.P.E. approach provides a way to achieve robustness to uncertainty, and to minimize the cost functional $J$, at conditions other than the nominal design condition [9].

For a $k$-th order compensator where $k < dim\mathcal{H} = \infty$, one wants to determine the operators $(A_c, B_c, C_c)$ such that the closed-loop system consisting of the controlled system

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u + Gw \tag{26}$$

where $x \in \mathcal{H}$, along with measurements

$$y = (C + \Delta C)x + \eta \tag{27}$$

with $y \in \Re^N$, and a finite-dimensional compensator described by

$$\dot{x}_c = A_c x_c + B_c y \tag{28}$$

$$u = C_c x_c \tag{29}$$

is exponentially stable for all perturbations $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, where $\mathcal{U}$ is the set of admissible operator triplets describing the perturbations to the operators $A$, $B$, and $C$ one wishes to consider. If, for instance, one only allows bounded perturbations, then $\mathcal{U}$ is the set of bounded linear operators which are bounded by some constant, say $D$. Through $\mathcal{U}$, one describes the robustness desired.

However, in addition to being exponentially stable, one also would like to minimize the cost functional associated with the optimal control problem at other than design conditions. This will be made clearer later, in Equation (37). The cost functional to be considered is denoted by $J$ and will be defined as

$$J(A_c, B_c, C_c) = \lim_{t \to \infty} \mathcal{E}[< Q_c x(t), x(t) > + < R_c u(t), u(t) >] \tag{30}$$

where $\mathcal{E}$ is the expectation operator and is defined as [12]

$$\mathcal{E}(x) = \int_\Psi x(\omega) dP \tag{31}$$

where $\Psi$ is the space of all possible $\omega$ that the random variable $x(t)$ maps to some Borel space over which a probability measure $P$ is defined. This cost functional is chosen instead of the one given by Curtain [3] since the objective of the O.P.E. approach is to achieve optimum steady state performance, and not necessarily optimum performance over the entire time interval.

For a reduced order compensator where $k < dim\mathcal{H}$, one wants to determine $(A_c, B_c, C_c)$ such that when the closed-loop system consisting of

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u + Gw \tag{32}$$

with noisy measurements given by

$$y = (C + \Delta C)x + \eta \tag{33}$$

is coupled with the compensator, the steady-state performance cost functional

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathcal{U}} \lim_{t \to \infty} \mathcal{E}[< Q_c x(t), x(t) > + < R_c u(t), u(t) >] \tag{34}$$

is minimized. In other words, one wants to design a controller so that the largest value that the cost functional can take on for all possible perturbations is minimized. In the development that follows, $G$ is assumed to be the identity operator (without loss of generality) in order to correspond to the development of Bernstein [1].

A control will be considered admissible only if it forces the cost functional $J$ to take on a finite value. To help simplify the notation, the closed-loop system can be written in terms of the augmented state-space $\tilde{\mathcal{H}} = \mathcal{H} \oplus \Re^k$. It is assumed that the noise terms $w$ and $\eta$ are independent. In terms of the augmented state vector

$$\tilde{x} = [x \quad x_c]^T \tag{35}$$

this yields

$$\dot{\tilde{x}} = (\tilde{A} + \Delta\tilde{A})\tilde{x} + \tilde{G}\zeta \tag{36}$$

where

$$\tilde{G} = \begin{bmatrix} I & 0 \\ 0 & B_c \end{bmatrix}$$

$$\zeta = \begin{bmatrix} w \\ \eta \end{bmatrix}$$

$$\mathcal{E}[\zeta(t)\zeta^T(t+\tau)] = \begin{bmatrix} Q_o & 0 \\ 0 & R_f \end{bmatrix} \delta(\tau)$$

$$\tilde{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}$$

$$\Delta\tilde{A} = \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c \Delta C & 0 \end{bmatrix}$$

and

$$\tilde{V} = \tilde{G} \begin{bmatrix} Q_o & 0 \\ 0 & R_f \end{bmatrix} \tilde{G}^*$$

or

$$\tilde{V} = \begin{bmatrix} Q_o & 0 \\ 0 & B_c R_f B_c^* \end{bmatrix}$$

Also, in terms of $\tilde{x}$, the performance cost functional can be written as

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathcal{U}} \lim_{t \to \infty} \mathcal{E}[< \tilde{R}\tilde{x}, \tilde{x} >] \tag{37}$$

where

$$\tilde{R} = \begin{bmatrix} Q_c & 0 \\ 0 & C_c^T R_c C_c \end{bmatrix}$$

The following lemma allows one to express the cost functional in terms of the second moment of $\tilde{x}(t)$. This will be needed so that an upper bound of the cost functional can be established in a theorem to follow.

**LEMMA 3.1**: For any given $(A_c, B_c, C_c)$, and $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, the performance cost functional can be expressed in terms of the covariance of $\tilde{x}(t)$, defined as:

$$\tilde{Q}_\Delta(t) = \mathcal{E}[(\tilde{x}(t) - \mathcal{E}\tilde{x}(t))(\tilde{x}(t) - \mathcal{E}\tilde{x}(t))^*]$$

Furthemore, if the system is stable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, then the performance cost functional can be expressed as

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathcal{U}} tr[\tilde{Q}_\Delta \tilde{R}]$$

where $tr$ denotes the trace operator, and where $\tilde{Q}_\Delta$ satisfies the equation

$$(\tilde{A} + \Delta\tilde{A})\tilde{Q}_\Delta + \tilde{Q}_\Delta(\tilde{A} + \Delta\tilde{A})^* + \tilde{V} = 0$$

**Proof**: Balakrishnan [13], page 317, defines the covariance of $\tilde{x}(t)$ as the nonnegative-definite operator $\tilde{Q}_\Delta(t)$ given above. Bernstein and Hyland [2], Lemma 4.1, page 137, prove that $J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathcal{U}} tr[\tilde{Q}_\Delta \tilde{R}]$ and Lemma 4.4, page 139 [2] proves that $\tilde{Q}_\Delta$ satisfies the last equation of Lemma 3.1. **Q.E.D.**

In the development to follow, the operator $\tilde{V}$ will be modified by a nonnegative operator $\Upsilon$. A result that will be needed is, if $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable, then so is $([\tilde{V} + \Upsilon]^{1/2}, \tilde{A} + \Delta\tilde{A})$ under certain conditions. The following theorem from Wonham [14] gives conditions under which the property of detectability is preserved.

**THEOREM 3.2**: Let $\mathcal{H}$ be a real Hilbert space, and let $M_m$ be a bounded linear operator mapping $\mathcal{H} \to \mathcal{H}$. If $M_m$ is a nonnegative operator and if $(M_m^{1/2}, A)$ is detectable, then for all nonnegative operators $N$, the pair $([M_m + N]^{1/2}, A)$ is detectable.

**Proof**: See Wonham's book [14] page 79, and let the operators of Theorem 3.6, $Q$ and $B$, be such that $Q = B = 0$. **Q.E.D.**

The next theorem is the main theorem of this section, and it provides sufficient conditions for robust stability and optimum performance. In the theorem to follow, the operator $\Omega$ is a positive self-adjoint operator that "bounds" the uncertainty described by the operator $\Delta\tilde{A}$. The operator $\Omega$ is part of a Lyapunov condition (which will be defined in more precision in the theorem) involving the nominal system operators. The operator $\underline{Q}$ is the bounded positive-semidefinite self-adjoint operator solution for the Lyapunov condition, and it is the only unknown in the Lyapunov equation. It is assumed that $\Omega$ and $\underline{Q}$ both exist. Satisfying the Lyapunov equation will ensure stability in the presence of perturbations described by the operator $\Delta\tilde{A}$. The next theorem demonstrates that the operator $\Omega$ is also a function of the operator $\underline{Q}$.

**THEOREM 3.3**: Let $\Omega : \mathcal{H} \to \mathcal{H}$ be a self-adjoint positive operator such that

$$i) \quad < \Delta\tilde{A}^* x, \underline{Q}x > + < \underline{Q}x, \Delta\tilde{A}^* x > \quad \leq \quad < \Omega x, x > \quad \forall x \in \mathcal{H}$$

for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$.

Also, for a given $(A_c, B_c, C_c)$, assume that there exists a $\underline{Q} \in \mathcal{S}$ (where $\mathcal{S}$ is the class of bounded, positive-semidefinite, self adjoint operators) such that $\underline{Q}$ satisfies the equation

$$ii) \quad \tilde{A}\underline{Q} + \underline{Q}\tilde{A}^* + \Omega + \tilde{V} \quad = \quad 0$$

on $D(\tilde{A})$.

In addition, assume that $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ (where $\Delta\tilde{A}$ is defined by Equation (36)) is detectable for all $(\Delta A, \Delta B, \Delta C) \in U$. Then, $\tilde{A} + \Delta\tilde{A}$ is exponentially stable for all $(\Delta A, \Delta B, \Delta C) \in U$. Also,

$$\tilde{Q}_\Delta \leq \underline{Q}$$

where $\tilde{Q}_\Delta$ satisfies Equation (ii) and

$$J(A_c, B_c, C_c) \leq tr[\underline{Q}\tilde{R}]$$

**Proof**: Following Bernstein [1, 9], and recalling Lemma 3.1, for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, Equation (ii) is equivalent to

$$(\tilde{A} + \Delta\tilde{A})\underline{Q} + \underline{Q}(\tilde{A}^* + \Delta\tilde{A}^*) + \tilde{V} + \Upsilon(\underline{Q}, B_c, C_c, \Delta\tilde{A}) = 0$$

where $\Upsilon = \Omega - (\Delta\tilde{A}\underline{Q} + \underline{Q}\Delta\tilde{A}^*)$.

Notice that, by Equation (i) of the theorem, $\Upsilon$ is nonnegative for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Since $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is assumed detectable, then by Theorem 3.2, it follows that $([\tilde{V} + \Upsilon]^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$. Bernstein [2] Lemma 4.1 gives that this detectability condition, along with the fact that the assumption that $\underline{Q}$ is bounded implies that $(\tilde{A} + \Delta\tilde{A})$ must be stable.

Next, subtracting

$$(\tilde{A} + \Delta\tilde{A})\tilde{Q}_\Delta + \tilde{Q}_\Delta(\tilde{A} + \Delta\tilde{A})^* + \tilde{V} = 0$$

(which is a result from Lemma 3.1) from the first equation in this proof,

$$(\tilde{A} + \Delta\tilde{A})\underline{Q} + \underline{Q}(\tilde{A}^* + \Delta\tilde{A}^*) + \tilde{V} + \Upsilon = 0$$

yields that

$$(\tilde{A} + \Delta\tilde{A})(\underline{Q} - \tilde{Q}_\Delta) + (\underline{Q} - \tilde{Q}_\Delta)(\tilde{A}^* + \Delta\tilde{A}^*) + \Upsilon = 0$$

Since Bernstein [2] Lemma 4.1 yields that $(\tilde{A} + \Delta\tilde{A})$ is stable, then Lemmas 4.4 and 4.1 of [2] allow one to write

$$\underline{Q} - \tilde{Q}_\Delta = \int_0^\infty T(t)\Upsilon T^*(t)dt \quad \geq \quad 0$$

where $T(t)$ is the semigroup generated by $(\tilde{A} + \Delta\tilde{A})$. Thus, $\underline{Q} \geq \tilde{Q}_\Delta$ and as a result

$$tr[\underline{Q}\tilde{R}] \geq tr[\tilde{Q}_\Delta \tilde{R}]$$

for all $(\Delta A, \Delta B, \Delta C) \in \mathcal{U}$, so that

$$J(A_c, B_c, C_c) \leq tr[\underline{Q}\tilde{R}]$$

Note that, since $\underline{Q}$ is bounded, then this last inequality places an upper bound on the $\sup tr[\tilde{Q}_\Delta \tilde{R}]$. **Q.E.D.**

Theorem 3.3 provides a sufficient condition that ensures robustness, and provides an upper bound on the value of the cost functional $J$. By choosing an operator $\Omega$ so that conditions (i) and (ii) of Theorem 3.3 are satisfied, one can achieve robustness. The difficulty with the theorem is the ability to find an operator $\Omega$ that satisfies the conditions. One would like to choose $\Omega$ so that it reflects uncertainty in a meaningful way, and this may be difficult to do. Also, the existence of a bounded operator $\underline{Q}$ is assumed.

## 4 Insights

Choosing $\Omega$ is based on the type of perturbations one wishes to consider, and depends on how one chooses to model uncertainty. Bernstein [1] gives one choice of $\Omega$ that works for finite-dimensional problems in which uncertainty is described in terms of stability radius. The only constraint on $\Omega$ is that it be chosen so that the hypotheses of Theorem 3.3 are satisfied. Assuming that is done, then using the operator $\Omega$ allows one to accomplish a procedure like that of Bernstein [2, 1]. Thus, by setting the Frechet derivatives of the cost functional

$$L(\underline{Q}, A_c, B_c, C_c) = tr[\lambda\underline{Q}\tilde{R} + (\tilde{A}\underline{Q} + \underline{Q}\tilde{A} + \Omega + \tilde{V})p]$$

(where $\lambda$ and $p$ are nonzero scalar Lagrange multipliers) to zero, one could derive conditions similar in form to those of Theorem 8.1 of [1] with the obvious differences due to operators being considered instead of matrices. The exact form of the resulting equations depends on the form chosen for $\Omega$.

The advantage of this approach is that one can possibly address perturbations other than bounded ones. The disadvantage is that an algorithm to solve the complex coupled equations is not readily available. However, the approach does allow a new interpretation of the LQG/LTR technique.

Let $\Delta B = 0$ and $\Delta C = 0$. Then one obtains that $\Delta\tilde{A} = \Delta A$. Following the development of Bernstein's Theorem 8.1 [1], one finds that one of the necessary A.R.E.s that must be satisfied is

$$AQ + QA^* + Q_o + \Omega - QC^*R_f^{-1}CQ + \tau_\perp QC^*R_f^{-1}CQ\tau_\perp^* = 0 \qquad (38)$$

Note that if $\Delta\tilde{A}$ and $\Delta\tilde{A}^*$ are bounded, then one can choose $\Omega = \beta^2 BVB^*$, and as $\beta \to \infty$ one gets that

$$\Omega \geq \Delta\tilde{A}\underline{Q} + \underline{Q}\Delta\tilde{A}^* \qquad (39)$$

Substituting $\Omega = \beta^2 BVB^*$ into this equation yields

$$BVB^* \geq \Delta\tilde{A}\frac{\underline{Q}}{\beta^2} + \frac{\underline{Q}}{\beta^2}\Delta\tilde{A}^* \qquad (40)$$

and the work of Matson [11] demonstrates that the operator $\frac{\underline{Q}}{\beta^2}$ monotonically decreases as $\beta \to \infty$ since $\underline{Q}$ is bounded. Therefore, one can achieve robustness to bounded perturbations by using a "tuning" procedure like LQG/LTR. Note that, by doing this, one does not asymptotically recover a loop transfer function with guaranteed margins, as was pointed out before. Also, this form for $\Omega$ will result in the other A.R.E. given by

$$A^*P + PA + R_1 - PBR_2^{-1}B^*P + \tau_\perp^* PBR_2^{-1}B^*P\tau_\perp = 0 \qquad (41)$$

which does not invlove $\Omega$. The only assumption is that Equation (ii) of Theorem 3.3 be satisfied in order to guarantee stability. In this way, $\underline{Q}$ will be admissible only if Equation (ii) is satisfied, which will also ensure that the cost functional $J$ is finite, since $\underline{Q}$ is bounded.

## 5 Summary

This paper demonstrates that the LQG/LTR technique can be viewed as a way to achieve robustness even under the constraint of a reduced order controller, even though one is not necessarily recovering a desired transfer function asymptotically. Also, note that if $\Delta\tilde{A}$ and $\Delta\tilde{A}^*$ are unbounded operators, then one cannot find a $\beta$ large enough to satisfy Theorem 5.3.3 when $\Omega$ is chosen to be $\beta^2 BVB^*$. This is similar to the problem in [15] where one needs to find a $\beta$ sufficiently large so that $K_\beta$ is uniformly bounded. The O.P.E. approach gives an expanded view of the LQG/LTR technique when the order of the controller is intentionally less than the order of the system design model. Also, the O.P.E. approach allows one to choose other forms for $\Omega$ which may give more flexibility as to how one models the system perturbations.

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