A Vector Generalized Finite Element-Boundary Integral Formulation for Scattering from Cavity-backed Apertures

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Introduction

Vector generalized finite element method (VGFEM) was recently developed for the solution of electromagnetic problems [2]. Theoretically, VGFEM is a powerful technique as it permits the use of a wide range of function spaces. Indeed, the method provides a rigorous framework for including approximation spaces seamlessly within a simulation. Doing so can provide better approximation, and perhaps, more efficient simulations. The method was successfully applied to canonical closed domain problems both in time and frequency domain. Since then, our goal has been to develop the accompanying suite of technologies that enable the simulation of more practical problems. This paper constitutes one such attempt, where we extend the application of the method to open domain problems by developing a hybrid VGFEM-Boundary Integral (VGFEM-BI) technique.

The formulation of VGFEM-BI compared to hybrid FEM-BI is not straightforward. Difficulties arise in implementation boundary conditions, as imposition of Dirichlet boundary conditions in VGFEM is not straightforward. The basis function space is not interpolatory. In addition, unlike FEM, auxiliary space of surface basis functions that is defined at the aperture can not be used with VGFEM as these basis functions do not satisfy the Babuska-Brezzi condition. This paper presents the means through which these difficulties can be overcome. The rest of the paper is organized as follows: In the next Section, we present a very brief overview of the the formulation. This is followed by a plethora of results that compare data obtained from VGFEM-BI against FEM-BI data and measurements. Finally, a summary of findings and directions for the future research are presented in the last Section.

Theory

Here, we formulate hybrid VGFEM-BI technique for the analysis of scattering from cavity-backed aperture in an infinite ground plane at $z = 0$. The problem domain is divided into 2 regions: free space region, Ω_f , above cavity ($z > 0$) and cavity region, Ω , for $(-c < z < 0)$, where c is the depth of cavity. We assume that the cavity region is source free and filled with a homogenous material having a relative permittivity ϵ_r and a relative permeability μ_r . Using equivalence principle and image theory, electromagnetic fields in these regions are decoupled and the fields in Ω are solved numerically by applying boundary conditions at the aperture. Electric field, $\mathbf{u}(\mathbf{r})$, in region Ω , whose boundary is denoted by $\partial\Omega := S = \bigcup_i S_i$, satisfies the vector Helmholtz equation

$$\begin{aligned} \nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{u}(\mathbf{r}) - k_0^2 \epsilon_r \mathbf{u}(\mathbf{r}) &= 0 \\ \mathfrak{B}_i\{\mathbf{u}(\mathbf{r})\} &= \mathbf{g}_i(\mathbf{r}), \forall \in S_i \end{aligned} \quad (1)$$

where k_0 is the free space wavenumber, $\mathfrak{B}_i$ is a differential operator, and $\mathbf{g}_i(\mathbf{r})$ is a boundary condition imposed on S_i . In order to solve Eq. (1) using VGFEM, the computation domain is first sampled and overlapping partition of unity domains are defined around each node such that $\Omega \subset \bigcup_i \Omega_i$. Then, a vector basis function space $\mathbf{V} = \text{span}\{\psi_i(\mathbf{r})\nabla \times (\hat{c}v_{i,m}^p(\mathbf{r})), \psi_i(\mathbf{r})\nabla \times \nabla \times (\hat{c}v_{i,n}^p(\mathbf{r}))\}$ is defined on each Ω_i as a product of PU function $\psi_i(\mathbf{r})$ and local approximation function $v_{i,n}^p$ [2]. As the problem is solved using the VGFEM with Galerkin's method, the following boundary condition [3] is imposed at the aperture,

$$\hat{\mathbf{z}} \times \frac{1}{\mu_r} [\nabla \times \mathbf{u}(\mathbf{r})] - 2k_0^2 \hat{\mathbf{z}} \times \int_{S_a} [\hat{\mathbf{z}} \times \mathbf{u}(\mathbf{r})] \cdot \bar{\bar{G}}_0(\mathbf{r}, \mathbf{r}') dS' = -2jk_0 Z_0 [\hat{\mathbf{z}} \times \mathbf{H}^i(\mathbf{r})], \quad (2)$$

where $\mathbf{H}^i(\mathbf{r})$ is the incident magnetic field, and k_0 and Z_0 are the free space wavenumber and impedance, respectively. The function $\bar{\bar{G}}_0$ is the free space dyadic green's function. For a smoother convergence, boundary condition on the normal components of the field

$$\hat{\mathbf{z}} \cdot \epsilon_r \mathbf{u}(\mathbf{r}) - 2\hat{\mathbf{z}} \cdot \int_{S_a} [(\mathbf{u}(\mathbf{r}) \times \hat{\mathbf{z}}) \times \nabla G_0(\mathbf{r}, \mathbf{r}')] dS' = 2\mathbf{E}^i(\mathbf{r}) \cdot \hat{\mathbf{z}}, \quad (3)$$

is **also** imposed, where $\mathbf{E}^i(\mathbf{r})$ is the incident electric field and G_0 is the free space green's function. In addition, Nitsche's method is applied for the Dirichlet boundary condition on the PEC walls. With these boundary conditions, the bilinear form of the VGFEM-BI is written as

$$\begin{aligned} & \int_{\Omega_i} (\nabla \times \mathbf{w}) \cdot \left(\frac{1}{\mu_r} \nabla \times \mathbf{u}\right) d\Omega - k_0^2 \epsilon_r \int_{\Omega_i} \mathbf{w} \cdot \mathbf{u} d\Omega + \beta \int_{\Omega_i} (\nabla \cdot \mathbf{w})(\nabla \cdot \mathbf{u}) d\Omega + \\ & \int_{S_{pec}} \left[(\mathbf{w} \times \hat{\mathbf{n}}) \cdot \left(\frac{1}{\mu_r} \nabla \times \mathbf{u}\right) + (\mathbf{u} \times \hat{\mathbf{n}}) \cdot \left(\frac{1}{\mu_r} \nabla \times \mathbf{w}\right) \right] dS + \\ & \beta_1 \int_{S_{pec}} (\mathbf{w} \times \hat{\mathbf{n}}) \cdot (\mathbf{u} \times \hat{\mathbf{n}}) dS - 2k_0^2 \int_{S_a} \left[(\mathbf{w} \times \hat{\mathbf{z}}) \cdot \int_{S_a} G_0(\mathbf{u} \times \hat{\mathbf{z}}) dS' \right] dS - \\ & 2 \oint_C \left[\int_{S_a} G_0 \nabla' \cdot (\mathbf{u} \times \hat{\mathbf{z}}) dS' - \oint_C G_0(\mathbf{u} \times \hat{\mathbf{z}}) \cdot \hat{\mathbf{n}}_l dl' \right] (\mathbf{w} \times \hat{\mathbf{z}}) \cdot \hat{\mathbf{n}}_l dl + \\ & 2 \int_{S_a} \left[\int_{S_a} G_0 \nabla' \cdot (\mathbf{u} \times \hat{\mathbf{z}}) dS' - \oint_C G_0(\mathbf{u} \times \hat{\mathbf{z}}) \cdot \hat{\mathbf{n}}_l dl' \right] \nabla \cdot (\mathbf{w} \times \hat{\mathbf{z}}) dS + \\ & \beta_2 \int_{S_a} (\mathbf{w} \cdot \hat{\mathbf{z}})(\epsilon_r \mathbf{u} \cdot \hat{\mathbf{z}}) dS - 2\beta_2 \int_{S_a} (\mathbf{w} \cdot \hat{\mathbf{z}}) \left[\hat{\mathbf{z}} \cdot \int_{S_a} [(\mathbf{u} \times \hat{\mathbf{z}}) \times \nabla G_0(\mathbf{r}, \mathbf{r}')] dS' \right] dS = \\ & 2jk_0 Z_0 \int_{S_a} (\mathbf{w} \times \hat{\mathbf{z}}) \cdot \mathbf{H}^i(\mathbf{r}) dS + 2\beta_2 \int_{S_a} (\mathbf{w} \cdot \hat{\mathbf{z}})(\mathbf{E}^i(\mathbf{r}) \cdot \hat{\mathbf{z}}), \end{aligned} \quad (4)$$

where $\mathbf{w}$ represents testing functions, Ω_i is the volume of the support domain of the testing function, S_{pec} and S_a are the surfaces of the domain of testing function on the PEC and aperture respectively, $\hat{\mathbf{n}}$ is the outward unit normal vector of the support domain, and $\hat{\mathbf{n}}_l$ is the outward normal vector of the contour C that bounds the

partition of unity domain. The constants β, β_1 and β_2 contribute uniform convergence [2]. Unlike FEM formulation, the line integrals appear in the formulation as the derivatives of the basis functions may not be continuous across the boundaries. In FEM-BI, auxiliary basis functions are defined at the aperture. However, it is well known that the definition of this auxiliary space is not possible in VGFEM as it does not satisfy the Babuska-Brezzi condition. Thus, volumetric vector basis functions are forced to satisfy boundary conditions. This is achieved by constructing the PU domains from the geometry brick elements such that there exists boundary PU domains that are defined around each boundary node. Note, the singular integrals in Eq. (4) are evaluated using rules presented in [4], and the other integrals are evaluated using Gauss-Legendre rule for the prescribed accuracy.

Results

We validate the VGFEM-BI by simulating scattering from cavity backed apertures. First, RCS of an empty cavity with the size of $1.5\lambda_0 \times 1.5\lambda_0 \times 0.6\lambda_0$ is simulated. Legendre polynomials is used on a cubic PU domain with an approximate edge length of $h_p = \lambda_0/2.5$. Fig. 1(a) compares RCS of the cavity computed by VGFEM-BI and FEM-BI [5] and shows the $-p$ convergence of VGFEM-BI. VGFEM-BI results with higher orders match with FEM-BI results well. Next, we simulate RCS of a cavity

Figure 1: Backscattered RCS of an empty and filled cavities for $E^i = \hat{\theta}e^{-j\mathbf{k}\cdot\mathbf{r}}$. RCS of filled cavity with $\epsilon_r = 4$ is compared against measurement [6] at 9.2GHz.

with the size of $2.89\text{ in} \times 2.1\text{ in} \times 0.057\text{ in}$ at 9.2 GHz. The cavity is filled with a material having $\epsilon_r = 4$. VGFEM-BI results are very close to the measurement results obtained in [6] as shown in Fig. 1(b). Finally, RCS of a $1\lambda_0 \times 0.25\lambda_0 \times 0.25\lambda_0$ cavity filled with a material having $\epsilon_r = 7 - j1.5$ and $\mu_r = 1.8 - j0.1$ is simulated. Cubic PU domain with edge length of approximately $h_p = \lambda_0/6$ and the polynomials order of $p = 2$ are used for VGFEM-BI. Figure 2 shows excellent agreement between VGFEM-BI and FEM-MoM results [7].

Conclusion

In this paper, we have developed a hybrid VGFEM-BI formulation for the analysis of scattering from filled cavities. Since surface basis functions are not supported in

Figure 2: Backscattered RCS of $1\lambda_0 \times 0.25\lambda_0 \times 0.25\lambda_0$ cavity filled with $\epsilon_r = 7 - j1.5$ and $\mu_r = 1.8 - j0.1$

the framework of VGFEM, volumetric vector basis functions are forced to impose boundary conditions at the aperture. This is achieved by constructing conformal PU domains from the geometry mesh. The method has been validated by comparing the simulated radar cross sections of filled cavities against measurement and FEM-BI data. Our current research is on the application of the method for large scattering and radiation problems using tailored basis functions, and these results will be presented at the conference.

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