Transient Scattering from Two Cavities Embedded in a Ground Plane

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Abstract—This work presents a numerical method for modeling electromagnetic scattering from two-dimensional cavities embedded in a perfectly conducting ground plane. First, the problem is discretized in time with Newmark's method, then solved using finite-element boundary-integral technique. A proof is presented to demonstrate that the discretized variational formulation is well-posed at each time step.

I. INTRODUCTION

Electromagnetic scattering from open cavities and cavity-backed apertures has been an active area of research with applications in aircraft design and radar cross-section (RCS) modeling. Past work on cavity scatterers has either focused on single-cavity environments, as in [4], [2], or the time-harmonic domain, as in [1], [3]. Transient applications are increasingly in demand because they may be applied wide-band and non-linear problems. This paper consolidates previous work by Van, Li, and Wood [4], [3] by presenting an application of the Newmark method to discretize and solve a two-cavity scattering problem. Additionally, it is proved that the formulation at each time step is well-posed.

Fig. 1. Two cavity environment

II. PROBLEM STATEMENT

A. Problem Environment

This work considers an infinite perfectly-conducting ground plane (PEC) with two fully-embedded two-dimensional cavities. In three dimensions, these cavities are z -invariant infinite grooves, or channels. The TM^z orientation is considered

here, indicating that the electric field will be fully z -directed, described as a function of x and y . Above the ground plane is considered free space $\mathbb{R}_+^2 = \{x, y \in \mathbb{R}^2 : y \geq 0\}$ with homogeneous parameters ϵ_0, μ_0 . The cavities, denoted Ω_1, Ω_2 , may be filled with non-magnetic, homogeneous, dielectric medium, with parameters $\mu_0, \epsilon_j, j = 1, 2, \epsilon_j \in \mathbb{C}, \text{Re}\{\epsilon_j\} > 0, \text{Im}\{\epsilon_j\} \geq 0$. PEC cavity walls are denoted S_1, S_2 and cavity apertures denoted Γ_1, Γ_2 , respectively for cavities 1 and 2. This environment is illustrated in Fig. 1.

B. Transient Scattering Problem

The problem is posed as an initial value problem. Given an incident TM^z electric field $u^{inc}(x, y, t)$ defined in $\mathbb{R}_+^2 \times t \geq 0$, find the total field $u(x, y, T)$ for a given time $T > 0$ and $(x, y) \in \mathbb{R}_+^2$ subject to:

$$\begin{cases} -\Delta u + \epsilon_r \frac{\partial^2 u}{\partial t^2} = 0 & \text{in } \mathbb{R}_+^2 \cup \Omega_1 \cup \Omega_2, \\ u = 0 & \text{on } \Gamma^c \cup S_1 \cup S_2, \\ \lim_{y \rightarrow 0+} \frac{\partial u}{\partial y} = \lim_{y \rightarrow 0-} \frac{\partial u}{\partial y} & \forall x : (x, 0) \in \Gamma_1 \cup \Gamma_2, \end{cases} \quad (1)$$

In the upper half plane, $u = u^{inc} + u^{ref} + u^s$, where u^{ref} is defined by a full reflection from a PEC ground plane. $u^s \in \mathbb{R}_+^2 \times [0, \infty)$ is purely outgoing, and subject to the standard radiation condition. Finally, we restrict the solution space to $H^1(\mathbb{R}_+^2 \cup \Omega_1 \cup \Omega_2)$ so the differential form of Maxwell's equations are applicable. In the proposed approach, which follows closely the method used in [4], the transient problem is first discretized in time, next the discretized problem is solved using a finite element-boundary integral (FE-BI) method. This process is then iterated using the Newmark method to progress in time.

C. Newmark Method

The Newmark method, is applied as in [4]. Using a fixed time step length h , the total number of time steps needed to approximate the fields at final time T is $N = T/h$. The notation, $u^n(x, y)$ and $\dot{u}^n(x, y)$ are used to denote the approximations of $u(x, y, nh)$ and $u_t(x, y, nh)$, respectively. For the TM^z case, the generated equations are:

$$\tilde{u}^{n+1} = u^n + h\dot{u}^n + \left(\frac{1}{2} - \beta\right)h^2\ddot{u}^n, \quad (2)$$

which is a prediction based on current time data, and

$$-\Delta u^{n+1} + \alpha^2 \varepsilon_r u^{n+1} = \alpha^2 \varepsilon_r \tilde{u}^{n+1}, \quad (3)$$

which is of similar form to the time-harmonic wave equation with a source term. Equation (3) is solved for u^{n+1} using the finite element method. Once a solution is obtained, the Newmark correction is applied:

$$\begin{aligned} \ddot{u}^{n+1} &= \alpha^2 (u^{n+1} - \tilde{u}^{n+1}) \\ \dot{u}^{n+1} &= \dot{u}^n + (1 - \gamma) h \ddot{u}^n + \gamma h \ddot{u}^{n+1} \end{aligned} \quad (4)$$

D. Exterior Solution

Given that u^{inc} is known for all time, then by applying image theory, u^{ref} is also known. That leaves u^s as the only unknown in the total exterior field u in $\mathbb{R}_+^2$. Define g_0 as u^{n+1} on $\Gamma_1 \cup \Gamma_2$ and zero elsewhere on the ground plane. Given u^{inc} and g_0 , the exterior fields can be solved analytically using the half-space Green's function, $G_\alpha(\vec{r}, \vec{r}')$, as demonstrated in [1], [4].

$$u^{ext}(\vec{r}) = u^{inc}(\vec{r}) - u^{inc}(\vec{r}_i) + u^s(\vec{r}). \quad (5)$$

$$u^s(\vec{r}) = \alpha^2 \int_{\mathbb{R}_+^2} G_\alpha \tilde{u}^s d\vec{r}' - \int_{\Gamma_1 \cup \Gamma_2} g_0 \frac{\partial G_\alpha}{\partial n} dx'. \quad (6)$$

$$G_\alpha(\vec{r}, \vec{r}') = \frac{1}{2\pi} \left[K_0(\alpha|\vec{r} - \vec{r}'|) - K_0(\alpha|\vec{r} - \vec{r}'_i|) \right]. \quad (7)$$

Where $\vec{r} = (x, y)$, $\vec{r}_i = (x, -y)$, and K_0 is the modified Bessel function of the second kind. By defining the exterior field in this way, we may construct the Neumann boundary condition for the discretized interior problem.

E. Interior Problem

The discretized interior problem is posed as a boundary value problem at each time step by combining information from equations (1) and (3). Thus, for each time step, we solve the following boundary value problem:

$$\begin{cases} -\Delta u^s + \alpha^2 \varepsilon_r u^s = \alpha^2 \varepsilon_r \tilde{u}^s & \text{in } \Omega_1 \cup \Omega_2, \\ \lim_{y \rightarrow 0^-} \frac{\partial u}{\partial y} = \lim_{y \rightarrow 0^+} \left(2 \frac{\partial u^{inc}}{\partial y} + \frac{\partial u^s}{\partial y} \right) & \text{on } \Gamma_1 \cup \Gamma_2, \\ u = 0 & \text{on } S_1 \cup S_2. \end{cases} \quad (8)$$

After applying Green's theorem and enforcing boundary and radiation conditions, the Neumann boundary condition on $\Gamma_1 \cup \Gamma_2$ can be summarized by:

$$\frac{\partial u^{n+1}}{\partial y} \Big|_{y=0}(x) = 2 \frac{\partial u^{inc, n+1}}{\partial y}(x, 0) + \tilde{H}^{n+1}(x) + [T_\alpha g](x), \quad (9)$$

$$\tilde{H}^{n+1}(x) = \alpha^2 \int_{\mathbb{R}_+^2} \frac{\partial G_\alpha}{\partial y}((x, 0), \vec{r}') \tilde{u}^{s, n+1}(\vec{r}') d\vec{r}',$$

$$[T_\alpha g](x) = \frac{1}{\pi} \int_{\Gamma_1 \cup \Gamma_2} \left[\frac{\partial^2}{\partial x^2} - \alpha^2 \right] K_0(\alpha|x - x'|) g_0(x') dx'.$$

We employ the finite element method to solve weakly the interior problem expressed in (8), deriving the variational formulation:

$$a(u^{n+1}, v) = b^{n+1}(v), \quad (10)$$

defined by:

$$\begin{aligned} a(u, v) &= \langle \nabla u, \nabla v \rangle_{\Omega_1 \cup \Omega_2} + \alpha^2 \langle \varepsilon_r u, v \rangle_{\Omega_1 \cup \Omega_2} - \langle T u, v \rangle_{\Gamma_1 \cup \Gamma_2}, \\ b(v) &= \alpha^2 \langle \varepsilon_r \tilde{u}, v \rangle_{\Omega_1 \cup \Omega_2} + \langle 2 \frac{\partial u^{inc}}{\partial y} + \tilde{H}, v \rangle_{\Gamma_1 \cup \Gamma_2}. \end{aligned}$$

III. RESULTS

The principal result of this work is the proof that the variational formulation (10) is well-posed. This proof builds upon the work in [4] which established that the one-cavity problem, similarly derived, is well-posed. We show that the bilinear operator $a(\cdot, \cdot)$ is coercive in $H^1(\Omega_1 \cup \Omega_2)$ and conclude by the Lax-Milgram theorem that a unique solution exists at each time step.

Theorem *The variational formulation (10) has a unique solution u in $H^1(\Omega_1 \cup \Omega_2)$ and satisfies $\|u\| \leq C\|g\|$ at time $T > 0$.*

Numerical implementation of the transient FE-BI method

Fig. 2. Continuous Wave Solution

models the transient near-field solution and is consistent with expected results. A sample problem, shown in Fig. 2, shows the maximum aperture fields generated by a continuous wave with wavelength of 1 m.

IV. CONCLUSION

This work presents a FE-BI method for modeling the transient scattering from two cavities embedded in the ground plane which has a well-posed variational formulation at each time step. This method is expected to be easily extended to multiple cavities embedded in a ground plane. Finally, the FE-BI method is in line with current applications used in the industry for cavity-backed apertures.

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