SCATTERING OF TRANSIENT WAVES BY A DISPERSIVE BODY

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1. Introduction

The scattering of transient electromagnetic waves by a non-dispersive medium was previously reduced to the solution of a weakly singular integral equation of the first kind for a single tangential vector field defined on the surface of the body [1]. The corresponding formulation for monochromatic waves was extended in [1] to conducting and dispersive media without any additional work, but a similar generalization for transient fields involves convolution products and other special considerations discussed in this presentation.

The incident pulse is specified by the fields $\underline{B}$ and $\underline{D}$ at the initial time $t=0$ in the region outside the scatterer, and it is determined at later times by integrations involving the Green function for the scalar equation. Two sets of auxiliary fields defined in both regions of space are expressed in terms of a tangential field which obeys an integral equation. The scattered and transmitted fields can be obtained from this tangential field by integrations. The Green function and some of the integrals are shown for the case of the simple conducting medium.

2. Equations for the Electromagnetic Fields

The relation between $\underline{E}$ and $\underline{D}$ in a dispersive medium is

$$\underline{D}(\underline{x}, t) = \epsilon_0 \underline{E}(\underline{x}, t) + \int_0^\infty dt' \chi(t') \underline{E}(\underline{x}, t-t'), \quad (1)$$

where $\chi(t)$ is the Fourier transform of the susceptibility given as a function of the frequency ω . Similarly, the current density in a dispersive medium is related to the field $\underline{E}$ by

$$\underline{j}(\underline{x}, t) = \int_0^\infty dt' \sigma(t') \underline{E}(\underline{x}, t-t'), \quad (2)$$

where $\sigma(t)$ is the Fourier transform of the conductivity $\sigma(\omega)$. The corresponding Maxwell equation reduces to

$$\nabla \times \underline{B}(\underline{x}, t) - \mu_0 \epsilon_0 \partial \underline{E}(\underline{x}, t) / \partial t - \int_0^\infty dt' \gamma_1(t') \underline{E}(\underline{x}, t-t') = 0, \quad (3)$$

where $\gamma_1 = \mu_0 [\epsilon_0 \partial \chi(t) / \partial t + \sigma(t)]$. The Green function G satisfies a related scalar wave equation with delta-function sources,

$$\epsilon_0 \mu_0 \partial^2 G(\underline{x}, t) / \partial t^2 + (\partial / \partial t) \int_0^\infty dt' \gamma_1(t') G(\underline{x}, t-t') - \nabla^2 G(\underline{x}, t) = \delta(\underline{x}) \delta(t). \quad (4)$$

The incident field can then be found from the initial values by

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$$\underline{E}^{\text{in}} = -\mu_0 \int dV' [\partial G(\underline{x}-\underline{x}', t) / \partial t' \underline{D}_0(\underline{x}') + \nabla' G(\underline{x}-\underline{x}', t) \times \underline{B}_0(\underline{x}')], \quad (5)$$

$$\underline{B}^{\text{in}} = -\int dV' [G'(\underline{x}-\underline{x}', t) \underline{B}_0(\underline{x}') - \epsilon_0 \mu_0 \nabla' G(\underline{x}-\underline{x}', t) \times \underline{D}_0(\underline{x}')], \quad (6)$$

where

$$G'(\underline{x}, t) = \epsilon_0 \mu_0 \partial G(\underline{x}, t) / \partial t + \int_0^\infty dt' \gamma_1(t') G(\underline{x}, t-t'). \quad (7)$$

An electromagnetic field that vanishes initially and at later times satisfies jump conditions $\Delta \underline{E} = \underline{\phi}$ and $\Delta \underline{B} = \underline{\eta}$ on the surface S can be expressed in terms of the jumps by

$$\begin{aligned} \underline{E}(\underline{x}, t) = & \int_0^t dt' \oint_S dS' [(\underline{n}' \times \underline{\phi}(\underline{x}', t')) \times \nabla' G(\underline{x}-\underline{x}', t-t') \\ & + \underline{n}' \cdot \underline{\phi}(\underline{x}', t') \nabla' G(\underline{x}-\underline{x}', t-t') + \underline{n}' \times \underline{\eta}(\underline{x}', t') \partial G(\underline{x}-\underline{x}', t-t') / \partial t'], \quad (8) \end{aligned}$$

$$\begin{aligned} \underline{B}(\underline{x}, t) = & \int_0^t dt' \oint_S dS' [(\underline{n}' \times \underline{\eta}(\underline{x}', t')) \times \nabla' G(\underline{x}-\underline{x}', t-t') \\ & + \underline{n}' \cdot \underline{\eta}(\underline{x}', t') \nabla' G(\underline{x}-\underline{x}', t-t') - \epsilon_0 \mu_0 \underline{n}' \times \underline{\phi}(\underline{x}', t') G'(\underline{x}-\underline{x}', t-t')], \quad (9) \end{aligned}$$

where $\underline{n}'$ is the unit normal to the surface S at $\underline{x}'$ pointing into the scatterer. Maxwell's equations imply that the normal components of the jumps are related to the tangential ones by

$$\begin{aligned} \epsilon_0 \mu_0 \partial \underline{n} \cdot \underline{\phi}(\underline{x}, t) / \partial t + \int_0^\infty dt' \gamma_1(t') \underline{n} \cdot \underline{\phi}(\underline{x}, t-t') \\ = -\nabla_s \cdot \underline{n} \times \underline{\eta}(\underline{x}, t), \quad (10) \end{aligned}$$

$$\partial \underline{n} \cdot \underline{\eta}(\underline{x}, t) / \partial t = \nabla_s \cdot \underline{n} \times \underline{\phi}(\underline{x}, t), \quad (11)$$

where $\nabla_s \cdot \underline{u}$ is the surface divergence of a field $\underline{u}$. Since the fields $\underline{\phi}$ and $\underline{\eta}$ vanish initially, Eqs. (10) and (11) give the normal components of these fields in terms of the tangential components; thus, the fields $\underline{E}$ and $\underline{B}$ in Eqs. (8) and (9) are functionals of these tangential fields. We write, symbolically,

$$\underline{E} = \underline{L}\{\underline{n} \times \underline{\phi}\} + \underline{M}\{\underline{n} \times \underline{\eta}\}, \quad (12)$$

$$\underline{B} = \underline{L}\{\underline{n} \times \underline{\eta}\} + \underline{M}'\{\underline{n} \times \underline{\phi}\}. \quad (13)$$

The operators $\underline{L}$, $\underline{M}$, and $\underline{M}'$ depend on the form of the Green function and, at least in simple cases, they involve weakly singular integrals that can be evaluated unambiguously by a limiting procedure similar to the definition of the principal part of an integral.

3. The Integral Equation

Since the permeability of both media is assumed to be μ_0 , the boundary conditions on the physical fields are the continuity of the tangential components of $\underline{E}$ and $\underline{B}$ across the surface S . To be able to use equations (8) and (9), the fields have to obey the same equations in both regions of space. Two such sets of

fields are defined. The auxiliary fields $\underline{E}_1$ and $\underline{B}_1$ are equal to the scattered fields outside the scatterer, they obey the same equations inside as outside the scatterer, and the tangential component of $\underline{E}_1$ is continuous across the surface. The auxiliary fields $\underline{E}_2$ and $\underline{B}_2$ are equal to the transmitted fields inside the scatterer and vanish on the outside. The one unknown tangential field to be determined from an integral equation is the jump $\underline{n}$ of $\underline{E}_1$ across the surface. The fields $\underline{E}_1$ and $\underline{B}_1$ are expressed in terms of $\underline{n}$ via Eqs. (12) and (13) by

$$\underline{E}_1 = \underline{M}_1\{\underline{n} \times \underline{n}\}, \quad (14)$$

$$\underline{B}_1 = \underline{L}_1\{\underline{n} \times \underline{n}\}. \quad (15)$$

The jumps in the fields $\underline{E}_2$ and $\underline{B}_2$ across the surface are determined from the boundary conditions to be

$$\Delta(\underline{n} \times \underline{E}_2) = \underline{n} \times \underline{E}_1^{\text{in}} + \underline{n} \times \underline{E}_1^{\text{out}}, \quad (16)$$

$$\Delta(\underline{n} \times \underline{B}_2) = \underline{n} \times \underline{B}_1^{\text{in}} + \underline{n} \times \underline{B}_1^{\text{out}}, \quad (17)$$

where $\underline{E}_1^{\text{in}}$ and $\underline{B}_1^{\text{in}}$ are the values of the fields in Eqs. (14) and (15) when the surface is approached from outside the scatterer. Then $\underline{E}_2$ and $\underline{B}_2$ are expressed in terms of $\underline{n}$ through these expressions by

$$\underline{E}_2 = \underline{L}_2\{\Delta(\underline{n} \times \underline{E}_2)\} + \underline{M}_2\{\Delta(\underline{n} \times \underline{B}_2)\}, \quad (18)$$

$$\underline{B}_2 = \underline{L}_2\{\Delta(\underline{n} \times \underline{B}_2)\} + \underline{M}_2\{\Delta(\underline{n} \times \underline{E}_2)\}. \quad (19)$$

These fields do not vanish outside the body for arbitrary $\underline{n}$. The condition that $\underline{n} \times \underline{E}_1^{\text{out}}$, for instance, vanish gives rise to the integral equation,

$$\begin{aligned} & \underline{n} \times (\underline{M}_1/2 - \underline{L}_2 \underline{M}_1 + \underline{M}_2/2 - \underline{M}_2 \underline{L}_1) \{\underline{n} \times \underline{n}\} \\ & + \underline{n} \times (\underline{E}_1^{\text{in}}/2 - \underline{L}_2 \{\underline{n} \times \underline{E}_1^{\text{in}}\} - \underline{M}_2 \{\underline{n} \times \underline{B}_1^{\text{in}}\}) = 0, \end{aligned} \quad (20)$$

where composite operators are defined as in

$$\underline{n} \times \underline{L}_2 \underline{M}_1 \{\underline{n} \times \underline{n}\} = \underline{n} \times \underline{L}_2 \{\underline{n} \times \underline{M}_1 \{\underline{n} \times \underline{n}\}\}. \quad (21)$$

4. The Simple Conductor

This is a medium where the permittivity ϵ and the conductivity σ are constants, and where dispersion comes from the conductivity. Although physically unrealizable, this medium can be a useful approximation to real materials. (In particular, Eq. (1) does not apply.) The Green function is

$$\begin{aligned} G(\underline{x}, t) &= \exp(-\sigma t/2\epsilon) [\delta(t-r/v)/4\pi r \\ &+ \sigma(8\pi\epsilon v/(t^2-r^2/v^2))^{-1} I_1(\sigma/(t^2-r^2/v^2)/2\epsilon) \theta(t-r/v)], \end{aligned} \quad (22)$$

where I_1 is the modified Bessel function and θ is the unit step function. The operators in Eqs. (12) and (13) are

$$\begin{aligned} \underline{L}\{\underline{n} \times \underline{\phi}\} = & (1/4\pi) P \delta_S dS' \exp(-\sigma R/2\varepsilon v) [\underline{n}' \times (\dot{\underline{\phi}}_{\text{ret}}/v \\ & + \sigma \underline{\phi}_{\text{ret}}/2\varepsilon v + \underline{\phi}_{\text{ret}}/R)] \times R/R^2 + \nabla \times \underline{I}\{\underline{n} \times \underline{\phi}\}, \end{aligned} \quad (23)$$

$$\begin{aligned} \underline{M}\{\underline{n} \times \underline{\eta}\} = & (1/4\pi) P \delta_S dS' \exp(-\sigma R/2\varepsilon v) [\underline{n}' \cdot (\dot{\underline{\phi}}_{\text{ret}}/v + \sigma \underline{\phi}_{\text{ret}}/2\varepsilon v \\ & + \underline{\phi}_{\text{ret}}/R) R/R^2 - \underline{n}' \times \dot{\underline{\eta}}_{\text{ret}}/R] + \nabla \underline{I}\{\underline{n} \cdot \underline{\phi}\} - \underline{I}\{\underline{n} \times \dot{\underline{\eta}}\}, \end{aligned} \quad (24)$$

$$\begin{aligned} \underline{M}'\{\underline{n} \times \underline{\phi}\} = & (1/4\pi) P \delta_S dS' \exp(-\sigma R/2\varepsilon v) [\underline{n}' \cdot (\dot{\underline{\eta}}_{\text{ret}}/v + \sigma \underline{\eta}_{\text{ret}}/2\varepsilon v \\ & + \underline{\eta}_{\text{ret}}/R) R/R^2 + \underline{n}' \times \dot{\underline{\phi}}_{\text{ret}}/v^2 R + \mu_0 \sigma \underline{n} \times \underline{\phi}_{\text{ret}}/R] \\ & + \nabla \underline{I}\{\underline{n} \cdot \underline{\eta}\} + \underline{I}\{\underline{n} \times \dot{\underline{\phi}}/v^2 + \mu_0 \sigma \underline{n} \times \underline{\phi}\}, \end{aligned} \quad (25)$$

where the subscript "ret" means that the field is a function of the retarded time $t' = t - R/v$, the "P" before an integral sign indicates that a kind of principal value has to be taken, the speed of propagation v is $\sqrt{1/\varepsilon\mu_0}$, the functional I stands for

$$\begin{aligned} \underline{I}\{\underline{\xi}\} = & (\sigma/8\pi\varepsilon v) \delta_S dS' \int_{R/v}^{\infty} dt' \exp(-\sigma t'/2\varepsilon) [1/\sqrt{(t'^2 - R^2/v^2)}] \\ & \cdot \underline{I}_1[\sigma\sqrt{(t'^2 - R^2/v^2)}/2\varepsilon] \underline{\xi}(\underline{x}', t-t'), \end{aligned} \quad (26)$$

and $\underline{I}$ is obtained from I by replacing ξ by a vector $\underline{\xi}$ in Eq.(26). The weakly singular integrals are similar to those that appear for the simple dielectric, and there is a damping factor in all integrals. If the damping is strong, the marching-in-time procedure would not involve many previous time values for the evaluation of the integrals.

Another medium that leads to expressions similar to those found for the simple conductor is the collisionless plasma. The corresponding Green function differs from the one given by Eq. (22) in that it has no damping factors and the modified Bessel function I_1 is replaced by the Bessel function J_1 .

5. Conclusions

Solutions of scattering of electromagnetic waves by homogeneous bodies can be extended to dispersive media without additional work for monochromatic waves. The modifications required for pulses involve the introduction of some convolution integrals, and the determination of the appropriate Green function from its Fourier transform. In particular, the formalism that reduces the scattering problem by penetrable bodies to a single integral equation can still be used.

Reference

- [1] E. Marx, J. Math. Phys. 23, 1057 (1982).