GUIDED WAVE PROPAGATION BY THE SUPERPOSITION OF PLANE WAVES IN TRIANGULAR WAVEGUIDES WITH PERFECTLY CONDUCTING WALLS

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Introduction

The mode solutions for the isosceles right triangular uniform waveguide are available in the literature [1]. However, solutions (other than numerical approximations) appear to be lacking for other triangular cross sections. Since the complete set of modes for a metallic rectangular guide can be found by superimposing four plane waves [2], it seems reasonable to apply the same approach to other guides of polygonal cross section. This paper is concerned with triangular waveguides of general shape.

General Procedure

Plane waves of the form $\exp(-jk_0 \hat{k} \cdot \vec{r})$ are assumed to be propagating down the guide by reflecting off the walls in accordance with Snell's law which in vector form [3] is

$$\hat{k}_r = \hat{k}_i - 2 \hat{N}_j (\hat{N}_j \cdot \hat{k}_i) \quad (1)$$

where $\hat{k}_r$ is the reflected unit propagation vector, $\hat{k}_i$ is the incident unit propagation vector, and $\hat{N}_j$ is a unit vector normal to the j^{th} waveguide wall. Where many reflections are required to form a complete set of propagation vectors, a more convenient form is

$$\begin{bmatrix} k_{rx} \\ k_{ry} \end{bmatrix} = \begin{bmatrix} 1-2N_{jx}^2 & -2N_{jx}N_{jy} \\ -2N_{jx}N_{jy} & 1-2N_{jy}^2 \end{bmatrix} \begin{bmatrix} k_{ix} \\ k_{iy} \end{bmatrix} \quad (2)$$

The waveguide cross section is in the xy -plane and propagation is in the z -direction. All of the propagation vectors $\hat{k}_r$ must have the same z -component, since the unit normals $\hat{N}_j$ have none. This is, in fact, physically required, since all of the superimposed plane waves must have the same phase velocity down the guide.

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To find the guided modes, an initial propagation vector

$$\hat{k}_1 = \alpha \hat{X} + \beta \hat{Y} \quad (3)$$

(+ $\gamma \hat{Z}$ understood, with α , β , and γ , the direction cosines of the vector) is assumed, and Equation (1) or (2) is applied repeatedly until the complete set of propagation vectors is found, or until it becomes evident that the set is infinite. While no difficult mathematics is involved, the procedure can be tedious and repetitive, with the attendant chance of a mistake. We have found computer algebra [4], specifically Mu math, to be of considerable help in the task.

Assuming a finite complete set of $\hat{k}_n$ can be found, the computation of the field distribution is straightforward. Set

$$A_z = [\exp(-jk_0 z)] \sum_{n=1}^N A_{0zn} \exp(-jk_0 \hat{k}_n \cdot \vec{R}) \quad (4)$$

where A_z is E_z for TM modes, H_z for TE modes, and $\vec{R} = x\hat{x} + y\hat{y}$. The other field components can be found from the usual relations. Since the superimposed plane waves are reflections of one another off the guide walls, all of the A_{0zn} must have equal magnitudes and differ only in phase. The phase relations can be found by applying the usual boundary conditions

$$E_z = 0 ; \quad \frac{\partial H_z}{\partial n} = 0 \quad (5)$$

at the guide walls.

Triangular Guide

Our work to date has concentrated on guides with a general right triangular or isosceles triangular cross section. Contrary to our initial expectations, in both cases there is no finite number of superimposed plane waves which yield a complete solution in the general case. However, such sets do exist for particular triangular cross sections.

For example, in the case of an equilateral triangle, six vectors given by

$$\begin{aligned}
\hat{k}_1 &= \alpha \hat{x} + \beta \hat{y} \\
\hat{k}_2 &= -\alpha \hat{x} + \beta \hat{y} \\
2\hat{k}_3 &= (\alpha + \beta\sqrt{3})\hat{x} + (\alpha\sqrt{3} - \beta)\hat{y} \\
2\hat{k}_4 &= (\alpha - \beta\sqrt{3})\hat{x} - (\alpha\sqrt{3} + \beta)\hat{y} \\
2\hat{k}_5 &= -(\alpha - \beta\sqrt{3})\hat{x} - (\alpha\sqrt{3} + \beta)\hat{y} \\
2\hat{k}_6 &= -(\alpha + \beta\sqrt{3})\hat{x} + (\alpha\sqrt{3} - \beta)\hat{y}
\end{aligned} \tag{6}$$

are required. In this case, the above set of vectors yields a complete set of TE and TM modes with transverse propagation constants of $N\pi/a\sqrt{3}$ and $M\pi/a\sqrt{3}$, where a is the length of a triangular side.

For right triangles complete sets are found when the two acute angles bear an integer relation, with 8 plane waves required for an isosceles right triangle, 12 for a 30° - 60° triangle, 16 for a 22.5° - 67.5° triangle, and so on.

Conclusions

Superimposing a finite number of plane waves allows the determination of the complete mode structure of metallic waveguides with certain triangular cross sections. The longitudinal electric and magnetic fields are then composed of a finite sum of rectangular harmonics. The general triangular cross section, however, requires an infinite sum of superimposed plane waves; hence, an infinite sum of rectangular harmonics is required for a solution. It is still an open question as to how many plane waves will yield a good approximation for a particular case. Perhaps closed-form solutions for the general triangular waveguide can be found in nonseparable solutions to the wave equation [5].

References

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