A COMPARISON OF THE CONTINUOUS GAUSSIAN ANTENNA WITH THE DISCRETE BINOMIAL ARRAY

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In this paper we would like to elaborate on an earlier article [1] in which we discussed the limiting case of the binomial antenna: The Gaussian antenna. Several examples of near Gaussian field patterns have been reported in the literature such as: the field across a single mode fiber optic transmission line (Marcuse [2]) or the field across a dielectric rod antenna (Yaghjian and Kornhauser [3]) or the field across the aperture of a corrugated horn (Clarricoats and Oliver [4]).

If the field across an aperture is given in terms of a coordinate, x , by a Gaussian distribution whose standard deviation is σ , e.g.

$$E_A(x) = E_0 \exp(-x^2/2\sigma^2) \quad (1)$$

then the far-field pattern can be obtained by the Fourier transform of this:

$$E_G(\theta) = \int_{-\infty}^{\infty} E_0 \exp(-x^2/2\sigma^2) \exp(jkx \cos\theta) dx$$

which reduces to:

$$E_G(\theta) = E_0 \exp\left\{-\frac{1}{2} (k^2 \cos^2\theta) / (\sigma^2)\right\} \quad (2)$$

where $k = 2\pi/\lambda$, λ is the wave length and θ is the angle with the x axis to the aperture. Viz: the far-field is also Gaussian, but the variance is inverted.

Utilizing standard definitions, the directivity of an aperture excited by a Gaussian distribution is:

$$D_G = \frac{4\pi}{\int_0^\pi \int_0^\pi \left\{ \exp\left[-\frac{1}{2} k^2 \sigma^2 \cos^2\theta\right] \right\}^2 \sin(\theta) d\theta d\phi} = \frac{2k\sigma}{\int_{-k\sigma}^{k\sigma} \exp(t^2) dt}$$

and, for very large values of $k\sigma$:

$$D_G \sim \frac{2k\sigma}{\sqrt{\pi}} \quad (3)$$

The half-power beam width is given by squaring the far-field and equating it to half the peak value,

$$E_0^2 (\exp [-\frac{1}{2} k^2 \sigma^2 \cos^2 \theta])^2 = \frac{1}{2} E_0^2 \quad (4)$$

This yields $\cos^2 \theta = (\ln 2)/(k^2 \sigma^2)$

The maximum field strength occurs at $\theta = \pi/2$ and setting $\phi = \pi/2 \pm \theta$, the half-power beam width, HPBW, is 2ϕ where:

$$\begin{aligned} \sin^2 \phi &= \frac{\ln 2}{k^2 \sigma^2} \\ \sin(\text{HPBW}) &= \frac{2\sqrt{\ln 2}}{k\sigma} \sqrt{1 - \frac{\ln 2}{k^2 \sigma^2}} \end{aligned} \quad (5)$$

If σ is very large compared to λ , the angle will be small and

$$\text{HPBS} \sim \frac{2\sqrt{\ln 2}}{k\sigma} \quad (6)$$

The corresponding results for the binomial antenna have been given by Ricardi [5]. The (complex) far field strength is given by

$$E_B(\theta) = \sum_{n=0}^{N-1} E_0 \frac{(N-1)!}{(N-n-1)! n!} [\exp(jkd \cos \theta)]^n \quad (7)$$

where $k = 2\pi/\lambda$, d is the (uniform) spacing between elements, λ (the wavelength) $= d/2$, and E_0 the field strength of the first of the $N-1$ elements. The directivity and half power beam width, as computed by Ricardi, are:

$$D_B = \sqrt{\pi(N-1)} \quad (8)$$

$$\text{HPBW} = \frac{1.06}{\sqrt{N-1}} \text{ (radians)} \quad (9)$$

To compare these expressions with the Gaussian antenna; sum (7), multiply by its complex conjugate and normalize the result to unity

$$|E_B(\theta)|^2 = \{\cos(\frac{1}{2} kd \cos \theta)\}^{N-1}$$

Both this result and that of equation (2) can be expanded in a power series;

$$E(\theta) = a_0 + a_2 \cos^2 \theta + a_4 \cos^4 \theta + a_6 \cos^6 \theta + \dots$$

with coefficients as show in the table:

	Binomial	Gaussian	Difference
a_0	1	1	0
a_2	$-\frac{N-1}{2!} \frac{k^2 d^2}{2^2}$	$-\frac{k^2 \sigma^2}{2}$	0
a_4	$\frac{(N-1)(3N-5)}{4!} \frac{k^4 d^4}{2^4}$	$\frac{k^4 \sigma^4}{2! 2^2}$	$\frac{k^4 d^4}{2! 2^6} [-2/3]$
a_6	$-\frac{(N-1)[15(N-1)^2 - 30(N-1) + 16]}{6!} \frac{k^6 d^6}{2^6}$	$-\frac{k^6 \sigma^6}{3! 2^3}$	$\frac{k^6 d^6}{3! 2^6} (N-1)[-2(N-1) + \frac{16}{15}]$

The column labelled "Difference" indicates the difference between the coefficients when $\sigma = d/2 \sqrt{N-1}$. These differences, as a fraction of the coefficient, are small and will get smaller as N (respectively: σ) gets larger. Let us emphasize that the Gaussian antenna is not an exact limiting case of the Binomial antenna, since

$$\lim_{N \rightarrow \infty} |E_B(\theta)/E_G(\theta)|$$

is zero except for $\theta = \pi/2$ where it is unity. Rather the equivalence of the field strengths should be understood as an asymptotic one in that, for large N (respectively: σ), the fields are approximately equal, but the range of values of θ for which this is true grows smaller with increasing N.

Ricardi's formulas for directivity (8) and half power beam width (9) may be compared with the Gaussian formulas (3) and (6) by setting $\sigma = 1/2d\sqrt{N-1}$. The results are identical if N (respectively: σ) is very large. Numerical calculations of the far field show that the patterns of a Gaussian antenna of a Binomial antenna are not significantly different for $N=5$.

BIBLIOGRAPHY

1. Krall, A. D., Jablonski, D. G., and Coughlin, J. P., "Radiation Properties of a Gaussian Antenna," *Microwave Journal*, Vol. 27, No. 25, May 1984.
2. Marcuse, D., "Radiation Losses of the Dominant Mode in Round Dielectric Waveguide," *Bell System Technical Journal*, Vol. 49, No. 8, Oct 1970.
3. Yaghjian, A. D., and Kornhausser, E. T., "A Modal Analysis of the Dielectric Rod Antenna," *IEEE Trans. Ant. Prop.*, Vol. AP20, No. 2, Mar 1972.
4. Clarricoats, P. J., and Olver, A. D., Corrugated Horns for Microwave Antennas, P. Peregrinus Ltd., London, UK, 1984.
5. Ricardi, L. J., "Radiation Properties of Binomial Array," *Microwave Journal*, Vol. 15, No. 12, Dec 1972.